On a p-adic invariant cycles theorem

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Abstract - For a proper semistable curve X over a DVR of mixed characteristics we reprove the "invariant cycles theorem" with trivial coefficients (see [Ch99]) i.e. that the group of elements annihilated by the monodromy operator on the first de Rham cohomology group of the generic fiber of X coincides with the first rigid cohomology group of its special fiber, without the hypothesis that the residue field of $\mathcal{V}$ is finite. This is done using the explicit description of the monodromy operator on the de Rham cohomology of the generic fiber of X with coefficients convergent F-isocrystals given in [CoIo10]. We apply these ideas to the case where the coefficients are unipotent convergent F-isocrystals defined on the special fiber (without log-structure): we show that the invariant cycles theorem does not hold in general in this setting. Moreover we give a sufficient condition for the non exactness.

1 Introduction

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics, K its fraction field and k the residue field, which we assume to be perfect. Let $W := W(k)$ denote the ring of Witt-vectors with coefficients in k seen as a subring of $\mathcal{V}$ and let K_0 denote its fraction field.

For a proper variety X over $\mathcal{V}$ with semistable reduction and special fiber X_k, via the theory of log schemes and the work of Hyodo-Kato one defines a monodromy operator on the de Rham comology groups of its generic fiber X_K. It has been known for some time now that associated to this operator there is an analogue of the classical invariant cycles sequence [Ch99]

$$H^i_{\mathrm{rig}}(X_k) \otimes_{K_0} K \to H^i_{\mathrm{dR}}(X_K) \to H^i_{\mathrm{dR}}(X_K).$$

The exactness of such a sequence is implied by the weight-monodromy conjecture [Ch99] if the residue field k is finite. Hence the above invariant cycles sequence is exact if X is a curve or a surface (which are the cases in which the weight-monodromy conjecture is known) and in this case the first map is even injective if $i = 1$ i.e. the following sequence is exact:

$$0 \to H^1_{\mathrm{rig}}(X_k) \otimes_{K_0} K \to H^1_{\mathrm{dR}}(X_K) \to H^1_{\mathrm{dR}}(X_K). \tag{1}$$

In these cases (i.e. in the cases in which the sequence (1) is exact) we obtain an interpretation of the part of the de Rham cohomology which is annihilated by the monodromy operator: it is the rigid cohomology group of the special fiber. On the other hand the same exact sequence gives us an interpretation à la Fontaine of the first rigid cohomology group, in fact we can translate the exactness as follows: since

$$D_{\mathrm{st}}(H^1_{\text{ét}}(X_K \times \overline{K}), \mathbb{Q}_p) = H^1_{\mathrm{log-crys}}(X_k) \otimes K$$

$$D_{\mathrm{st}}^{N=0} = D_{\mathrm{crys}},$$

then

$$H^1_{\mathrm{rig}}(X_k) = D_{\mathrm{crys}}(H^1_{\text{ét}}(X_K \times \overline{K}), \mathbb{Q}_p)$$

In [CoIo10] it was given a new definition of a monodromy operator in the case X is a curve with semistable reduction using the combinatorics of the curve together with the use of the analytic spaces associated to the generic fiber. The authors also considered the case of cohomology with coefficients and generalized the definition of the monodromy operator on the de Rham cohomology with coefficients non trivial log-F-isocrystals and they showed that it coincides with the previous definition given by Faltings [Fa]. Using this definition of the monodromy operator we are able (see §5) to re-prove the exactness of the invariant cycles sequence (1) without any hypothesis on the finiteness of the residue field. It is then natural to ask, when the log-F-isocrystals are induced from convergent F-isocrystals on the special fiber, if such an invariant cycles sequence (1) is still exact. This is one of the aims of the present article. As a matter of fact, the invariant cycles sequence (1) involves the trivial convergent F-isocrystal on the special fiber of X and its rigid cohomology. Hence we start with coefficients which a priori do not have singularities being convergent on the special fiber without any log structure. But, even for the simplest non-trivial coefficients on a curve (i.e. the unipotent ones) the sequence fails sometimes to be exact and we give a sufficient condition for such a behavior (see Theorem 10). Underlying our work, of course, is the aim of giving a cohomological interpretation for the part of the cohomology on which the monodromy operator acts as zero.

Of course the invariant cycles theorem can be studied also in the ℓ-adic and respectively the complex settings, where it is known for semi-simple perverse sheaves or $\mathcal{D}$-modules of geometric origin and it follows from the decomposition theorem ([BBD] corollaire 6.2.8, [Mo] theorem 19.47, [Sa], theorem 1, [DeMi], theorem section 1.7). Our p-adic setting deals with unipotent, non-trivial coefficients, which are therefore not semi-simple. We did not find any evidence of a similar result for reducible coefficients in the ℓ-adic or complex settings, although we believe that such results should hold.

Here it is the plan of our article. In §2 we introduce notations and recall results on rigid spaces which will be used in the article, in the third paragraph we recall some properties of the monodromy operator on the de Rham cohomology with coefficients on a curve as introduced by Coleman and Iovita and of the associated invariant cycles sequence. In §4 we give some properties of such a monodromy operator: in particular for general convergent F-isocrystals we prove that the rigid cohomology of the convergent F-isocrystal injects on the part of the de Rham cohomology of the associated log-F isocrystal where the monodromy acts as zero. In §5, we then re-prove ([Ch99]) the invariant cycles theorem for trivial coefficients in a combinatorial way along the lines of the work in [CoIo10]. In §6 we study the invariant cycles sequence for unipotent convergent F-isocrystals and we prove a sufficient conditions for the non exactness of the sequence. Finally we give an explicit example of this on a Tate curve.

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2 Notation and Settings. A Mayer-Vietoris exact sequence

We assume the notations in section 1. Let X be a proper curve over $\mathcal{V}$ that is semistable, which means that locally for the Zariski topology there is an étale map to $\mathrm{Spec}(V[x,y]/xy-\pi)$ and we suppose that the special fiber, union of smooth irreducible components, has at least two components. We denote by X_k the special fiber of X which we suppose connected, by X_K its generic fiber and by X_K^{rig} the rigid analytic generic fiber. By theorem 2.8, in [Li] X being a proper, regular curve over V is in fact a projective V-scheme.

Following [CoIo99] we associate to X_k a graph $Gr(X_k)$ whose definition we now recall. To every irreducible component C_v of X_k we associate a vertex v and if v, w are vertices, an oriented edge $e = [v, w]$ with origin v and end w corresponds to an intersection point C_e of the components C_v and C_w. We denote by $\mathscr{V}$ the set of vertices and by $\mathscr{E}$ the set of oriented edges.
Then we have the specialization map

$$\mathrm{sp} : X_K^{\mathrm{rig}} \to X_k$$

defined in [Be].
For every $v \in \mathscr{V}$ we define

$$X_v := sp^{-1}(C_v)$$

and for every $e \in \mathscr{E}$

$$X_e := sp^{-1}(C_e).$$

The set X_e is an open annulus in X_K^{rig} and X_v is what is called a wide open subspace in ([Co89] proposition 3.3), that means an open of X_K^{rig} isomorphic to the complement of a finite number of closed disks, each contained in a residue class, in a smooth proper curve over K with good reduction. If C_v and C_w intersect in C_e, then $X_v \cap X_w = X_e$.
One can prove that $\{X_v\}_{v\in\mathscr{V}}$ is an admissible covering of X_K^{rig} ([Co89]) and that wide opens are Stein spaces so that we can use the covering $\{X_v\}_{v\in\mathscr{V}}$ to calculate the de Rham cohomology of X_K^{rig} using a Čech complex. Moreover one can prove that the first de Rham cohomology of a wide open is finite ([Co89] theorem 4.2) proving a comparison theorem with the de Rham cohomology of an algebraic curve minus a finite set of points.Let $(\mathcal{E}, \nabla)$ be a module with integrable connection on X_K^{rig}.
Given the admissible covering $\{X_v\}_{v\in\mathscr{V}}$ that is such that their elements intersect only two by two, we can write the Mayer-Vietoris sequence:

$$\begin{tikzcd} \oplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)) \arrow[r, "\alpha"] & \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)) \arrow[r] & H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)) \arrow[dll, rounded corners] \\ & \oplus_{v\in\mathscr{V}} H^1_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)) \arrow[r, "\beta"] & \oplus_{e\in\mathscr{E}} H^1_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)). \end{tikzcd} \tag{2}$$

Let us remark that every cohomology group that appears in the long exact sequence except for $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla))$ can be calculated as the cohomology of the global sections of the de Rham complex, due to the fact that every wide open is Stein.
From the equation (2) we can deduce the short exact sequence

$$0 \longrightarrow H^1(Gr(X_k),\mathcal{E}) \xrightarrow{\gamma} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla))) \longrightarrow \mathrm{Ker}(\beta) \longrightarrow 0. \tag{3}$$

where $H^1(Gr(X_k),\mathcal{E}) := \mathrm{Coker}(\alpha)$.

3 The monodromy operator and the rigid cohomology

We consider again a proper and semistable curve X, its generic fiber X_K and its associated rigid space X_K^{rig}. We recall the construction of the monodromy operator in [CoIo10] section 2.2.
By our assumptions there is a proper scheme P over W, smooth around X_k and such that we have a global embedding $X \hookrightarrow P \times_{\mathrm{Spec(W)}} \mathrm{Spec}(\mathcal{V}) = P_{\mathcal{V}}$. Let us denote by P_k its special fiber and by $P_{K_0}^{\mathrm{rig}}$ and P_K^{rig} the rigid analytic spaces associated to P, and $P_{\mathcal{V}}$ then one has the following diagram:

$$\begin{tikzcd} & & P_{K_0}^{\mathrm{rig}} \arrow[dl, "\mathrm{sp}_P"'] \arrow[d] \\ X_k \arrow[r] & P_k \arrow[r] & P \end{tikzcd}$$

where the map between $P_{K_0}^{\mathrm{rig}}$ and P_k is the specialization map that we denote by sp_P. We also have a specialization map $\mathrm{sp}_{P_{\mathcal{V}}} : P_K^{\mathrm{rig}} \longrightarrow P_k$. One can consider the tubes $\mathrm{sp}_P^{-1}(X_k) :=]X_k[_P$ and $Y_K := sp_{P_{\mathcal{V}}}^{-1}(X_k) =]X_k[_{P_{\mathcal{V}}}$. Let now, E, be a convergent F-isocrystal on X_k. It has a realization on $]X_k[_P$: $(\mathcal{E},\nabla)$ and we denote by $(\mathcal{E},\nabla)_K$ its base change to K. It is a module with connection on Y_K. We will denote by the same symbol its restriction to X_K^{rig}. We may then define the first rigid cohomology group with coefficients in E as

$$H^1_{\mathrm{rig}}(X_k,E) := H^1_{\mathrm{dR}}(]X_k[_P,(\mathcal{E},\nabla)),$$

which is a finite dimensional K_0-vector space. We also consider

$$H^1_{\mathrm{rig}}(X_k,E)_K := H^1_{\mathrm{dR}}(]X_k[_{P_{\mathcal{V}}},(\mathcal{E},\nabla)_K) = H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K).$$

On the other hand we can proceed as before and take X_K^{rig} as the rigid analytic space associated to X_K, we then have

$$\varphi : X_K^{\mathrm{rig}} \longrightarrow Y_K$$

given by the immersion of X into $P_{\mathcal{V}}$ that induces a map in cohomology

$$\varphi^* : H^1_{\mathrm{rig}}(X_k,E)_K := H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)_K). \tag{4}$$

In the notations above we define following [CoIo10] a K-linear map

$$N : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)_K) \to H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)_K).$$

Due to the fact that wide opens are Stein spaces, every element $[\omega]$ in $H^1_{\mathrm{dR}}(X_K,(\mathcal{E},\nabla)_K)$ can be described as a hypercocycle $((\omega_v)_{v\in\mathscr{V}},(f_e)_{e\in\mathscr{E}})$, with (ω_v) in $\Omega^1_{X_v}\otimes\mathcal{E}_{X_v}$ and f_e in $\mathcal{E}_{X_e}$ that verifies that $\omega_{v|X_e}-\omega_{w|X_e} = \nabla(f_e)$ if $e=[v,w]$.
Let us remember that every X_e is an ordered open annulus; we can define a residue map

$$\mathrm{Res} : H^1_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K) \to H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K)$$

as follows. The module with connection $(\mathcal{E},\nabla)_K$ has a basis of horizontal sections $e_1,\ldots,e_n$ on X_e because X_e is a residue class (lemma 2.2 of [CoIo10]). Hence if z is an ordered uniformizer of the ordered annulus X_e every differential form $\mu_e\in H^1_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K)$ can be written as $\mu_e=\sum_{i=1}^n(e_i\otimes\sum_j a_{i,j}z^j dz)$ with $a_{i,j}\in K$. Then $\mathrm{Res}(\mu_e)=\sum_{i=1}^n a_{i,-1}e_i$, and it is an isomorphism of vector spaces.

For a cohomology class $[\omega]$ represented as before by $((\omega_v)_{v\in\mathscr{V}},(f_e)_{e\in\mathscr{E}})$ we define N as the composition of the following maps:

$$\tilde{N} : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)_K) \longrightarrow \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K))$$

$$\tilde{N} : [\omega] \mapsto (\mathrm{Res}(\omega_{v|X_e})_{e=[v,w]})$$

and the map

$$i : \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K) \longrightarrow \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K)/\oplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)_K) \xrightarrow{\ \gamma\ } H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}},(\mathcal{E},\nabla)_K)$$

$$i : (f_e)_{e\in\mathscr{E}} = \Big(0, f_e/\mathrm{Im}(\alpha)\Big)_{v\in\mathscr{V},e\in\mathscr{E}}$$

and γ the same map as in (3).
Hence N is defined as $N=i\circ\tilde{N}$. Note that $N^2=0$.

In order to give an interpretation of the monodromy operator on the de Rham cohomology defined above we'll introduce the log formalism. The curve X can be equipped with a log structure, associated to the special fiber X_k which is a divisor with normal crossing and $\mathrm{Spec}(\mathcal{V})$ with the log structure given by the closed point. Pulling them back to X_k and to $\mathrm{Spec}(k)$ respectively, we may consider X_k and $\mathrm{Spec}(k)$ as log schemes, and when we want to treat them as log schemes we denote them by $X_k^\times$ and $\mathrm{Spec}(k)^\times$. The log structure on $\mathrm{Spec}(\mathcal{V})$ induces a log structure on $\mathrm{Spf}(\mathcal{V})$, and again when we want to treat it as a log formal scheme we denote it by $\mathrm{Spf}(\mathcal{V})^\times$. We note that in the case of the trivial isocrystal by [HK] the de Rham cohomology groups of the generic fiber coincide with the log-crystalline ones of $X_k^\times$, base-changed to K . This result holds also in our case with coefficients. In fact if we start with a convergent F-isocrystal on X_k, then one can associate to it a log-convergent F-isocrystal on $X_k^\times$ and then a log(-crystalline) F-isocrystal on $X_k^\times$ (([Sh1] theorem 5.3.1): we again denote it by E.

Proposition 1. *In the previous hypothesis and notations if we start with a convergent F-isocrystal E on X_k and we denote by $(\mathscr{E}, \nabla)$ its realization on $]X_k[_P$, then the cohomology of the restriction $H^i_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathscr{E}, \nabla)_K)$ coincide with the log-crystalline cohomology of the associated log-F-isocrystal on $X_k^\times$, $H^i_{\log-\mathrm{crys}}(X_k^\times, E) \otimes_{K_0} K$. The monodromy operators coincide as well.*

Proof. We are in the case of [Fa]. The Frobenius structure will imply that the relative log cohomology arising from the deformation gives a locally free module, but it will guarantee also that the exponents of the associated Gauss-Manin differential system are non-Liouville numbers: hence we may trivialize the system by the transfer theorem [Cr]. For the coincidence of the monodromy operators we refer to [CoIo10]. $\square$

Using φ^* of (4) and the monodromy operator N we can form the following sequence

$$H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) \xrightarrow{\varphi^*} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K) \xrightarrow{N} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K). \tag{5}$$

In [Ch99] it is proven the following theorem when k is finite and for varieties of dimensions 1 and 2 and X_k projective.

Theorem 2. *In the sequence (5) if E is the trivial isocrystal, then the map φ^* is injective and* $\mathrm{Imm}(\varphi^*) = \mathrm{Ker}(N)$.

In the next paragraph we are going to prove that if E is not necessarily the trivial isocrystal, then in the sequence (5) the map φ^* is injective and $\mathrm{Im}(\varphi^*) \subset \mathrm{Ker}(N)$. Moreover if E is the trivial isocrystal we will give a new proof of theorem 2 using the explicit description of the monodromy operator as introduced before.

Remark 3. *According to [CoIo10] for the definition of the monodromy operator on the de Rham cohomology we didn't need either the Frobenius structure or an isocrystal: we just needed a connection on the generic fiber. In general we don't know the interpretation of such an operator in terms of the integral structures.*

4 The behavior of the monodromy operator

We would like to study the properties of the monodromy operator as defined in the previous section and, in particular, the exactness of the sequence (5).

As in section 2 let us consider the graph $Gr(X_k)$ associated to X_k, with vertices in $\mathscr{V}$ and edges in $\mathscr{E}$. For $v \in \mathscr{V}$ we denote by $X_v := \mathrm{sp}_X^{-1}(C_v)$ and by $Y_v := \mathrm{sp}_P^{-1}(C_v)$; because the definition of φ, we have that $\varphi(X_v) \subset Y_v$. In the same way we denote by $X_e := \mathrm{sp}_X^{-1}(C_e)$ and by $Y_e := \mathrm{sp}_P^{-1}(C_e)$; because the definition of φ, we have that $\varphi(X_e) \subset Y_e$.
Let us note that Y_e is a polidisk because P is smooth. We choose the admissible covering of X_K^{rig} given by $\{X_v\}_{v\in\mathscr{V}}$ to calculate the de Rham cohomology using Čech complexes.
As before let E be an F-convergent isocrystal on X_k, we can also use the Mayer-Vietoris spectral sequence for rigid cohomology with coefficients in E ([Tsu] theorem 7.1.2). We pick as finite close covering of X_k the

covering given by $\{C_v\}$. Since every intersection of three distinct components is empty the spectral sequence degenerates in a Mayer-Vietoris long exact sequence ([Go] theorem 4.6.1)

$$\begin{tikzcd}
\oplus_{v\in\mathscr{V}} H^0_{\mathrm{rig}}(C_v,E) \arrow[r, "\alpha"] & \oplus_{e\in\mathscr{E}} H^0_{\mathrm{rig}}(C_e,E) \arrow[r] & H^1_{\mathrm{rig}}(X_k,E) \arrow[dl, rounded corners] \\
& \oplus_{v\in\mathscr{V}} H^1_{\mathrm{rig}}(C_v,E) \arrow[r, "\sigma"] & \oplus_{e\in\mathscr{E}} H^1_{\mathrm{rig}}(C_e,E).
\end{tikzcd} \tag{6}$$

whose base-change to K can be described in terms of the de Rham cohomology of Y_K as

$$\begin{tikzcd}
\oplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K) \arrow[r, "\alpha"] & \oplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(Y_e,(\mathcal{E},\nabla)_K) \arrow[r] & H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K) \arrow[dl, rounded corners] \\
& \oplus_{v\in\mathscr{V}} H^1_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K) \arrow[r, "\sigma"] & \oplus_{e\in\mathscr{E}} H^1_{\mathrm{dR}}(Y_e,(\mathcal{E},\nabla)_K).
\end{tikzcd} \tag{7}$$

Now we study the exactness property of the sequence (5).

Lemma 4. *If E is a convergent isocrystal and $(\mathcal{E},\nabla)$ is the coherent module with integrable connection induced by it, then the map φ^* in the sequence (5) is injective.*

Proof. We fix an irreducible component C_v of X_k, we want to prove that the following sequence is exact:

$$0 \longrightarrow H^1_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K) \longrightarrow H^1_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)_K) \longrightarrow \bigoplus_{e\in\mathscr{E}_v} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K); \tag{8}$$

where the last map is the residue map and $\mathscr{E}_v := \{e$ such that there exists a vertex w with $e=[v,w]\}$. As C_v is proper and smooth the above sequence will be isomorphic to the following sequence:

$$0 \longrightarrow H^1_{\mathrm{crys}}(C_v,E)\otimes K \longrightarrow H^1_{\mathrm{log-crys}}(C_v^{\times\times},E)\otimes K \longrightarrow \bigoplus_{e\in\mathscr{E}_v} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K), \tag{9}$$

where $C_v^{\times\times}$ is the log scheme given by the component C_v with the log structure induced by the divisor given by the intersection points of C_v with the other components. The two sequence are isomorphic because $H^1_{\mathrm{crys}}(C_v,E)\otimes K \cong H^1_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K)$ since C_v is proper and smooth, $H^1_{\mathrm{log-crys}}(C_v^{\times\times},E)\otimes K \cong H^1_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)_K)$ by [CoIo10] lemma 5.2. Moreover the second one is exact because is the Gysin sequence for rigid cohomology.
In fact the Gysin sequence for rigid cohomology is the following (proposition 2.1.4 of [ChLeS]):

$$0 \longrightarrow H^1_{\mathrm{rig}}(C_v,E)\otimes K \longrightarrow H^1_{\mathrm{rig}}(U_v,E)\otimes K \longrightarrow \bigoplus_{e\in\mathscr{E}_v} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K), \tag{10}$$

where U_v is the complement in C_v of all the points of C_v that intersect the other components of X_k.
The isomorphism $H^1_{\mathrm{rig}}(U_v,E) \cong H^1_{\mathrm{log-crys}}(C_v^{\times\times},E)$ follows from [Sh02] paragraph 2.4 and [Sh02] theorem 3.1.1. Moreover $H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K) \cong H^0_{\mathrm{dR}}(Y_e,(\mathcal{E},\nabla)_K)$ because Y_e and X_e are residue classes and E has a basis of horizontal sections on each residue class, which means that both $H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K)$ and $H^0_{\mathrm{dR}}(Y_e,(\mathcal{E},\nabla)_K)$ are isomorphic to K^d where d is the rank of $\mathcal{E}$ as $\mathcal{O}_{X_K}$-module. Moreover by the Gysin isomorphism in degree zero (proposition 2.1.4 of [ChLeS]), with the same notations as before, we have $H^0_{\mathrm{rig}}(C_v,E) \cong H^0_{\mathrm{rig}}(U_v,E)$, which implies that $H^0_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)_K) \cong H^0_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K)$, using the same techniques as before.
Using the Mayer-Vietoris long exact sequence for rigid cohomology (7), we can pass to the following short exact sequence

$$0 \longrightarrow H^1(Gr(X_k),\mathcal{E}_K) \xrightarrow{\delta} H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K)) \longrightarrow \mathrm{Ker}(\sigma) \longrightarrow 0. \tag{11}$$

where $H^1(Gr(X_k), \mathcal{E}_K) := \mathrm{Coker}(\alpha)$.
Putting together Mayer-Vietoris sequences for the coverings $\{X_v\}$ and $\{Y_v\}$ respectively we obtain the following diagram

$$
\begin{tikzcd}
& & \oplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) \arrow[r] \arrow[dl, "\theta"'] & \oplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) & \\
0 \arrow[r] & H^1(Gr(X_k), \mathcal{E}_K) \arrow[r, "\gamma"] & H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) \arrow[u, "Res"] \arrow[r, "\pi_X"] & \oplus_v H^1_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)_K) \arrow[u] & \\
0 \arrow[r] & H^1(Gr(X_k), \mathcal{E}_K) \arrow[u] \arrow[r, "\delta"] & H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) \arrow[u, "\varphi^*"] \arrow[r, "\pi_Y"] & \oplus_v H^1_{\mathrm{dR}}(Y_v, (\mathcal{E}, \nabla)_K) \arrow[u, "\varphi^*"] \arrow[r] & 0 \\
& & & 0 \arrow[u] &
\end{tikzcd} \tag{12}
$$

and by the snake lemma one can conclude that $\varphi^* : H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) \to H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ is injective. □

Remark 5. *Let us note that in (12) the monodromy operator on $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ acts as $N = \gamma \circ \theta \circ Res$.*

Lemma 6. *If E is a convergent F-isocrystal and $(\mathcal{E}, \nabla)$ is the coherent module with integrable connection induced by it, then in the sequence (5)*

$$N \circ \varphi^* = 0.$$

Proof. Let us consider $[\omega] \in H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)$. Then $\varphi^*[\omega]$, which is an element of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$, can be represented by an hypercocycle $((\alpha_v)_{v \in \mathscr{V}}, (g_e)_{e \in \mathscr{E}})$ where $\alpha_v \in \Omega^1_{X_v} \otimes \mathcal{E}_{X_v}$ and g_e in $\mathcal{E}_{X_e}$ and they verify that $\alpha_{v|X_e} - \alpha_{w|X_e} = \nabla(g_e)$ if $e = [v, w]$. We want to calculate $N(\varphi^*([\omega]))$. We now look at the diagram (12). By the definition of N one can see that

$$N(\varphi^*([\omega])) = \gamma \circ \theta \circ Res(\varphi^*([\omega])) = \gamma \circ \theta \circ Res_{|X_e}(\pi_X(\varphi^*[\omega])).$$

By the commutativity of the diagram (12) $Res_{|X_e}(\pi_X(\varphi^*[\omega])) = Res_{|X_e}(\varphi^*(\pi_Y([\omega]))$.
If we denote by $\omega_v = \pi_Y([\omega])$, then we have to compute $Res_{|X_e}(\varphi^*(\omega_v))$:

$$Res_{|X_e}(\varphi^*(\omega_v)) = Res(\varphi^*(\omega_v)_{|X_e}) = Res(\varphi^*(\gamma_e))$$

where $\gamma_e \in \mathcal{E}_{Y_e} \otimes \Omega^1_{Y_e}$, but as Y_e is an open polydisc we have that $H^1_{\mathrm{dR}}(Y_e, (\mathcal{E}, \nabla)_K) = 0$ and so $Res(\phi^*(\gamma_e)) = 0$ as claimed. □

From the above lemma we can conclude that $\mathrm{Im}(\varphi^*) \subset \mathrm{Ker}(N)$. Now we'd like to characterize in terms of residues the elements of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ which are in the image of φ^*.
Let us take $[\omega] \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$. As before we can choose a representative $\omega = ((\omega_v)_{v \in \mathscr{V}}, (f_e)_{e \in \mathscr{E}})$, with (ω_v) in $\mathcal{E}_{X_e} \otimes \Omega^1_{X_v}$ and f_e in $\mathcal{E}_{X_e}$ which verifies that $\omega_{v|X_e} - \omega_{w|X_e} = \nabla(f_e)$ if $e = [v, w]$.
In the next lemma we prove a necessary and sufficient condition for an element of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ to be in the image of the map φ^*.

Lemma 7. *Let us take $[\omega] \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ and a representative $\omega = ((\omega_v)_{v \in \mathscr{V}}, (f_e)_{e \in \mathscr{E}})$ as above. Then $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ for every $e \in \mathscr{E}$ if and only if $[\omega] \in \mathrm{Im}(\varphi^*)$.*

Proof. Let us see first that if $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ for every $e \in \mathscr{E}$, then $[\omega] \in \mathrm{Im}(\varphi^*)$. If $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$, then thanks to the exact sequence (8) there exists $\gamma_v \in H^1_{\mathrm{dR}}(Y_v, (\mathcal{E}, \nabla)_K)$ such that $\varphi^*(\gamma_v) = \omega_v$ for every $v \in \mathscr{V}$. As the map π_Y in (12) is surjective there exists $\alpha \in H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)$ such that $\pi_Y(\alpha) = (\omega_v)_{v \in \mathscr{V}}$. Now $\pi_X([\omega] - \varphi^*(\alpha)) = 0$, hence, looking again at diagram (12), there exists $c \in H^1(Gr(X_k), \mathcal{E}_K)$ such that

$[\omega]-\varphi^*(\alpha)=\gamma(c)$. By the commutativity of diagram (12) there exists an element $\mu\in H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K)$ such that $\varphi^*(\mu)=\gamma(c)$. (One can choose $\mu=\delta(c)$.)
Viceversa if $[\omega]=\varphi^*(\alpha)$ for $\alpha\in H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K)$, then $(\omega_v)_{v\in\mathscr{V}}=\varphi^*(\pi_Y(\alpha)):=\varphi^*(\alpha_v)_{v\in\mathscr{V}}$. Hence $Res_{|X_e}(\omega_v)=Res_{|X_e}(\varphi^*(\alpha_v))$ for every $v\in\mathscr{V}$. But as in the proof of lemma 6 one can prove that from this it follows that $Res_{|X_e}(\omega_v)=0$ for every $v\in\mathscr{V}$. □

5 The constant coefficients case

In this paragraph we show that if E is the trivial convergent F-isocrystal, then the condition in lemma 7 is fulfilled. This will imply that the sequence in (5) is exact and it will give a new proof of theorem 2 i.e. the exactness of the invariant cycles sequence under the assumption that k is perfect instead of finite. The realization of E on X_K^{rig} is the structure sheaf with trivial connection $(\mathcal{O}_{X_K},d)$.

We'd like to prove that if $[\omega]\in H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}})$ is such that $N([\omega])=0$, then one can find a hypercocycle (ω_v,f_e) representing it such that $\mathrm{Res}_{X_e}(\omega_{v|X_e})=0$: hence we may apply lemma 7 and conclude.
Let (ω_v,f_e) be a hypercocycle representing $[\omega]$ and consider $\mathrm{Res}_{X_e}(\omega_{v|X_e})$; if $[\omega]$ is in $\mathrm{Ker}(N)$, then $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e=0$ in $H^1(Gr(X_k),\mathcal{O}_K)$, that means that $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e\in\mathrm{CoKer}(\oplus_{v\in\mathscr{V}}H^0_{\mathrm{dR}}(X_v)\to\oplus_{e\in\mathscr{E}}H^0_{\mathrm{dR}}(X_e))$.
On the other hand, thanks to the residue theorem on wide opens (proposition 4.3 of [Co89]), for every irreducible component C_v in X_k, the family $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e$ verifies that

$$\sum_{e\in\mathscr{E}_v}\mathrm{Res}_{X_e}(\omega_{v|X_e})=0,\tag{13}$$

where the notation $\mathscr{E}_v$ refers to the set $\{e$ such that there exists a vertex w with $e=[v,w]\}$.
Hence to prove that $\mathrm{Res}_{X_e}(\omega_{v|X_e})=0$ we are left to prove that if $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e\in\mathrm{CoKer}(\oplus_{v\in\mathscr{V}}H^0_{\mathrm{dR}}(X_v)\to\oplus_{e\in\mathscr{E}}H^0_{\mathrm{dR}}(X_e))$ and for every v it verifies that $\sum_{e\in\mathscr{E}_v}\mathrm{Res}_{X_e}(\omega_{v|X_e})=0$, then $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e=0$ for all e.
So we are reduced to a linear algebra and graph theory problem, which we can translate as follows.
Let F be a field of characteristic 0. Let G be a connected graph with n vertices and m edges. Let us denote by $\mathscr{V}$ the set of all vertices and by $\mathscr{E}$ the set of all oriented edges. We use the notation $e=[v,w]$ to indicate an edge between the vertex v and the vertex w. We associate to G a vector space $V=\oplus_{e\in\mathcal{E}}F$ with the convention that if $e=[v,w]$ and $\bar{e}=[w,v]$ then $a_e=-a_{\bar{e}}$. Then there is a map

$$\phi:\oplus_{v\in\mathscr{V}}F\to\oplus_{e\in\mathscr{E}}F$$

$$(a_v)_{v\in\mathscr{V}}\mapsto(a_e)_{e\in\mathscr{E}}$$

where $a_e=a_v-a_w$ if $e=[v,w]$. We consider two vector subspaces W and T of $\oplus_{e\in\mathscr{E}}F$ where

$$W=\{(a_e)_{e\in\mathscr{E}}|(a_e)_{e\in\mathscr{E}}\in\mathrm{Im}(\phi)\}$$

$$T=\{(a_e)_{e\in\mathscr{E}}|\forall v\in\mathscr{V}\sum_{e\in\mathscr{E}_v}a_e=0\}.$$

Proposition 8. *With notations as before we have $W\cap T=0$*

Proof. An element $(a_e)_{e\in\mathscr{E}}$ which belongs to W and to T is described by the following equations

$$a_e=a_v-a_w$$

$$\forall v\in\mathscr{V}\quad\sum_{e\in\mathscr{E}_v}a_e=0.$$

We can rewrite the equations as follows:

$$\forall v\in\mathscr{V}\ \ \deg(v)a_v=a_{w_1}+\cdots+a_{w_{s_v}}\tag{14}$$

where $w_1, \ldots, w_{s_v}$ are the vertices connected to v by an edge and by $\deg(v) := s_v$ we denote the cardinality of the set of the vertices connected to v. Requiring that $W \cap T = 0$ is equivalent to requiring that the linear system in (14) has a 1-dimensional space of solutions, generated by the vector $(1, \ldots, 1)$. This is equivalent to requiring that the matrix associated to the system in (14) has rank $n-1$, i.e. that there exists at least one minor of rank $n-1$ whose determinant is non-zero.
This last condition is independent of the field F, hence to prove that $W \cap T = 0$ it is enough to prove that the equations in (14) imply that $a_v = a_{w_i}$ for all w_i and for all v assuming that F is a totally ordered field. We assume in what follows that F is a totally ordered field of characteristic 0. Let us suppose by absurd that the equations in (14) do not imply that $a_v = a_w$ for all w. Let us call

$$a_{v_0} = \min_{v \in \mathscr{V}} a_v$$

which exists because our assumption that our field K is totally ordered; then $a_{v_0} \leq a_v$ for all $v \in \mathscr{V}$. If $a_{v_0} = a_v$ for all $v \in \mathscr{V}$ we are done, if not there exists v_1 such that $a_{v_0} < a_{v_1}$. Moreover we can suppose that v_1 is connected to v_0 by an edge because if not, then this means that $a_{v_0} = a_v$ for all v connected to v_0 by an edge. Then if we now fix a $v \neq v_0$ that is connected to v_0, we can consider all the w that are connected to it by an edge; if $a_v = a_w$ for all these w we can go on as before. In the end we will find that all the a_v are equal for all $v \in \mathscr{V}$ which proves the claim.
Hence we suppose that there exists v_1 such that $a_{v_0} < a_{v_1}$ for v_1 connected to v_0 by an edge. We consider the equation (14) for $v = v_0$ and we get the contradiction

$$\deg(v_0) a_{v_0} < a_{w_1} + \ldots a_{w_{s_v}}.$$

□

With this proposition we end the proof of the exactness of the invariant cycles sequence for trivial coefficients.

Remark 9. We'd like now to give another proof of proposition 8 more in the spirit of graph theory: it uses proposition 4.3, proposition 4.8 of [Bi], and lemma 13.1.1 of [GoRo].

Proof. The matrix associated to the linear system in (14) is an $n \times n$ matrix $A = (a_{i,j})$, where for $i \neq j$ $a_{i,j} = -1 h_{i,j}$ if there are $h_{i,j}$ edges between the vertex v_i and v_j and 0 otherwise, and $a_{i,i} = \deg(v_i)$. We will prove that the rank of the matrix A is $n-1$.
The matrix A is called the Laplacian matrix associated to the graph G; we will see that $A = DD^t$ and that D is an $n \times m$ matrix with rank $n-1$.
The following are equivalent:

- (i) there exists an $(n-1) \times (n-1)$ minor of A with determinant different from zero,
- (ii) the rank of A is $(n-1)$ dimensional,
- (iii) the dimension of the Kernel of A is 1,
- (iv) $\mathrm{Kernel}(D^t) = \mathrm{Kernel}(A)$.

Assertion (i) is independent from the field K, so we can suppose that K is the field $\mathbb{R}$.
We will prove assertion (iv).
Let us suppose that z is a vector in $\mathbb{R}^n$ that is in the $\mathrm{Kernel}(A)$, we want to prove that $z \in \mathrm{Kernel}(D^t)$. Being $z \in \mathrm{Kernel}(A)$, then

$$Az = 0,$$

$$DD^t z = 0$$

$$z^t DD^t z = 0.$$

But the last equality implies that the vector $D^t z$ has inner product with itself in $\mathbb{R}^n$ equal to zero, that means that $D^t z$ is the zero vector, i.e. $z \in \mathrm{Kernel}(D^t)$, as we wanted.
We are left to prove that $A = DD^t$ and that D is an $n \times m$ matrix with rank $n-1$.
We consider the matrix D associated to the graph G defined as follows: D is $n \times m$ matrix such that $(D)_{i,j} = 1$ if the vertex v_i is such that $e_j = [v_i, -]$, $(D)_{i,j} = -1$ if the vertex v_i is such that $e_j = [-, v_i]$, and $(D)_{i,j} = 0$ otherwise.
Now if we consider $(DD^t)_{i,j}$, this is the inner product of the rows $\mathbf{d}_i$ and $\mathbf{d}_j$. They have a non zero entry in the same column if and only if there is an edge between v_i and v_j, and these entries are one -1 and one $+1$, hence $(DD^t)_{i,j}$ is given by -1 times the number of edges between v_i and v_j. Moreover $(DD^t)_{i,i}$ is the number of entries in $\mathbf{d}_i$ different from zero, which means the degree of v_i. This proves that $A = DD^t$.
Let us see now that D has rank $n-1$.
On every column there is a $+1$ and a -1, hence the sum of all the elements on the columns are zero, hence the rank of D is less or equal to $n-1$. Let us suppose to have a linear relation

$$\sum_i a_i \mathbf{d}_i = 0, \tag{15}$$

where as before $\mathbf{d}_i$ is the row corresponding to the vertex v_i and suppose that not all the a_i are zero. Choose a row $\mathbf{d}_k$ for which $a_k \neq 0$. This row has non zero entries in the columns corresponding to the edges that intersect v_i. For every such column there is only on other row $\mathbf{d}_l$ with a non zero entry in that column. Hence we should have that $a_l = a_k$, hence $a_l = a_k$ for all vertices v_l adjacent to v_k. Hence all the a_k are equal, being the graph G connected, and the equation in (15) is a multiple of $\sum_i \mathbf{d}_i = 0$. But $(a_1, \dots, a_n)$ that verifies (15) is in $\mathrm{Kernel}(D^t)$, hence we have proven that $\mathrm{Kernel}(D^t)$ is 1-dimensional and generated by $(1, \dots, 1)$, the rank is $(n-1)$-dimensional and as well as the rank of D.

□

6 Unipotent coefficients

In this section we study the sequence in (5) when the coefficients are unipotent F-isocrystals. In particular we prove that, unlike the case of constant coefficients, the sequence in (5) is not necessarily exact. We give a sufficient condition for non exactness.

Let E be a unipotent convergent F-isocrystal for which the sequence in (5) is exact and let us consider the following extension in the category of convergent F-isocrystals

$$0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} \mathcal{O} \to 0 \tag{16}$$

where $\mathcal{O}$ is the trivial F-isocrystal. Let us also consider the element $x \in H^1_{\mathrm{rig}}(X_k, E)$ corresponding to the class of this extension (x is then fixed by the Frobenius operator; see propositions 1.3.1 and 3.2.1 of [ChLeS])
Let us suppose that $x \neq 0$.
In the sequel we use sequence (5) for the isocrystals E, F and $\mathcal{O}$; to avoid confusion we denote the first maps by $\phi^*_{\mathcal{E}}$, $\phi^*_{\mathcal{F}}$ and $\phi^*_{\mathcal{O}}$ respectively and the monodromy operators by $N_{\mathcal{E}}$, $N_{\mathcal{F}}$ and $N_{\mathcal{O}_X}$ respectively.

Our assumptions imply that $H^1_{\mathrm{rig}}(X_k, E) \otimes K$ is isomorphic via $\varphi^*_{\mathcal{E}}$ to $\mathrm{Ker}(N_{\mathcal{E}})$, and this last group contains the image of $N_{\mathcal{E}}$, as this operator has square zero.

Theorem 10. *If $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y)$ for $y \in H^1_{\mathrm{dR}}(X_K, (E, \nabla_E)_K)$, then if we denote by $\alpha_{\mathrm{dR}} : H^1_{\mathrm{dR}}(X_K, (E, \nabla_E)_K) \to H^1_{\mathrm{dR}}(X_K, (E, \nabla_E)_K)$ the map induced by α in the sequence , the following holds:*

$$\mathrm{Kernel}(N_{\mathcal{F}}) = \left(H^1_{\mathrm{rig}}(X_k, F) \otimes K\right) \oplus K\alpha_{\mathrm{log-crys}}(y)$$

Proof. Let us consider the following commutative diagram

$$\begin{tikzcd}
H^0_{\mathrm{rig}}(X_k)\otimes K \arrow[r, "i^0_{\mathcal{O}}"] \arrow[d, "\delta^0_{\mathrm{rig}}"] & H^0_{\mathrm{dR}}(X_K) \arrow[r, "N^0_{\mathcal{O}}"] \arrow[d, "\delta^0_{\mathrm{log-crys}}"] & H^0_{\mathrm{dR}}(X_K) \arrow[d, "\delta^0_{\mathrm{dR}}"] \\
H^1_{\mathrm{rig}}(X_k,E)\otimes K \arrow[r, "\varphi^*_{\mathcal{E}}"] \arrow[d, "\alpha_{\mathrm{rig}}"] & H^1_{\mathrm{dR}}(X_K,(\mathcal{E},\nabla_{\mathcal{E}})_K) \arrow[r, "N_{\mathcal{E}}"] \arrow[d, "\alpha_{\mathrm{dR}}"] & H^1_{\mathrm{dR}}(X_K,(\mathcal{E},\nabla_{\mathcal{E}})_K) \arrow[d, "\alpha_{\mathrm{dR}}"] \\
H^1_{\mathrm{rig}}(X_k,F)\otimes K \arrow[r, "\varphi^*_{\mathcal{F}}"] \arrow[d, "\beta_{\mathrm{rig}}"] & H^1_{\mathrm{dR}}(X_K,(\mathcal{F},\nabla_{\mathcal{F}})_K) \arrow[r, "N_{\mathcal{F}}"] \arrow[d, "\beta_{\mathrm{dR}}"] & H^1_{\mathrm{dR}}(X_K,(\mathcal{F},\nabla_{\mathcal{F}})_K) \arrow[d, "\beta_{\mathrm{dR}}"] \\
H^1_{\mathrm{rig}}(X_k)\otimes K \arrow[r, "\varphi^*_{\mathcal{O}}"] \arrow[d, "\gamma_{\mathrm{rig}}"] & H^1_{\mathrm{dR}}(X_K) \arrow[r, "N_{\mathcal{O}}"] \arrow[d, "\gamma_{\mathrm{dR}}"] & H^1_{\mathrm{dR}}(X_K)\otimes K \arrow[d, "\gamma_{\mathrm{dR}}"] \\
H^2_{\mathrm{rig}}(X_k,E)\otimes K \arrow[r, "i^2_{\mathcal{E}}"] \arrow[d] & H^2_{\mathrm{dR}}(X_K,(\mathcal{E},\nabla_{\mathcal{E}})_K) \arrow[r, "N^2_{\mathcal{E}}"] \arrow[d] & H^2_{\mathrm{dR}}(X_K,(\mathcal{E},\nabla_{\mathcal{E}})_K) \arrow[d] \\
H^2_{\mathrm{rig}}(X_k,F)\otimes K \arrow[r] \arrow[d] & H^2_{\mathrm{dR}}(X_K,(\mathcal{F},\nabla_{\mathcal{F}})_K) \arrow[r] \arrow[d] & H^2_{\mathrm{dR}}(X_K,(\mathcal{F},\nabla_{\mathcal{F}})_K) \arrow[d] \\
H^2_{\mathrm{rig}}(X_k)\otimes K \arrow[r] & H^2_{\mathrm{dR}}(X_K) \arrow[r] & H^2_{\mathrm{dR}}(X_K).
\end{tikzcd} \tag{17}$$

Let $\varphi^*_{\mathcal{E}}(x\otimes 1)\in\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k,E)\otimes K)=\mathrm{Ker}(N_{\mathcal{E}})$, with $\varphi^*_{\mathcal{E}}(x\otimes 1)=N_{\mathcal{E}}(y)$ and $y\in H^1_{\mathrm{dR}}(X_K,(\mathcal{E},\nabla_{\mathcal{E}})_K)$. One can notice that the class of 1 in $H^0_{\mathrm{rig}}(X_k)\otimes K=K$ is sent to $x\otimes 1$ in $H^1_{\mathrm{rig}}(X_k,E)\otimes K$ by the map δ^0_{rig}.
Let us prove first that $N_{\mathcal{F}}(\alpha_{\mathrm{dR}}(y))=0$.
By the commutativity of the diagram (17) we have that

$$N_{\mathcal{F}}(\alpha_{\mathrm{dR}}(y))=\alpha_{\mathrm{dR}}(N_{\mathcal{E}}(y))=\alpha_{\mathrm{dR}}(\varphi^*_{\mathcal{E}}(x\otimes 1))=\alpha_{\mathrm{dR}}(\delta^0_{\mathrm{dR}}(1))=0,$$

hence $\alpha_{\mathrm{dR}}(y)\in\mathrm{Ker}(N_{\mathcal{F}})$.
We claim that $z=\alpha_{\mathrm{dR}}(y)\notin\varphi^*_{\mathcal{F}}\big(H^1_{\mathrm{rig}}(X_k,F)\otimes K\big)$. Let us suppose that $z=\alpha_{\mathrm{dR}}(y)=\varphi^*_{\mathcal{F}}(b)$, with $b\in H^1_{\mathrm{rig}}(X_k,F)\otimes K$, then

$$\varphi^*_{\mathcal{O}}(\beta_{\mathrm{rig}}(b))=\beta_{\mathrm{dR}}(\varphi^*_{\mathcal{F}}(b))=\beta_{\mathrm{dR}}(z)=\beta_{\mathrm{dR}}(\alpha_{\mathrm{dR}}(y))=0.$$

As $\varphi^*_{\mathcal{O}}$ is injective we have $\beta_{\mathrm{rig}}(b)=0$, hence $b\in\mathrm{Ker}(\beta_{\mathrm{rig}})=\mathrm{Im}(\alpha_{\mathrm{rig}})$, i.e. there exists $a\in H^1_{\mathrm{rig}}(X_k,E)\otimes K$ such that $\alpha_{\mathrm{rig}}(a)=b$. So

$$z=\alpha_{\mathrm{dR}}(y)=\varphi^*_{\mathcal{F}}(b)=\varphi^*_{\mathcal{F}}(\alpha_{\mathrm{rig}}(a))=\alpha_{\mathrm{dR}}(\varphi^*_{\mathcal{E}}(a)),$$

from which it follows that

$$y-\varphi^*_{\mathcal{E}}(a)\in\mathrm{Ker}(\alpha_{\mathrm{dR}})=\mathrm{Im}(\delta^0_{\mathrm{dR}}).$$

But the image of δ^0_{dR} is generated by $\varphi^*_{\mathcal{E}}(x\otimes 1)$, as vector space, hence $y-\varphi^*_{\mathcal{E}}(a)=m\varphi^*_{\mathcal{E}}(x\otimes 1)$ for some $m\in K$.
Now

$$N_{\mathcal{E}}(y)-N_{\mathcal{E}}(\varphi^*_{\mathcal{E}}(a))=N_{\mathcal{E}}(m\varphi^*_{\mathcal{E}}(x\otimes 1))=0,$$

hence

$$N_{\mathcal{E}}(y)=N_{\mathcal{E}}(\varphi^*_{\mathcal{E}}(a))=0,$$

but

$$N_{\mathcal{E}}(y) = \varphi^*_{\mathcal{E}}(x \otimes 1) = 0,$$

which is absurd.
We are left to prove that $\forall \alpha \in \mathrm{Ker} N_{\mathcal{F}}$ there exists $\beta \in H^1_{\mathrm{rig}}(X_k, F) \otimes K$ and $t \in K$ such that $\alpha = \varphi^*_{\mathcal{F}}(\beta) + t\alpha_{\mathrm{dR}}(y)$. Let us calculate

$$N_{\mathcal{O}}(\beta_{\mathrm{dR}}(\alpha)) = \beta_{\mathrm{dR}}(N_{\mathcal{F}}(\alpha))) = 0,$$

hence

$$\beta_{\mathrm{dR}}(\alpha) \in \mathrm{Ker}(N_{\mathcal{O}}) = \mathrm{Im}(\varphi^*_{\mathcal{O}}),$$

so that there exists $\gamma \in H^1_{\mathrm{rig}}(X_k) \otimes K$ such that $\varphi^*_{\mathcal{O}}(\gamma) = \beta_{\mathrm{dR}}(\alpha)$. By lemma 11 we have $\gamma_{\mathrm{rig}}(\gamma) = 0$. Hence there exists $\beta \in H^1_{\mathrm{rig}}(X_k, F) \otimes K$ such that $\beta_{\mathrm{rig}}(\beta) = \gamma$. Let us consider now the element $\alpha - \varphi^*_{\mathcal{F}}(\beta)$; it is in the Kernel of β_{dR}, because

$$\beta_{\mathrm{dR}}(\alpha - \varphi^*_{\mathcal{F}}(\beta)) = \beta_{\mathrm{dR}}(\alpha) - \varphi^*_{\mathcal{O}}(\beta_{\mathrm{rig}}(b)) = \beta_{\mathrm{dR}}(\alpha) - \varphi^*_{\mathcal{O}}(\gamma) = 0.$$

Hence there exists $u \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla_{\mathcal{E}})_K)$ such that $\alpha_{\mathrm{dR}}(u) = \alpha - \varphi^*_{\mathcal{F}}(\beta)$. Now

$$\alpha_{\mathrm{dR}}(N_{\mathcal{E}}(u)) = N_{\mathcal{F}}(\alpha_{\mathrm{dR}}(u)) = N_{\mathcal{F}}(\alpha - \varphi^*_{\mathcal{F}}(\beta)) = 0$$

because $\alpha \in \mathrm{Ker}(N_{\mathcal{F}})$ and $N_{\mathcal{F}}(\varphi^*_{\mathcal{F}}(\beta)) = 0$ by lemma 6. Then $N_{\mathcal{E}}(u) \in \mathrm{Ker}(\alpha_{\mathrm{dR}}) = \mathrm{Im}\,\delta^0_{\mathrm{dR}}$, i.e $N_{\mathcal{E}}(u) = t\varphi^*_{\mathcal{E}}(x \otimes 1) = tN_{\mathcal{E}}(y)$, for some $t \in K$ and $u - ty \in \mathrm{Ker}(N_{\mathcal{E}}) = \varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$. Hence there exists $\beta' \in H^1_{\mathrm{rig}}(X_k, E) \times K$ such that $u = ty + \varphi^*_{\mathcal{E}}(\beta')$. So

$$\alpha - \varphi^*_{\mathcal{F}}(\beta) = \alpha_{\mathrm{dR}}(u) = \alpha_{\mathrm{dR}}(ty + \varphi^*_{\mathcal{E}}(\beta')) = t\alpha_{\mathrm{dR}}(y) + \alpha_{\mathrm{dR}}(\varphi^*_{\mathcal{E}}(\beta'))$$

which means that

$$\alpha = \varphi^*_{\mathcal{F}}(\beta) + t\alpha_{\mathrm{dR}}(y) + \alpha_{\mathrm{dR}}(\varphi^*_{\mathcal{E}}(\beta')),$$

but $\varphi^*_{\mathcal{F}}(\beta) + \alpha_{\mathrm{dR}}(\varphi^*_{\mathcal{E}}(\beta')) = \varphi^*_{\mathcal{F}}(\beta) + \varphi^*_{\mathcal{F}}(\alpha_{\mathrm{rig}}(\beta'))$, hence we are done. □

Lemma 11. *With the same hypothesis and notations as in the previous theorem, the co-boundary map* $\gamma_{\mathrm{rig}} : H^1_{\mathrm{rig}}(X_k) \otimes K \longrightarrow H^2_{\mathrm{rig}}(X_k, E) \otimes K$ *induced by the exact sequence (16) is the zero map.*

Proof. Clearly, the vanishing of γ_{rig} is equivalent to the fact that the map $j : H^2_{\mathrm{rig}}(X_k, E) \otimes K \longrightarrow H^2_{\mathrm{rig}}(X_k, F) \otimes K$ is **injective**.

Let us first make more explicit the group $H^2_{\mathrm{rig}}(X_k, G) \otimes K$, where G is any one of the isocrystals $E, F, \mathcal{O}$ and $(\mathcal{G}, \nabla)$ is the module with integrable connection that it induces. Let us recall the notations of section 3: we consider the diagram

$$X_k \hookrightarrow P_k \overset{sp_{P_{\mathcal{V}}}}{\longleftarrow} P_K$$

with P_k smooth and let $Y_K := sp^{-1}_{P_{\mathcal{V}}}(X_k)$. Then $H^i_{\mathrm{rig}}(X_k, G) \otimes K = H^i_{\mathrm{dR}}(Y_K, (\mathcal{G}, \nabla)_K)$.

The relevant part of the Mayer-Vietoris exact sequence for the admissible covering $\{Y_v\}_v$ of Y_K then reads

$$\oplus_e H^1_{\mathrm{dR}}(Y_e, (\mathcal{G}, \nabla)_K) \longrightarrow H^2_{\mathrm{dR}}(Y_K, (\mathcal{G}, \nabla)_K) \longrightarrow \oplus_v H^2_{\mathrm{dR}}(Y_v, (\mathcal{G}, \nabla)_K) \longrightarrow \oplus_e H^2_{\mathrm{dR}}(Y_e, (\mathcal{G}, \nabla)_K).$$

As Y_e is a wide open polydisk, $H^i_{\mathrm{dR}}(Y_e, (\mathcal{G}, \nabla)_K) = 0$ for $i \geq 1$, therefore we have a natural isomorphism $H^2_{\mathrm{dR}}(Y_K, (\mathcal{G}, \nabla)_K) \cong \oplus_v H^2_{\mathrm{dR}}(Y_v, (\mathcal{G}, \nabla)_K)$.
Moreover, as C_v which is the irreducible component of X_k corresponding to v was supposed smooth it follows that we have canonical isomorphisms $H^i_{\mathrm{dR}}(Y_v, (\mathcal{G}, \nabla)_K) \cong H^i_{\mathrm{crys}}(C_v, G) \otimes K$. In particular, if we denote by Z_v a smooth proper curve over K whose reduction is C_v and which contains the wide open X_v, then the isocrystal G can be evaluated on Z_v to give a sheaf with connection which we'll denote again by $(\mathcal{G}, \nabla)$. Then $H^i_{\mathrm{dR}}(Y_v, (\mathcal{G}, \nabla)_K) \cong H^i_{\mathrm{dR}}(Z_v, (\mathcal{G}, \nabla)_K)$ for all $i \geq 0$.

Therefore we have a natural isomorphism $H^2_{\rm rig}(X_k, G) \otimes K \cong \oplus_v H^2_{\rm dR}(Z_v, (\mathcal{G}, \nabla))$.

For every vertex v we denote as before by $\mathscr{E}_v := \{e$ such that there exists a vertex w with $e = [v, w]\}$. For every v and $e \in \mathscr{E}_v$ we denote by D_e the residue disk of the point in C_v corresponding to e in Z_v. Let us then remark that the family $\{X_v, D_e\}_{e \in \mathscr{E}_v}$ is an admissible covering of Z_v and $X_v \cap D_e = X_e$ for every $e \in \mathscr{E}_v$. We will represent classes in $H^2_{\rm dR}(Z_v, (\mathcal{G}, \nabla)_K)$ by hypercocycles for the above covering.

We now prove the injectivity of $j : H^2_{\rm rig}(X_k, E) \otimes K \longrightarrow H^2_{\rm rig}(X_k, F) \otimes K$.

Let $z \in H^2_{\rm rig}(X_k, E) \otimes K = \oplus_v H^2_{\rm dR}(Z_v, (\mathcal{E}, \nabla)_K)$ such that $j(z) = 0$. Let $z_v \in H^2_{\rm dR}(Z_v, (\mathcal{E}, \nabla)_K)$ be the v-component of z and $j_v : H^2_{\rm dR}(Z_v, (\mathcal{E}, \nabla)_K) \longrightarrow H^2_{\rm dR}(Z_v, (\mathcal{F}, \nabla)_K)$ be the v component of j. Obviously $j_v(z_v) = 0$ and it would be enough to show that this implies $z_v = 0$ for every v.

Let $\big(\omega_e\big)_{e \in \mathcal{E}_v}$ be a 2-hyper cocycle representing z_v, where $\omega_e \in H^0(X_e, \mathcal{E}_K \otimes \Omega^1_{Z_v})$ for all e. Then $j_v(z_v)$ will be represented by the 2-hyper cocycle $\big(\alpha(\omega_e)\big)_{e \in \mathscr{E}_v}$, where let us recall α is defined by the exact sequence of isocrystals on X_k below

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{O} \longrightarrow 0.$$

As extension on X_K this is given by the class $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y) \in H^1_{\rm dR}(X_K, (\mathcal{E}, \nabla)_K)$ and therefore, for every v, the sequence

$$0 \longrightarrow H^0(X_v, \mathcal{E}_K) \xrightarrow{\alpha} H^0(X_v, \mathcal{F}_K) \xrightarrow{\beta} H^0(X_v, \mathcal{O}_{X_K}) \longrightarrow 0,$$

is exact because X_v are wide opens and moreover, it is naturally split as an exact sequence of $\mathcal{O}_{X_v}$-modules with connections because $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y)$ can be represented by $(0_v, f_e)$ with $f_e \in H^0_{\rm dR}(X_e, (\mathcal{E}, \nabla)_K)$. Let $s : H^0(X_v, \mathcal{O}_{X_K}) \longrightarrow H^0(X_v, \mathcal{F}_K)$ be such a section of β. We remark that it is determined by $s(1)$, which is an element of $H^0_{\rm dR}(X_v, (\mathcal{F}, \nabla)_K)$ such that $\beta(s(1)) = 1$.

Therefore, s determines, for every $e \in \mathscr{E}_v$, a splitting of the exact sequence

$$0 \longrightarrow H^0(X_e, \mathcal{E}_K) \xrightarrow{\alpha_e} H^0(X_e, \mathcal{F}_K) \xrightarrow{\beta_e} H^0(X_e, \mathcal{O}_{X_K}) \longrightarrow 0$$

which will also be called s_e (it is determined by the element $s_e(1) = s(1)|_{X_e}$).

Now the sequence

$$0 \longrightarrow H^0_{\rm dR}(X_e, (\mathcal{E}, \nabla)_K) \xrightarrow{\alpha_e} H^0_{\rm dR}(X_e, (\mathcal{F}, \nabla)_K) \xrightarrow{\beta_e} H^0_{\rm dR}(X_e, (\mathcal{O}_{X_K}, d)) \longrightarrow 0 \tag{18}$$

is exact and s_e induces a natural splitting of it.

The isocrystal G (which is any one of $E, F, \mathcal{O}$ regarded as a sheaf with connection on Z_v) has a basis of horizontal sections on D_e, for every $e \in \mathscr{E}_v$. Therefore the natural restriction map $H^0_{\rm dR}(D_e, (\mathcal{G}, \nabla)_K) \longrightarrow H^0_{\rm dR}(X_e, (\mathcal{G}, \nabla)_K)$ is an isomorphism. Thus the exact sequence (18) implies that the sequence

$$0 \longrightarrow H^0_{\rm dR}(D_e, (\mathcal{E}, \nabla)_K) \xrightarrow{\alpha_e} H^0_{\rm dR}(D_e, (\mathcal{F}, \nabla)_K) \xrightarrow{\beta_e} H^0_{\rm dR}(D_e, (\mathcal{O}_{X_K}, d)) \longrightarrow 0 \tag{19}$$

is exact and naturally split, where we denote the splitting by s_e. By tensoring (19) with $\Omega^1_{D_e}$ we obtain that the sequence

$$0 \longrightarrow H^0(D_e, \mathcal{E}_K \otimes \Omega^1_{D_e}) \xrightarrow{\alpha_e} H^0(D_e, \mathcal{F}_K \otimes \Omega^1_{D_e}) \xrightarrow{\beta_e} H^0(D_e, \Omega^1_{D_e}) \longrightarrow 0$$

is exact, naturally split as sequence of $\mathcal{O}_{D_e}$-modules with connection and everything is compatible with restriction to X_e.

Using these splittings, we write $H^0(X_v, \mathcal{F}_K \otimes \Omega^1_{X_v}) = H^0\big(X_v, \mathcal{E}_K \otimes \Omega^1_{X_v}\big) \oplus H^0\big(X_v, \Omega^1_{X_v}\big)$ and similarly for sections over X_e and D_e.

Now we go back to proving that j_v is injective for all v. Suppose that $j_v(z_v) = 0$, i.e. for every $e \in \mathscr{E}_v$, $\alpha_e(\omega_e) = \eta_v|_{X_e} - \rho_e|_{X_e} - \nabla(f_e)$, where $\eta_v \in H^0\big(X_v, \mathcal{F}_K \otimes \Omega^1_{X_v}\big)$, $\rho_e \in H^0\big(D_e, \mathcal{F}_K \otimes \Omega^1_{D_e}\big)$, $f_e \in H^0\big(X_e, \mathcal{F}_K\big)$.

Using the decompositions above we write (uniquely): $\eta_v = \eta_{v,E} + \eta_{v,\mathcal{O}}$, $\rho_e = \rho_{e,E} + \rho_{e,\mathcal{O}}$ and $f_e = f_{e,E} + f_{e,\mathcal{O}}$, with $\eta_{v,E} \in H^0(X_v, \mathcal{E}_K \otimes \Omega^1_{X_v})$, $\rho_{e,E} \in H^0\big(D_e, \mathcal{E}_K \otimes \Omega^1_{D_e}\big)$ etc.

Using the fact that the decompositions respect the connections and the restrictions to X_e, we obtain:

$$\omega_e - \Big(\eta_{v,E}|_{X_e} - \rho_{e,E}|_{X_e} - \nabla(f_{e,E})\Big) = \eta_{v,\mathcal{O}_X}|_{X_e} - \rho_{e,\mathcal{O}_X}|_{X_e} - d_X(f_{e,\mathcal{O}}).$$

As the decomposition is a direct sum decomposition the LHS and the RHS are 0.

Therefore $\omega_e = \eta_{v,E}|_{X_e} - \rho_{e,E}|_{X_e} - \nabla(f_{e,E})$ for every $e \in \mathscr{E}_v$ and we have z_v=0. □

A An example for a Tate curve

In this paragraph we use explicit calculations to confirm theorem 10, i.e. that the sequence (5) is not exact for a certain non-trivial unipotent F-isocrystal E on a specific Tate curve.

Let X be a Tate elliptic curve over K with invariant q, where $q \in m_{\mathcal{V}}$. We consider $x \in H^1_{\mathrm{rig}}(X_k)$. Thanks to what we said before $\varphi^*_{\mathcal{O}}(x \otimes 1)$ in $H^1_{\mathrm{dR}}(X_K)$ is such that $N(\varphi^*_{\mathcal{O}}(x \otimes 1)) = 0$; since $H^1_{\mathrm{dR}}(X_K)$ is a 2-dimensional K-vector space, then $\mathrm{Im}(N) = \mathrm{Ker}(N)$, hence $\varphi^*_{\mathcal{O}}(x \otimes 1) \in \mathrm{Im}(N)$. This means that in this case the hypothesis of the theorem 10 are satisfied.

Every element in $H^1_{\mathrm{rig}}(X_k)$ corresponds to an extension of the trivial F-isocrystal by itself (proposition 1.3.1 of [ChLeS]), hence the element x corresponds to the following exact sequence

$$0 \to \mathcal{O} \to E \to \mathcal{O} \to 0.$$

As before we consider $\varphi^*_{\mathcal{O}}(x \otimes 1) \in H^1_{\mathrm{dR}}(X_K)$ and the exact sequence of modules with connections induced by the one above:

$$0 \to (\mathcal{O}_{X_K}, d) \to (\mathcal{E}, \nabla) \to (\mathcal{O}_{X_K}, d) \to 0.$$

We suppose from now on that $\mathrm{ord}_\pi q = 3$. Then the graph associate to X is a triangle with vertices I, II, III and edges $[I, II], [II, III], [I, III]$.

The element $\varphi^*_{\mathcal{O}}(x \otimes 1)$, as hypercocycle, can be written as $(0_v, g_e)$ with $g_e \in H^0(X_e)$; in particular $d(g_e) = 0$, so $g_e \in K$. Moreover since E is an F-isocrystal, the class x is fixed by the Frobenius of $H^1_{\mathrm{rig}}(X_k)$ ([ChLeS] prop 3.2.1), in particular we can take $g_e \in \mathbb{Q}_p$ for every e.

The $\mathcal{O}_{X_K}$-module $\mathcal{E}$ is locally free: on X_v it has a basis given by $e_{1,v}, e_{2,v}$ and on X_w it has a basis given by $e_{1,w}, e_{2,w}$. If on X_e we choose $e_{1,v}, e_{2,v}$ as basis, then the changing basis matrix is given by

$$\begin{pmatrix} 1 & g_e \\ 0 & 1 \end{pmatrix}$$

and the connection on X_e is given by the direct sum of the two trivial connections.

Now we consider $(\omega_v, f_e) \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla))$, then

$$\omega_v = h_{1,v} e_{1,v} + h_{2,v} e_{2,v},$$

$$\omega_w = h_{1,w} e_{1,w} + h_{2,w} e_{2,w},$$

$$\omega_{w_{|X_e}} = (h_{1,w} + g_e h_{2,w}) e_{1,v} + h_{2,w} e_{2,w},$$

with $h_{1,v}$ and $h_{2,v}$ elements of $\Omega^1_{X_v}$ and $h_{1,w}$ and $h_{2,w}$ elements of $\Omega^1_{X_w}$. Let us suppose now that $(\omega_v, f_e) \in \mathrm{Kernel}(N_{\mathcal{E}})$, which means that

$$N_{\mathcal{E}}(\omega_v, f_e) = (0, Res_{|X_e}\omega_v) = 0 \ \text{ in } \ H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)),$$

but as the map from $H^1(Gr, \mathcal{E})$ to $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla))$ is injective, we have that $Res_{|X_e}\omega_v$ is zero as element of $H^1(Gr, \mathcal{E})$.

Let us write the system which tells us that an element $a_e = (a_e^1, a_e^2) \in H^1(Gr, \mathcal{E}) = \frac{\oplus_e H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla))}{\oplus_v H^0_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla))}$, written in coordinates with respect to the basis $e_{v,1}, e_{v,2}$, is zero:

$$\begin{cases} a^1_{[I,II]} = a^1_I - a^1_{II} - g_{[I,II]} a^2_{II} \\ a^2_{[I,II]} = a^2_I - a^2_{II} \end{cases}$$

$$\begin{cases} a^1_{[II,III]} = a^1_{II} - a^1_{III} - g_{[II,III]} a^2_{III} \\ a^2_{[II,III]} = a^2_{II} - a^2_{III} \end{cases}$$

$$\begin{cases} a^1_{[I,III]} = a^1_I - a^1_{III} - g_{[I,III]} a^2_{III} \\ a^2_{[I,III]} = a^2_I - a^2_{III} \end{cases}$$

Moreover from the Gysin sequence ([ChLeS] proposition 2.1.4), applied to every component C_v of X_k (on every wide open X_v $(\mathcal{E}, \nabla)$ is the direct sum of two copies of $(\mathcal{O}_X, d)$), we can derive the following equations:

$$\begin{cases} a^1_{[I,II]} + a^1_{[I,III]} = 0 \\ a^1_{[II,III]} + a^1_{[II,I]} = 0 \\ a^1_{[III,I]} + a^1_{[III,II]} = 0 \end{cases}$$

$$\begin{cases} a^2_{[I,II]} + a^2_{[I,III]} = 0 \\ a^2_{[II,III]} + a^2_{[II,I]} = 0 \\ a^2_{[III,I]} + a^2_{[III,II]} = 0 \end{cases}$$

Putting together the previous equations and writing a linear system in terms of the a_v's, we find the following matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & -g_{[I,II]} & -1 & -g_{[I,III]} \\ 0 & 2 & 0 & -1 & 0 & -1 \\ -1 & -g_{[II,I]} & 2 & 0 & -1 & -g_{[II,III]} \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & -g_{[III,I]} & -1 & -g_{[III,II]} & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{pmatrix}$$

where $g_{[I,II]} = -g_{[II,I]}$, $g_{[II,III]} = -g_{[III,II]}$ and $g_{[I,III]} = -g_{[III,I]}$. The matrix A has determinant equal to zero and dimension of the rank equal to 4. Two generators of the Kernel are the following vectors:

$$K_1 = (1, 0, 1, 0, 1, 0)$$

$$K_2 = (\frac{1}{3}g_{[I,II]} + \frac{2}{3}g_{[I,III]} + \frac{1}{3}g_{[II,III]}, 1, -\frac{1}{3}g_{[I,II]} + \frac{1}{3}g_{[I,III]} + \frac{2}{3}g_{[II,III]}, 1, 0, 1).$$

If we now write K_1 and K_2 as elements of $H^1(Gr, \mathcal{E})$, i.e. as elements of $\oplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla))$, we find the following vectors:

$$H_1 = (0, 0, 0, 0, 0, 0),$$

$$H_2 = (-\frac{1}{3}g_{[I,II]} - \frac{1}{3}g_{[II,III]} + \frac{1}{3}g_{[I,III]}, 0, -\frac{1}{3}g_{[I,II]} - \frac{1}{3}g_{[II,III]} + \frac{1}{3}g_{[I,III]}, 0, \frac{1}{3}g_{[I,II]} + \frac{1}{3}g_{[II,III]} - \frac{1}{3}g_{[I,III]}, 0).$$

These computations show that the Kernel of $N_{\mathcal{E}}$ consists of the $(\omega_v, f_e) \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla))$ such that $Res_{|X_e}\omega_v$ equals H_1 or H_2. The elements (ω_v, f_e) of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla))$ which are such that $Res_{|X_e}\omega_v = H_1$ are the elements that come from $H^1_{\mathrm{rig}}(X_k, E) \otimes K$.

Let us consider now the subvector space

$$V = \{(\omega_v, f_e) | Res_{|X_e}\omega_v = tH_2, \text{with}\, t \in K\}$$

Clearly the elements of $\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$ are contained in V and one can see that $V/\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$ is a 1-dimensional vector space, in fact two elements in V are multiples one of the other modulo an element of $\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$.

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