Overconvergent modular sheaves and modular forms for $\mathbf{GL}_{2/F}$

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1 Introduction

Let $p \geq 3$ be a prime integer, K a finite extension of $\mathbb{Q}_p$ and $N \geq 4$ a positive integer not divisible by p. We fix once for all an algebraic closure $\overline{K}$ of K and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We denote by $\mathbb{C}_p$ the completion of $\overline{K}$ and let v be the p-adic valuation of $\mathbb{C}_p$ normalized such that $v(p) = 1$. We write $\tau \colon \mathbb{F}_p^* \to \mu_{p-1} \subset \mathbb{Q}_p^*$ for the Teichmüller character.

Let F be a totally real number field of degree $g \geq 1$ over $\mathbb{Q}$ in which p is unramified and let us denote by $\mathcal{W}_F$ the rigid analytic space over $\mathbb{Q}_p$ called *the weight space*, which is the rigid space attached to the noetherian complete algebra $\mathbb{Z}_p[[(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times]]$. Let us fix an $\mathcal{O}_K$-algebra A which is a p-adically complete and separated integral domain, formally smooth and topologically of finite type over $\mathcal{O}_K$ and a weight $k \in \mathcal{W}_F(A_K)$. Here $A_K := A \otimes_{\mathcal{O}_K} K$ and we may view k as a continuous group homomorphism $k : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \longrightarrow A_K^\times$. We choose w a rational number such that $0 \leq w < 2/p^{r-1}$ if $p > 3$ and $0 \leq w < 1/3^r$ if $p = 3$, where r is a certain constant determined by k (see section §3.3.) We call such a w *adapted* to k.

The main purpose of this article is to attach to the data (A, F, N, w, k) an *overconvergent modular sheaf* $\omega_A^{\dagger,k}$ and to this modular sheaf the A_K-module of *overconverget Hilbert modular forms* of weight k, tame level μ_N and degree of overconvergence w denoted $\overline{M}(k, \mu_N, w)$.

The first remark is that if $A = \mathcal{O}_K$ and $k \in \mathcal{W}_F(K)$ then $\overline{M}(k, \mu_N, w)$ is the K-vector space of what would usually be called overconvergent Hilbert modular forms of weight k. If, on the other hand we choose A such that $U := \mathrm{Spm}(A_K)$ is an admissible affinoid open of $\mathcal{W}_F$ and choose $k \in \mathcal{W}_F(A_K)$ to be the universal character attached to A_K, i.e. $k : (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \longrightarrow A_K^\times$ is defined by $t^k(\alpha) := t^\alpha$ for all $\alpha \in U \hookrightarrow \mathcal{W}_F$, then $\overline{M}(k, \mu_N, w)$ is the A_K-module usually called the *p-adic families of Hilbert modular forms over* U.

We show that the above mentioned constructions of overconvergent modular sheaves and overconvergent Hilbert modular forms are functorial in A and w and that we have natural Hecke operators $T_\mathfrak{q}$ and U_p on $\overline{M}(k, \mu_N, w)$, for every $\mathfrak{q}$ prime ideal of $\mathcal{O}_F$ not dividing Np. Moreover we show compatibility (as Hecke modules) of these overconvergent Hilbert modular forms with the classical Hilbert modular forms of tame level μ_N (for $A = \mathcal{O}_K$ and k a classical weight) and with Katz-type p-adic modular forms.

For $g = 1$ we show that the overconvergent modular forms defined in this article coincide (as Hecke modules) with the ones previously defined by R. Coleman in [C1].

Let us quickly sketch the new concept of *overconvergent modular sheaf* and how such objects define overconvergent Hilbert modular forms. For the precise definitions and details of proofs see chapters 3 and 4.

Let the notations be as at the beginning of this section. We define the following categories and functors.

• The category $\mathbf{FSchemes}_A$. It is the category whose objects are normal p-adic formal schemes topologically of finite type over $S := \mathrm{Spf}(A)$ and the morphisms are morphisms of formal schemes over S.

• The category $\mathbf{Sheaves}_K$. It is the category whose objects are pairs $(\mathcal{U}, \mathcal{F})$ where $\mathcal{U}$ is an object of $\mathbf{FSchemes}_A$ and $\mathcal{F}$ is a sheaf of $\mathcal{O}_\mathcal{U} \otimes_{\mathcal{O}_K} K$-modules on $\mathcal{U}$ such that there is a coherent $\mathcal{O}_\mathcal{U}$-module F on $\mathcal{U}$ with the property that $\mathcal{F} \cong F \otimes_{\mathcal{O}_K} K$. The morphisms in this category are defined in section §3.1.

• The category $\mathbf{Hilb}(\mu_N)_A^w$. It is the category whose objects are sequences $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$

where

– $\mathcal{U}$ is an object of $\mathbf{FSchemes}_A$ and $G \longrightarrow \mathcal{U}$ is a formal abelian scheme.

– ι denotes real multiplication by $\mathcal{O}_F$ on $G/\mathcal{U}$.

–λ is an identification of the sheaf of symmetric $\mathcal{O}_F$-homomorphisms from G to its dual, $G^\vee$, and the notion of positivity given by the sub-sheaf of polarizations with a representative of a class of the strict class group of F.

–ψ_N is a level μ_N-structure on $G/\mathcal{U}$.

–finally Y is a p^w-growth condition.

We send the reader to section §3.1 for the details and also for the definition of the morphisms in this category. We have natural functors $\mathbf{Hilb}(\mu_N)^w_A \longrightarrow \mathbf{FSchemes}_A$ (and $\mathbf{Sheaves}_K \longrightarrow \mathbf{FSchemes}_A$) defined on object by $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big) \to \mathcal{U}$ (respectively $(\mathcal{U}, \mathcal{F}) \to \mathcal{U}$.)

We prove that if $k \in \mathcal{W}_F(A_K)$ is a weight such that w is adapted to it, then:

1) There exists a cartesian functor $\omega^{\dagger,k}_A\colon \mathbf{Hilb}(\mu_N)^w_A \longrightarrow \mathbf{Sheaves}_K$ over $\mathbf{FSchemes}_A$ which we call an overconvergent modular sheaf of weight k.

2) The category $\mathbf{Hilb}(\mu_N)^w_A$ has a final object denoted $\underline{G}^{\rm univ}$ consisting of the abelian scheme $G^{\rm univ} \longrightarrow \mathfrak{M}(A, \mu_N)(w)$ with its extra structure, where $\mathfrak{M}(A, \mu_N)(w)$ is an appropriate formal model of the strict neighborhood of width p^w of the ordinary locus in the fine moduli space of abelian schemes over A with real multiplication by $\mathcal{O}_F$ and μ_N-level structure, denoted $\mathfrak{M}(A, \mu_N)$, and $G^{\rm univ}$ is the pull-back of the universal abelian scheme (see section §3.1 and lemma 3.2).

We denote

$$M(k, \mu_N, w) := H^0\big(\mathfrak{M}(A, \mu_N)(w), \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\big)$$

and call them *weakly holomorphic overconvergent modular forms of weight* k.

3) By construction there are natural Hecke operators U_p and $T_\mathfrak{q}$ on $M(k, \mu_N, w)$ for every $\mathfrak{q}$ prime ideal of $\mathcal{O}_F$ not dividing Np.

4) Let $\overline{\mathfrak{M}}(A, \mu_N)(w)$ denote the normalization of $\mathfrak{M}(A, \mu_N)(w)$ in a smooth, projective toroidal compactification $\overline{\mathfrak{M}}(A, \mu_N)$ of $\mathfrak{M}(A, \mu_N)$. We show that $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ extends uniquely to a coherent, locally free $\mathcal{O}_{\overline{\mathfrak{M}}(A,\mu_N)(w)} \otimes K$-module of rank 1 on $\overline{\mathfrak{M}}(A, \mu_N)(w)$ which will be also denoted $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$.

Moreover the A_K-module

$$\overline{M}(k, \mu_N, w) := H^0\big(\overline{\mathfrak{M}}(A, \mu_N)(w), \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\big)$$

which will be called the module of overconvergent Hilbert modular forms of weight k, is independent of the toroidal compactification.

5) If T is a Hecke operator on $M(k, \mu_N, w)$ then $T\big(\overline{M}(k, \mu_N, w)\big) \subset \overline{M}(k, \mu_N, w)$.

6) For certain *accessible* weights k we construct integral models of our overconvergent modular forms of weight k.

Finally let us comment on what we do not do in this article. First we do not treat the case $p = 2$ as the main results on canonical subgroups in the paper of Andreatta-Gasbarri [AGa], which we use in Appendix A and apply in Chapters 2 and 3, do not hold for $p = 2$.

We do not deal here with the case in which p is ramified in F, this is done in [AIP]. In that article we also prove that every finite slope overconvergent Hilbert cusp form sits in a p-adic family of finite slope cusp forms and therefore one can define the cuspidal part of the Hilbert modular eigenvariety.

We'd like to point out that Vincent Pilloni independently has constructed in [P], using a different idea, sheaves like our $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ in the case $g = 1$. In Appendix B we briefly recall his definition and prove that those overconvergent modular forms are the same as the ones defined in this article. Pilloni's definition goes through the construction of a "partial Igusa tower" over certain strict neighborhoods of the ordinary locus in $X_1(N)^{\rm rig}$ and it is a very natural, finite slope extension of Hida's construction of overconvergent ordinary modular forms. Our method is different, at least at the first sight: we emphasise certain universal torsors which we construct using p-adic Hodge theory. This method provides good integral structures for the Banach spaces of overconvergent modular forms. In [AIP1] we combined the two ideas and constructed the cuspidal part of the Eigenvariety for Siegel modular forms.

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Notations Throughout this article we'll use the following notations: if $u \in \mathbb{Q}$, we'll denote by p^u an element of $\mathbb{C}_p$ of valuation u. If R is an $\mathcal{O}_K$-algebra, $p^u \in \mathcal{O}_K$ and M is an object over R (an R-module, an R-scheme or formal scheme) then we denote by $M_u := M \otimes_R R/p^uR$. In particular $M_1 = M \otimes_R R/pR$.

2 The Hodge-Tate sequence for abelian schemes

We fix N, p and w where N is a positive integer, $p \geq 3$ is a prime integer and w is a rational number such that $0 \leq w < 1/p$. We denote by K a finite extension of $\mathbb{Q}_p$ containing an element of valuation w. Let k be its residue field.

Let us denote by R an $\mathcal{O}_K$-algebra which is a noetherian, p-adically complete and separated normal integral domain and let $A \longrightarrow \mathcal{U} := \mathrm{Spec}(R)$ be an abelian scheme of relative dimension g.

Let $\mathbb{K}$ be an algebraic closure of the fraction field of R which contains $\overline{K}$ and let $\eta = \mathrm{Spec}(\mathbb{K})$ denote the respective geometric generic point of $\mathcal{U}$. Let $T := T_p(A_\eta)$, where A_η is the fiber of A at η.

Our **standard local assumptions** will be the following.

(i) We suppose we are given a fixed R-subalgebra $\overline{R} \subseteq \mathbb{K}$ such that $\overline{R}_K$ is Galois over R_K, and we let $\mathcal{G}$ denote the Galois group.

(ii) We suppose the natural action of $\mathrm{Gal}\big(\mathbb{K}/\mathrm{Frac}(R)\big)$ on T factors through $\mathcal{G}$.

(iii) We suppose $(\widehat{\overline{R}})^{\mathcal{G}} = R$, where $\widehat{\overline{R}}$ denotes the p-adic completion of $\overline{R}$.

There are two cases of interest for us in this article as well as for future applications in which these local assumptions are satisfied:

a) The situation when R is a *small* $\mathcal{O}_K$*-algebra*, i.e. there is an $\mathcal{O}_K$-algebra W and $\mathcal{O}_K$-algebra morphisms

$$\mathcal{O}_K\{T_1, T_2, ..., T_d\}/(T_1T_2, ..., T_j - \pi^a) \longrightarrow W \longrightarrow R$$

where $1 \leq j \leq d$, $a \geq 0$ and the first morphism is étale and the second is finite and becomes étale after inverting p.

In this case we take $\overline{R}$ to be the inductive limit of all R-sub-algebras of $\mathbb{K}$ which are normal and finite and étale after inverting p. Then the theory of almost étale extensions ([F1], [GR]) guarantees the desired properties.

b) Suppose that R' is a small $\mathcal{O}_K$-algebra (as in (a) above) and that the abelian scheme A is defined over $\mathrm{Spf}(R')$. Then, (i)–(ii) hold for $R' \subset \overline{R}'$ thanks to (a). Let $\mathcal{U}' := \mathrm{Spf}(R')$ and let $\mathcal{U} = \mathrm{Spf}(R)$ denote the normalization of a formal open affine of an admissible blow-up of $\mathcal{U}$. In this case we take $\overline{R}$ to be the image of $\overline{R}' \otimes_{R'} R$ in $\mathbb{K}$.

Lemma 2.1. *The local assumptions (i), (ii) and (iii) hold in cases (a) and (b).*

Proof. In (a) one constructs an intermediate extension R_∞ equal to the normalization of the image of $R \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\infty}[T_1^{\frac{1}{p^\infty}}, \ldots, T_d^{\frac{1}{p^\infty}}]$ in $\overline{R}$ with Galois group Γ over R_K after inverting p. Here $K \subset K_\infty$ is the extension obtained by adding all p^n–roots of 1. Faltings' Almost Purity Theorem [F1, Thm. 4] guarantees that for every finite and étale extension $R_\infty[p^{-1}] \subset S$, the normalization S_∞ of R_∞ in S is almost Śhirley. Arguing as in the proof of [Br, Prop. 3.1.1] we get that $p(\widehat{\overline{R}})^{\mathcal{G}} \subset \big(\widehat{R}_\infty\big)^\Gamma$. One then proves that the latter is contained in $R \cdot \frac{1}{p}$ using the explicit action of Γ on the variables $T_i^{\frac{1}{p^n}}$; see the proof of [Br, Lemme 3.1.4& Prop. 3.1.8]. Thus, $p^2(\widehat{\overline{R}})^{\mathcal{G}} \subset R$ so that we have inclusions $R \subset (\widehat{\overline{R}})^{\mathcal{G}} \subset R \cdot \frac{1}{p^2}$. In particular, as R is noetherian and normal, to deduce the equality $R = (\widehat{\overline{R}})^{\mathcal{G}}$ it suffices to check it after localizing at height one prime ideals $\mathcal{P}$ of R over p. Replacing R with $R_{\mathcal{P}}$ we may then assume that R is a dvr in which case the inclusions $R \subset (\widehat{\overline{R}})^{\mathcal{G}} \subset R \cdot \frac{1}{p^2}$ imply that $R = (\widehat{\overline{R}})^{\mathcal{G}}$.

In (b) let R_∞ be the image of $R \otimes_{R'} R'_\infty$ in $\overline{R}$ with R'_∞ as above. Consider a normal extension $R'_\infty \subset S' \subset \overline{R}'$, finite and Śhirley after inverting p. Let S be the normalization of the image of $S' \otimes_{R'} R$ in $\overline{R}$. We claim that it is an almost étale extension of R_∞. This means that given the canonical idempotent $e_S \in S \otimes_{R_\infty} S[p^{-1}]$, see loc. cit., for every element p^δ in the maximal ideal of $\mathcal{O}_{K_\infty}$ we have that $p^\delta e_S \in S \otimes_{R_\infty} S$. As e_S is the image of the the canonical idempotent $e_{S'} \in S' \otimes_{R'_\infty} S'[p^{-1}]$ and $R'_\infty \subset S'$ is almost étale, this condition holds for $e_{S'}$ and, hence, for e_S. We conclude that $\overline{R}$ is the direct limit of almost étale extensions of R_∞. Arguing as in loc. cit. we deduce that the map $\big(\widehat{R_\infty}\big)^\Gamma \to (\widehat{\overline{R}})^{\mathcal{G}}$ has cokernel annihilated by multiplication by p so that $p(\widehat{\overline{R}})^{\mathcal{G}} \subset \big(\widehat{R}_\infty\big)^\Gamma$. Note that Γ acts as a subgroup of automorphisms of $R \subset R_\infty$ so that $p\widehat{R}_\infty^\Gamma \subset R$ because of the explicit action of Γ on the variables $T_i^{\frac{1}{p^n}}$ and on $\mathcal{O}_{K_\infty}$. Thus we have inclusions $R \subset (\widehat{\overline{R}})^{\mathcal{G}} \subset R \cdot \frac{1}{p^2}$ and we conclude that $R = (\widehat{\overline{R}})^{\mathcal{G}}$ arguing as in case (a). □

¿From now on we assume that R is such that the local assumptions are satisfied. We denote by $\omega_{A/R} := \pi_*(\Omega^1_{A/R})$ and will assume that $\omega_{A/R}$ is a free $\mathcal{O}_{\mathcal{U}}$-module of rank g.

We now consider the relative Frobenius morphism $\varphi_A\colon R^1\pi_*(\mathcal{O}_{A_1}) \longrightarrow R^1\pi_*(\mathcal{O}_{A_1})$, and let $\det(\varphi_A)$ denote the ideal of R/pR generated by the determinant of φ_A in a basis of $R^1\pi_*(A_1)$. We assume that there exists $0 \le w < 1/p$ such that $p^w \in \det(\varphi_A)$ and $p^w \in \det(\varphi_{A^\vee})$. By [AGa] it follows that there exists a canonical subgroup $C \subset A_K[p]$ of A_K defined over $\mathcal{U}_K$. We let $D \subset A_K[p]^\vee \cong (A^\vee)_K[p]$ be the Cartier dual of $A_K[p]/C$ over $\mathcal{U}_K$. It is proven by Fargue in [Fa2] that $A_K^\vee$ also has a canonical subgroup and that it coincides with D.

Let us remark that as a consequence of the local assumptions we have an isomorphism as $\mathcal{G}$-modules $T \cong \lim\limits_{\leftarrow, n} A[p^n](\overline{R})$.

We'd now like to recall a classical construction which will be essential for the rest of this article, namely the map dlog. Let G be a finite and locally free group scheme over $\mathcal{U}$ annihilated by p^m, let $G^\vee$ denote its Cartier dual and we denote by $\omega_{G^\vee/R}$ the R-module of global invariant differentials on $G^\vee$. We fix an affine, noetherian, normal scheme p-torsion free $S \to \mathcal{U}$ and define the map

$$\mathrm{dlog}_{G,S}\colon G(S_K) \longrightarrow \omega_{G^\vee/R} \otimes_R \mathcal{O}_S/p^m\mathcal{O}_S$$

as follows. Let x be an S_K-point of G. Since S is normal, affine and p-torsion free and G is finite and flat over $\mathcal{U}$, x extends uniquely to an S-valued point of G, abusively denoted x. Such a point corresponds to a group scheme homomorphism over S, $f_x\colon G^\vee \longrightarrow \mathbf{G}_{\mathrm{m}}$ and we set $\mathrm{dlog}(x) = f_x^*(dT/T)$ where dT/T is the standard invariant

Lemma 2.2. *The map* dlog *is functorial with respect to $\mathcal{U}, G$ and S as follows:*

a) Let $\mathcal{U}' \longrightarrow \mathcal{U}$ be a morphism of schemes and let $G \longrightarrow \mathcal{U}$ be a finite locally free group scheme. We denote by $G' \longrightarrow \mathcal{U}'$ the base change of G to $\mathcal{U}'$ and let $S \longrightarrow \mathcal{U}'$ be a morphism with S normal, noetherian, affine and flat over $\mathcal{O}_K$. Then the natural diagram commutes

$$\begin{array}{ccc} G'(S_K) & \xrightarrow{\mathrm{dlog}_{G',S}} & \omega_{(G')^\vee/R'} \otimes_{R'} \mathcal{O}_S/p^m\mathcal{O}_S \\ \downarrow & & \downarrow \\ G(S_K) & \xrightarrow{\mathrm{dlog}_{G,S}} & \omega_{G^\vee/R} \otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \end{array}$$

b) Let G and G' be group schemes, finite and locally free over $\mathcal{U} = Spec(R)$ and $G' \longrightarrow G$ a homomorphism of group schemes over $\mathcal{U}$. As before we fix a morphism $S \longrightarrow \mathcal{U}$ with S normal, noetherian, affine and flat over $\mathcal{O}_K$. Then, we have a natural commutative diagram

$$\begin{array}{ccc} G'(S_K) & \xrightarrow{\mathrm{dlog}_{G',S}} & \omega_{(G')^\vee/R} \otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \\ \downarrow & & \downarrow \\ G(S_K) & \xrightarrow{\mathrm{dlog}_{G,S}} & \omega_{G^\vee/R} \otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \end{array}$$

c) Finally let us suppose that we have a morphism of normal, noetherian, affine schemes $S' \longrightarrow S$ over $\mathcal{U}$, which are flat over $\mathcal{O}_K$. Then we have a natural commutative diagram

$$\begin{array}{ccc} G(S_K) & \xrightarrow{\mathrm{dlog}_{G,S}} & \omega_{(G)^\vee/R} \otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \\ \downarrow & & \downarrow \\ G(S'_K) & \xrightarrow{\mathrm{dlog}_{G,S'}} & \omega_{G^\vee/R} \otimes_R \mathcal{O}_{S'}/p^m\mathcal{O}_{S'} \end{array}$$

Proof. The proof is standard and we leave it to the reader. □

Applying the construction above to the group schemes $A^\vee[p^n] \cong (A[p^n])^\vee$ for $n \geq 1$ over the tower of normal R-algebras S whose union is $\overline{R}$ (see above) we obtain compatible $\mathcal{G}$-equivariant maps (for varying n)

$$\mathrm{dlog}_n \colon A^\vee[p^n](\overline{R}_K) \longrightarrow \omega_{A[p^n]/R} \otimes_R \overline{R}/p^n\overline{R} \cong \omega_{A/R} \otimes_R \overline{R}/p^n\overline{R}.$$

By taking the projective limit of these maps we get the morphism of $\mathcal{G}$-modules

$$\mathrm{dlog}_{A^\vee} \colon T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \longrightarrow \omega_{A/R} \otimes_R \widehat{\overline{R}}.$$

We also have the analogous map for A itself, i.e., a map $\mathrm{dlog}_A \colon T_p(A_\eta) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \longrightarrow \omega_{A^\vee/R} \otimes_R \widehat{\overline{R}}$. The Weil pairing identifies $T_p(A_\eta)$ with the $\mathcal{G}$-module $T_p(A_\eta^\vee)^\vee(1)$ so that the $\widehat{\overline{R}}$-dual of dlog_A provides a map $a \colon \omega_{A^\vee/R}^\vee \otimes_R \widehat{\overline{R}}(1) \longrightarrow T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}$. We thus obtain the following sequence of $\widehat{\overline{R}}$-modules compatible with the semi-linear action of $\mathcal{G}$ which we call "the Hodge-Tate sequence attached to A"

$$(*) \quad 0 \longrightarrow \omega_{A^\vee/R}^\vee \otimes_R \widehat{\overline{R}}(1) \xrightarrow{a} T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \xrightarrow{\mathrm{dlog}} \omega_{A/R} \otimes_R \widehat{\overline{R}} \longrightarrow 0.$$

Since $\mathrm{H}^0\big(\mathcal{G}, \widehat{\overline{R}}(-1)\big) = 0$ we have that $\mathrm{dlog} \circ a = 0$, i.e., this sequence is in fact a complex. We prove (proposition 6.1):

Theorem 2.3. *The cokernel of the map* dlog *in the sequence* $(*)$*, which we call the Hodge-Tate sequence attached to A, is annihilated by* p^v *for* $v = w/(p-1)$*.*

Remark 2.4. It follows from [Br] that p is not a zero divisor in $\widehat{\overline{R}}$, therefore the morphism a above is injective.

G. Faltings proved that the homology of the given sequence is annihilated by $p^{\frac{1}{p-1}}\mathcal{D}_{K/K_0}$ where $\mathcal{D}_{K/K_0}$ is the different of K over the fraction field K_0 of $\mathrm{W}(k)$. L. Fargues improved this result showing that it is annihilated by $p^{\frac{1}{p-1}}$. One can prove that A is ordinary if and only if the theorem holds with $v = 0$; see [AGa, §13.6]. The general case is a consequence of proposition 5.1 of the appendix and proposition 2.5 (below).

¿From now on whenever we write D, $A^\vee[p]$ and $A^\vee[p]/D$ we mean the $\mathcal{G}$-representations $D(\overline{R}_K)$, $A^\vee[p](\overline{R}_K)$ and $\big(A^\vee[p]/D\big)(\overline{R}_K)$ respectively. Let us denote by $F^0 := \mathrm{Im}(\mathrm{dlog})$ and $F^1 = \mathrm{Ker}(\mathrm{dlog})$. They are $\widehat{\overline{R}}$-modules and because dlog is $\mathcal{G}$-equivariant it follows that F^0 and F^1 have natural continuous actions of $\mathcal{G}$.

Proposition 2.5. *The* $\widehat{\overline{R}}$*-modules* F^0 *and* F^1 *are free of rank* g *and we have a commutative diagram with exact rows and vertical isomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1/p^{1-v}F^1 & \longrightarrow & T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \overline{R}/p^{1-v}\overline{R} & \longrightarrow & F^0/p^{1-v}F^0 & \longrightarrow & 0 \\ & & \downarrow\cong & & || & & \downarrow\cong & & \\ 0 & \longrightarrow & D \otimes \overline{R}/p^{1-v}\overline{R} & \longrightarrow & A^\vee[p] \otimes \overline{R}/p^{1-v}\overline{R} & \longrightarrow & (A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R} & \longrightarrow & 0. \end{array}$$

Moreover, the cohomology of the Hodge-Tate sequence $(*)$ *is annihilated by* p^v*.*

Proof. We divide the proof in three steps.

Step 1: F^0 is a free $\widehat{\overline{R}}$-module of rank g and p^v annihilates $\mathrm{Coker}(d\log)$. It follows from proposition 6.1 that the mod p reduction of dlog factors via a map

$$\alpha\colon \big(A^\vee[p]/D\big)\otimes_{\mathbb{F}_p}\overline{R}/p\overline{R}\longrightarrow \omega_{A/R}\otimes_R \overline{R}/p\overline{R}.$$

Choose elements $\{\tilde{f}_1,\ldots,\tilde{f}_g\}$ of $T_p(A_\eta^\vee)/pT_p(A_\eta^\vee)$ which provide a basis of D over $\mathbb{F}_p$. We also fix a basis $\{\omega_1,\omega_2,\ldots,\omega_g\}$ of $\omega_{A/R}$. If $t_1,t_2,\ldots,t_g$ are elements of a group we denote by $\underline{t}$ the column vector with those coefficients and by $\underline{\tilde{t}}$ the reduction of $\underline{t}$ modulo p. Let us denote by $\underline{\tilde{\delta}}\in M_{g\times g}(\overline{R}/p\overline{R})$ the matrix with the property that $\alpha(\underline{\tilde{f}})=\underline{\tilde{\delta}}\cdot\underline{\tilde{\omega}}$. Let us now denote by $\underline{\delta}\in M_{g\times g}(\widehat{\overline{R}})$ any matrix such that the image of $\underline{\delta}$ under the natural projection $M_{g\times g}(\widehat{\overline{R}})\longrightarrow M_{g\times g}(\overline{R}/p\overline{R})$ is $\underline{\tilde{\delta}}$. Let $G^0\subset\omega_{A/R}\otimes_R\widehat{\overline{R}}$ be the $\widehat{\overline{R}}$-module generated by the vectors $\underline{\delta}\cdot\underline{\omega}$. Thus Step 1 follows if we prove:

Lemma 2.6. *(1) There exists a matrix $\underline{s}\in M_{g\times g}(\widehat{\overline{R}})$ such that $\underline{\delta}\cdot\underline{s}=\underline{s}\cdot\underline{\delta}=p^v\mathrm{Id}$.*

(2) The $\widehat{\overline{R}}$-module G^0 is free of rank g and it contains $p^v\omega_{A/R}\otimes_R\widehat{\overline{R}}$.

(3) The $\widehat{\overline{R}}$-module G^0 coincides with F^0.

In particular, F^1 is a finite and projective $\widehat{\overline{R}}$-module of rank g.

Proof. The last statement follows from the others.

(1) We use proposition 6.1: as $p^v\mathrm{Coker}(\mathrm{dlog})=0$ and $\underline{\delta}\cdot\underline{\omega}$ generates the image of dlog modulo p, it follows that there are matrices A and $B\in M_{g\times g}(\widehat{\overline{R}})$ such that

$$p^v\underline{\omega}=\mathrm{dlog}(d)=pA\cdot\underline{\omega}+B\underline{\delta}\cdot\underline{\omega}.$$

Therefore, we have $p^v\big(\mathrm{Id}-p^{1-v}A\big)=B\underline{\delta}$ and as $\mathrm{Id}-p^{1-v}A$ is invertible we obtain that $\underline{s}\cdot\underline{\delta}=p^v\mathrm{Id}$. Let us now recall that p is a non-zero divisor in $\widehat{\overline{R}}$, therefore the natural morphism $M_{g\times g}(\widehat{\overline{R}})\longrightarrow M_{g\times g}(\widehat{\overline{R}}[1/p])$ is injective and in the target module the matrices $\underline{\delta}$ and $p^{-v}\underline{s}$ are inverse one to the other. Therefore we obtain the relation $\underline{\delta}\cdot\underline{s}=p^v\mathrm{Id}$ first in $M_{g\times g}(\widehat{\overline{R}}[1/p])$ and then even in $M_{g\times g}(\widehat{\overline{R}})$.

(2) By construction G^0 is generated by the g vectors $\underline{\delta}\cdot\underline{\omega}$. Let $\underline{a}\in\big(\widehat{\overline{R}}\big)^g$ be a row vector such that $\underline{a}\cdot\underline{\delta}\cdot\underline{\omega}=0$. Since $\underline{\delta}$ is invertible after inverting p and $\underline{\omega}$ is a basis of $\omega_{A/R}$, we have that $\underline{a}=0$ in $\widehat{\overline{R}}[p^{-1}]^g$. Since p is not a zero divisor in $\widehat{\overline{R}}$ and $\underline{\delta}$ is invertible after inverting p, we conclude that $\underline{a}=0$. Moreover, we have $p^v\underline{\omega}=\underline{s}\cdot\underline{\delta}\cdot\underline{\omega}$. Hence, the last claim follows.

(3) For every $d\in T_p(A_\eta^\vee)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}$ there exists A and $B\in M_{g\times g}(\widehat{\overline{R}})$ such that $\mathrm{dlog}(d)=A\underline{\delta}\cdot\underline{\omega}+pB\underline{\omega}$. Since $p^v\omega_{A/R}\otimes_R\widehat{\overline{R}}$ is contained in G^0, we conclude that $F^0\subset G^0$. Similarly, since $p^v\mathrm{Coker}(\mathrm{dlog})=0$ we have that $p^v\omega_{A/R}\otimes_R\widehat{\overline{R}}\subset F^0$. The vectors $\underline{\delta}\cdot\underline{\omega}$ are contained in $F^0+p\omega_{A/R}\otimes_R\widehat{\overline{R}}$ which is contained in F^0. The conclusion follows.

□

Step 2: We prove that we have a commutative diagram

$$\begin{array}{ccccccccc} & & \omega^\vee_{A^\vee/R}\otimes_R \overline{R}/p\overline{R} & \stackrel{\tilde a}{\longrightarrow} & T_p(A^\vee_\eta)\otimes \overline{R}/p\overline{R} & \stackrel{\widetilde{\rm dlog}}{\longrightarrow} & \omega_{A/R}\otimes \overline{R}/p\overline{R} & & \\ & & \beta\downarrow & & || & & \uparrow\alpha & & \\ 0 & \longrightarrow & D\otimes \overline{R}/p\overline{R} & \longrightarrow & A^\vee[p]\otimes \overline{R}/p\overline{R} & \longrightarrow & \big(A^\vee[p]/D\big)\otimes \overline{R}/p\overline{R} & \longrightarrow & 0 \end{array}$$

Let H^0 and H^1 be the image and respectively the kernel of the map $\mathrm{dlog}\colon T_p(A_\eta)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\longrightarrow \omega_{A^\vee/R}\otimes_R\widehat{\overline{R}}$. Since also $A^\vee$ admits a canonical subgroup, we know from Step 1 that H^0 is a free $\widehat{\overline{R}}$-module of rank g and H^1 is a finite and projective $\widehat{\overline{R}}$-module of rank g. It follows from [F, §3, lemma 2] that H^1 and F^1 are orthogonal with respect to the perfect pairing $\big(T_p(A_\eta)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\big)\times\big(T_p(A^\vee_\eta)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\big)\longrightarrow\widehat{\overline{R}}(1)$ defined by extending $\widehat{\overline{R}}$-linearly the Weil pairing. In particular, via the isomorphism

$$h\colon T_p(A^\vee_\eta)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\longrightarrow T_p(A^\vee_\eta)^\vee\otimes\widehat{\overline{R}}(1),$$

induced by the pairing, we have $h(F^1)\subset (H^0)^\vee(1)$. Thus, h induces a morphism $h'\colon F^0\longrightarrow (H^1)^\vee(1)$. Since H^1 is a projective $\widehat{\overline{R}}$-module, the map $T_p(A^\vee_\eta)^\vee\otimes\widehat{\overline{R}}(1)\to (H^1)^\vee(1)$ is surjective so that h' is a surjective morphism of finite and projective $\widehat{\overline{R}}$-modules of the same rank and, hence, it must be an isomorphism. This implies that h induces an isomorphism $F^1\cong (H^0)^\vee(1)$. Since $(H^0)^\vee/p(H^0)^\vee)(1)\subset T_p(A^\vee_\eta)^\vee\otimes\widehat{\overline{R}}/p\widehat{\overline{R}}(1)$ is identified with $D\otimes\overline{R}/p\overline{R}\subset A^\vee[p]\otimes\overline{R}/p\overline{R}$ via h, we get the claim in Step 2.

Step 3: End of proof. From Step 1 (applied to the abelian scheme A/R) we have that $p^v\big(\omega_{A^\vee/R}\otimes_R\widehat{\overline{R}}\big)\subset H^0$ and from Step 2 we have an isomorphism $F^1\cong\big(H^0\big)^\vee(1)$. Consider the map

$$\gamma\colon F^1\subset T_p(A^\vee_\eta)\otimes\overline{R}\longrightarrow A^\vee[p]\otimes\overline{R}/p\overline{R}\longrightarrow\big(A^\vee[p]/D\big)\otimes\overline{R}/p\overline{R}.$$

Note that that $\omega^\vee_{A/R}\otimes_R\overline{R}/p\overline{R}$ goes to zero in $\big(A^\vee[p]/D\big)\otimes\overline{R}/p\overline{R}$ by Step 2 and the latter is a free $\overline{R}/p\overline{R}$-module. Since $\overline{R}$ is p-torsion free, the subset of elements of $\overline{R}/p\overline{R}$ annihilated by p^v coincides with $p^{1-v}\overline{R}/p\overline{R}$. We conclude that the image of the map γ is contained in $p^{1-v}\big(A^\vee[p]/D\big)\otimes\overline{R}/p\overline{R}$ so that it is zero modulo p^{1-v}. In particular, the map $T_p(A^\vee_\eta)\otimes\overline{R}\longrightarrow A^\vee[p]\otimes\overline{R}/p^{1-v}\overline{R}\longrightarrow\big(A^\vee[p]/D\big)\otimes\overline{R}/p^{1-v}\overline{R}$ induces a map $F^0/p^{1-v}F^0\longrightarrow\big(A^\vee[p]/D\big)\otimes\overline{R}/p^{1-v}\overline{R}$. This is a surjective morphism of free $\overline{R}/p^{1-v}\overline{R}$-modules of the same rank. Hence, it must be an isomorphism. More concretely, it is defined by sending the reduction of $\underline{\delta}\cdot\underline{\omega}$ modulo p^{1-v}, to the basis $\underline{\tilde f}$ of D. Since F^0 is a free $\widehat{\overline{R}}$-module, we can find a (non canonical) splitting $T_p(A^\vee_\eta)\otimes\overline{R}=F^0\oplus F^1$. In particular, $F^1/p^{1-v}F^1$ injects into $T_p(A^\vee_\eta)\otimes\overline{R}/p^{1-v}\overline{R}$ and it must coincide with $D\otimes\overline{R}/p^{1-v}\overline{R}$. This provides the diagram in the statement of proposition 2.5.

Note that $D\otimes\overline{R}/p^{1-v}\overline{R}$ is a free $\overline{R}/p^{1-v}\overline{R}$-module of rank g. Since F^1 is a projective $\widehat{\overline{R}}$-module of rank g, any lift of a basis of $D\otimes\overline{R}/p^{1-v}\overline{R}$ to elements of F^1 provides a basis of the latter as $\widehat{\overline{R}}$-module. We conclude that also F^1 is a free $\widehat{\overline{R}}$-module of rank g as claimed.

□

Let us now suppose that $(A^\vee[p]/D)(\overline{R})=(A^\vee[p]/D)(R)$ and therefore, using 6.2, it follows that $\underline{\tilde\delta}\in M_{g\times g}(R/pR)$. We have the following

Proposition 2.7. *Let $\underline{\delta}_0 \in M_{g\times g}(R)$ be any lift of $\underline{\tilde{\delta}}$ in $M_{g\times g}(R)$. Let us denote by $G_0 \subset \omega_{A/R}$ the R-sub-module generated by the entries of $\underline{\delta}_0 \cdot \underline{\omega}$.*

1) Then, G_0 is a free R-module of rank g with basis $\underline{\delta}_0 \cdot \underline{\omega}$ and $G_0 \otimes_R \widehat{\overline{R}} \cong F^0$.

2) The R-module $F_0 := (F^0)^{\mathcal{G}} \subset \omega_{A/R}$ coincides with G_0. In particular, $F_0 \otimes_R \widehat{\overline{R}} \cong F^0$.

3) We have a natural isomorphism $F_0/p^{1-v}F_0 \cong (A^\vee[p]/D) \otimes R/p^{1-v}R$ whose base change via $R \to \widehat{\overline{R}}$ provides the isomorphism $F^0/p^{1-v}F^0 \cong (A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R}$ in 2.5 via the isomorphism $F_0 \otimes_R \widehat{\overline{R}} \cong F^0$.

4) Let $F_1 := (F_0)^\vee(1)$, then we have a natural $\mathcal{G}$-equivariant isomorphism of $\widehat{\overline{R}}$-modules: $F_1 \otimes_R \widehat{\overline{R}} \cong F^1$.

Proof. (1) As $\underline{\delta}_0(\bmod p\widehat{\overline{R}}) = \underline{\tilde{\delta}}$ lemma 2.6 implies that there is an $\underline{s}_0 \in M_{g\times g}(\widehat{\overline{R}})$ such that $\underline{\delta}_0 \cdot \underline{s}_0 = \underline{s}_0 \cdot \underline{\delta}_0 = p^v\mathrm{Id}$. In $M_{g\times g}\big(\widehat{\overline{R}}[1/p]\big)$ we have $\underline{s}_0 = p^{-v}\underline{\delta}_0 \in M_{g\times g}\big(R[1/p]\big) \cap M_{g\times g}(\widehat{\overline{R}})$. But $R[1/p] \cap \widehat{\overline{R}} = R$ because R is normal. Hence, $\underline{\delta}_0 \cdot \underline{s}_0 = \underline{s}_0 \cdot \underline{\delta}_0 = p^v\mathrm{Id}$. This implies that G_0 is a free R-module. By lemma 2.6 the $\widehat{\overline{R}}$-sub-module generated by $\underline{\delta}_0 \cdot \underline{\omega}$ is F^0. This concludes the proof of (1).

(2) It follows from (1) that $G_0 \subset F^0$ and that $G_0 \subset (F^0)^{\mathcal{G}}$. Let now $x \in (F^0)^{\mathcal{G}}$. Then $x = \underline{u}^t \cdot (\underline{\delta}_0 \cdot \underline{\omega})$ for some column vector $\underline{u}$ with coefficients in $\widehat{\overline{R}}$. As x and $\underline{\delta}_0 \cdot \underline{\omega}$ are $\mathcal{G}$-invariant and the elements of $\underline{\delta}_0 \cdot \underline{\omega}$ are R-linearly independent (as $\underline{\delta}_0$ is in $\mathbf{GL}_g(R[1/p])$), it follows that $\underline{u}$ is $\mathcal{G}$-invariant. Since $\big(\widehat{\overline{R}}\big)^{\mathcal{G}} = R$ by our assumption on R, we have $x \in G_0$. This proves (2).

(3) By construction the map $F^0 \longrightarrow (A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R}$ in 2.5 sends the basis $\underline{\delta}_0 \cdot \underline{\omega}$ to the given basis of $A^\vee[p]/D$; see Step 3 of the proof of 2.5. Extending R-linearly and reducing modulo p^{1-v} we get the claimed isomorphism $F_0/p^{1-v}F_0 \to (A^\vee[p]/D) \otimes R/p^{1-v}R$.

(4) This follows from the natural isomorphism as $\widehat{\overline{R}}$-modules which is $\mathcal{G}$-equivariant $F^1 \cong (F^0)^\vee(1)$ (see Step 2 of the proof of proposition 2.5.) □

We now claim that the free R-module F_0 defined in proposition 2.7, and which will be now denoted $F_0(A/R)$ is functorial both in A/R. More precisely, we have:

Lemma 2.8. *a) Let $R \longrightarrow R'$ be a morphism of $\mathcal{O}_K$-algebras of the type defined at the beginning of this section, let A/R be an abelian scheme such that the assumptions of 2.7 hold and let A'/R' be the base change of A to R'. Then we have a natural isomorphism of R'-modules $F_0(A/R) \otimes_R R' \cong F_0(A'/R')$ compatible with the isomorphism $\omega_{A/R} \otimes_R R' \cong \omega_{A'/R'}$.*

b) Let us assume that we have a morphism of abelian schemes $A \longrightarrow B$ over R and that for both A and B the assumptions of 2.7 hold. Then we have a natural morphism of R-modules $F_0(B/R) \longrightarrow F_0(A/R)$ compatible with the morphism $\omega_{B/R} \longrightarrow \omega_{A/R}$.

Proof. We have similar statements for F^0: in case (a) we have a morphism of $\widehat{\overline{R}}'$-modules $F^0(A/R) \otimes_{\widehat{\overline{R}}} \widehat{\overline{R}}' \longrightarrow F^0(A'/R')$, compatible with the isomorphism of invariant differentials, and in case (b) we have a natural morphism of $\widehat{\overline{R}}$-modules $F^0(B/R) \longrightarrow F^0(A/R)$ compatible with the morphism on invariant differentials. These two statements follow from the functoriality of dlog; see 2.8. Taking Galois invariants we immediately get (b) and also that we have a morphism

$f\colon F_0(A/R)\otimes_R R' \longrightarrow F_0(A'/R')$. Since f is an isomorphism modulo p^{1-v} by 2.7 and since it is a linear morphism of free R'-modules of the same rank, it must be an isomorphism as claimed. $\square$

2.1 The Hodge-Tate sequence for semi-abelian schemes

We need a generalization of proposition 2.5 to the case of semi-abelian schemes in order to deal with the cusps when implementing our constructions to moduli spaces of abelian varieties. We follow [F, §3.e] providing more details.

Let $S\subset \mathcal{U}$ be a simple normal crossing divisor, transversal to the special fiber of $\mathcal{U}$. We write $\mathcal{U}^o := \mathcal{U}\backslash S$ and let $R[S^{-1}]$ be the underlying ring. Assume that its p-adic completion $\widehat{R[S^{-1}]}$ is an integral normal domain. As before we fix a geometric generic point $\eta=\operatorname{Spec}(\mathbb{K})$ of $\widehat{\mathcal{U}}^o$. Let A be an abelian scheme over $\mathcal{U}^o$ and assume that there exists an étale sheaf X over $\mathcal{U}$ of finite and free $\mathbb{Z}$-modules, a semi-abelian scheme G over $\mathcal{U}$, extension of an abelian scheme B by a torus T, and a 1-motive $M:=[X\to G_{\mathcal{U}^o}]$ over $\mathcal{U}_K^o$ such that $T_p(A_\eta)$ is isomorphic to the Tate module $T_p(M_\eta)$ as $\mathcal{G}^o$-module. This is the case for generalized Tate objects used to define compactifications of moduli spaces.

By analogy with our discussion at the beginning of §2, we assume that we are given an R-subalgebra $\overline{R}$ of $\mathbb{K}$, which is an inductive limit of normal R-algebras $M\subset\mathbb{K}$ for which each $R[S^{-1},p^{-1}]\subset M[S^{-1},p^{-1}]$ is finite and étale, and which satisfies the following *local assumptions*:

(i) We suppose $\overline{R}_K[S^{-1}]$ is Galois over $R_K[S^{-1}]$ and let $\mathcal{G}^o$ denote the Galois group.

(ii) We suppose the natural action of $\operatorname{Gal}\big(\mathbb{K}/\operatorname{Frac}(R)\big)$ on T factors through $\mathcal{G}^o$.

(iii) We suppose $(\widehat{\overline{R}})^{\mathcal{G}}=R$, where $\widehat{\overline{R}}$ denotes the p-adic completion of $\overline{R}$.

Let $M^\vee:=[Y\to H_{\mathcal{U}^o}]$ be the dual one-motive; here, Y is the character group of T and H is a semi-abelian scheme over $\mathcal{U}$, extension of $B^\vee$ by the torus T' with character group X. Then, $T_p(A_\eta^\vee)$ is isomorphic to the p-adic Tate module of the dual motive $M^\vee$. In particular, it admits a decreasing filtration $W_{-i}T_p(A_\eta^\vee)$ for $i=0$, 1 and 2 by $\mathcal{G}^o$-sub-modules such that (1) $W_{-1}T_p(A_\eta^\vee)=T_p(H_\eta^\vee)$; (2) $\operatorname{gr}_0 T_p(A_\eta^\vee)\cong Y\otimes\mathbb{Z}_p$; (3) $\operatorname{gr}_{-2}T_p(A_\eta^\vee)\cong T_p(T')$; (4) $\operatorname{gr}_{-1}(T)\cong T_p(B_\eta^\vee)$.

Consider the map $\operatorname{dlog}\colon T_p(A_\eta^\vee)\otimes\widehat{\overline{R}[S^{-1}]}\longrightarrow \omega_{A/\mathcal{U}^o}\otimes_R\widehat{\overline{R}[S^{-1}]}$. Since the p-adic completion of $\omega_{A/\mathcal{U}^o}$ can be expressed in terms of the p-divisible subgroup of A and the latter can be expressed using M we have $\omega_{A/\mathcal{U}^o}\otimes_R\widehat{\overline{R}[S^{-1}]}\cong\omega_{G/\mathcal{U}}\otimes_R\widehat{\overline{R}[S^{-1}]}$. The latter sits in the following exact sequence of sheaves on $\mathcal{U}$:

$$0\longrightarrow\omega_{B/\mathcal{U}}\longrightarrow\omega_{G/\mathcal{U}}\longrightarrow\omega_{T/\mathcal{U}}\longrightarrow 0.$$

We then define a filtration on $\omega_{G/\mathcal{U}}$ by setting $W_{-2}\omega_{G/\mathcal{U}}:=0$, $W_{-1}\omega_{G/\mathcal{U}}:=\omega_{B/\mathcal{U}}$ and $W_{-2}\omega_{G/\mathcal{U}}:=\omega_{G/\mathcal{U}}$.

Lemma 2.9. *The map* dlog *extends uniquely to a morphism*

$$\operatorname{dlog}\colon T_p(A_\eta^\vee)\otimes\widehat{\overline{R}}\longrightarrow\omega_{G/\mathcal{U}}\otimes_R\widehat{\overline{R}},$$

which is trivial on $\operatorname{gr}_{-2}T_p(A_\eta^\vee)$*, induces an isomorphism on* $\operatorname{gr}_0 T_p(A_\eta^\vee)\otimes\widehat{\overline{R}}\longrightarrow\omega_{T/\mathcal{U}}\otimes_R\widehat{\overline{R}}$ *and coincides with the map* $\operatorname{dlog}\colon T_p(B_\eta^\vee)\otimes\widehat{\overline{R}}\longrightarrow\omega_{B/\mathcal{U}}\otimes_R\widehat{\overline{R}[S^{-1}]}$ *on* B *via the identification* $\operatorname{gr}_{-1}T_p(A_\eta^\vee)\cong T_p(B_\eta^\vee)$.

This extension is functorial in R and in A. More precisely, assume that we have a morphism of abelian schemes $f\colon A \to A'$ over $\mathcal{U}^o$ such that the associated p-adic Tate modules are the Tate modules of one-motives $M = [X \to G]$ and $M' = [X' \to G']$ as above and the map induced from f on Tate modules arises from a morphism of one-motives $M \to M'$. Then, the following diagram is commutative

$$\begin{array}{ccc} T_p((A'_\eta)^\vee) \otimes \widehat{\overline{R}} & \stackrel{\mathrm{dlog}_{(A')^\vee}}{\longrightarrow} & \omega_{G'/R} \otimes \widehat{\overline{R}} \\ \downarrow & & \downarrow \\ T_p(A_\eta^\vee) \otimes \widehat{\overline{R}} & \stackrel{\mathrm{dlog}_{A^\vee}}{\longrightarrow} & \omega_{G/R} \otimes \widehat{\overline{R}} \end{array}$$

Proof. The uniqueness follows from the fact that dlog is a map of free $\widehat{\overline{R}}$-modules and the map $\widehat{\overline{R}} \to \widehat{\overline{R}[S^{-1}]}$ is injective. This also implies that the displayed diagram commutes since it commutes after base change to $\widehat{\overline{R}[S^{-1}]}$ by functoriality of dlog.

For every $n \in \mathbb{N}$ let $R \subset R_n \subset \overline{R}$ be a finite and normal extension such that $A_\eta^\vee[p^n] \cong M_\eta^\vee[p^n]$ is trivial as Galois module over $R_n[S^{-1}, p^{-1}]$. This allows to split the filtration on $M_\eta[p^n]$ so that $M_\eta[p^n] \cong B_\eta^\vee[p^n] \oplus T'_\eta[p^n] \oplus Y/p^nY$ as representations of the Galois group of $\overline{R}$ over R_n. Note that the Galois module $B_\eta^\vee[p^n] \oplus T'_\eta[p^n] \oplus Y/p^nY$ is associated to the group scheme $B^\vee[p^n] \oplus T'[p^n] \oplus Y/p^nY$ over R_n. By functoriality of the map dlog we deduce that dlog modulo p^n extends to all of $\mathrm{Spec}(R_n)$ and coincides with the sum of the maps dlog of these group schemes. Passing to the limit the conclusion follows. □

Applying the same argument to the dual abelian scheme and one-motive we deduce that also in this case we have a Hodge-Tate sequence attached to A:

$$0 \longrightarrow \omega_{H^\vee/R}^\vee \otimes_R \widehat{\overline{R}}(1) \stackrel{a}{\longrightarrow} T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \stackrel{\mathrm{dlog}}{\longrightarrow} \omega_{G/R} \otimes_R \widehat{\overline{R}} \longrightarrow 0.$$

As before we denote by F^0 the image of dlog and by F^1 its kernel. *We assume* that Frobenius $\varphi_B\colon R^1\pi_*(B_1) \longrightarrow R^1\pi_*(B_1)$ and $\varphi_{B^\vee}\colon R^1\pi_*(B_1^\vee) \longrightarrow R^1\pi_*(B_1^\vee)$ have determinant ideals containing p^w for some $0 \le w < 1/p$. Let $C \subset B[p]$ be the Galois submodule associated to the canonical subgroup and $D_B \subset B^\vee[p]$ to be the Cartier dual of $B[p]/C$. We define $D \subset A^\vee[p]$ (as Galois modules) to be the inverse image of D_B in $H[p] \to B^\vee[p]$ which we view in $A^\vee[p]$ via the inclusion $H[p] \subset A^\vee[p]$. Note that the kernel of $D \to D_B$ is $T'[p]$. Then,

Corollary 2.10. *The $\widehat{\overline{R}}$-modules F^0 and F^1 are free of rank g and we have a commutative diagram with exact rows and vertical isomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1/p^{1-v}F^1 & \longrightarrow & T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \overline{R}/p^{1-v}\overline{R} & \longrightarrow & F^0/p^{1-v}F^0 & \longrightarrow & 0 \\ & & \downarrow\cong & & || & & \downarrow\cong & & \\ 0 & \longrightarrow & D \otimes \overline{R}/p^{1-v}\overline{R} & \longrightarrow & A^\vee[p] \otimes \overline{R}/p^{1-v}\overline{R} & \longrightarrow & (A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R} & \longrightarrow & 0 \end{array}$$

Moreover, if $A^\vee[p]/D$ is a constant group over $\mathcal{U}^o$, the R-module $F_0 := (F^0)^{\mathcal{G}} \subset \omega_{A/R}$ is free of rank g, we have $F_0 \otimes_R \widehat{\overline{R}} \cong F^0$ and we have a natural isomorphism $F_0/p^{1-v}F_0 \cong (A^\vee[p]/D) \otimes R/p^{1-v}R$ whose base change via $R \to \overline{R}$ provides the isomorphism $F^0/p^{1-v}F^0 \cong A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R}$.

Eventually, the construction of F_0 is functorial in $\mathcal{U}$ and A (see 2.9 for the meaning of the functoriality in A).

Proof. The statement concerning F_0 follows from the first arguing as in 2.7. The statement about the functoriality is the analogue of 2.8 and is proven as in loc. cit. It is deduced from the functoriality of F^0 using the functoriality of the map dlog proven in 2.9.

For the first statement one argues that, using the filtration given in 2.9 and the description of dlog on such a filtration, it suffices to prove the claim for $B^\vee$ and this is the content of 2.5. The details are left to the reader. $\square$

3 Overconvergent modular sheaves

3.1 Definition of overconvergent modular sheaves

We start by fixing a p-adically complete and separated, formally smooth and topologically of finite type $\mathcal{O}_K$-algebra A. We write $S := \mathrm{Spf}(A)$ for the associated formal scheme and $S_K = \mathrm{Spm}(A_K)$ for the associated rigid analytic fiber. We let $||\_||$ be the Gauss norm on A_K so that $A = \{x \in A_K \quad | \quad ||x|| \leq 1\}$. We define the following categories and functors.

a) $\mathbf{FSchemes}_A$. It is the category whose objects are p-adic formal schemes $\mathcal{U}$ over S such that $\mathcal{U}$ has a finite affine open covering $\{\mathcal{U}_i = \mathrm{Spf}(R_i)\}_{i \in I}$ with the property that R_i is a p-adically complete and separated normal domain, topologically of finite type over A, for every $i \in I$. The morphisms in **FSchemes** are morphisms as formal schemes over S.

For any object $\mathcal{U}$ we denote by $\mathcal{U}_K \to S_K$ the corresponding morphism of rigid analytic spaces.

b) Given an object $\mathcal{U}$ of $\mathbf{FSchemes}_A$ we define $\mathrm{Coh}(\mathcal{O}_\mathcal{U} \otimes_{\mathcal{O}_K} K)$ to be the category of sheaves of $\mathcal{O}_\mathcal{U} \otimes_{\mathcal{O}_K} K$-modules $\mathcal{F}$ on $\mathcal{U}$, with the property that there is a coherent $\mathcal{O}_\mathcal{U}$-module F on $\mathcal{U}$ such that $\mathcal{F} = F \otimes_{\mathcal{O}_K} K$. We let $\mathbf{Sheaves}_K$ be the category whose objects are pairs $\big(\mathcal{U}, \mathcal{F}\big)$ where $\mathcal{U}$ is an object of $\mathbf{FSchemes}_A$ and $\mathcal{F}$ is an object of $\mathrm{Coh}(\mathcal{O}_\mathcal{U} \otimes_{\mathcal{O}_K} K)$.

If $\big(\mathcal{U}, \mathcal{F}\big)$ and $\big(\mathcal{U}', \mathcal{F}'\big)$ are two objects of $\mathrm{Sheaves}_K$ then a morphism $\big(\mathcal{U}, \mathcal{F}\big) \longrightarrow \big(\mathcal{U}', \mathcal{F}'\big)$ is a pair $\big(\phi, \phi^+\big)$ where $\phi \colon \mathcal{U} \longrightarrow \mathcal{U}'$ is a morphism of formal schemes over S and $\phi^+ \colon \phi_K^*(\mathcal{F}') \longrightarrow \mathcal{F}$ is a morphism of $\mathcal{O}_\mathcal{U} \otimes_{\mathcal{O}_K} K$-modules over $\mathcal{U}$.

We have a natural functor $\mathbf{Sheaves}_K \longrightarrow \mathbf{FSchemes}_A$ defined at the level of objects by $\big(\mathcal{U}, \mathcal{F}\big) \longrightarrow \mathcal{U}$ and at the level of morphisms $\big(\phi, \phi^+\big) \longrightarrow \phi$. For every object $\mathcal{U}$ of $\mathbf{FSchemes}_A$ we let $\mathbf{Sheaves}_K|_\mathcal{U}$ be the full subcategory of $\mathbf{Sheaves}_K$ consisting of pairs of type $\big(\mathcal{U}, \mathcal{F}\big)$ with morphisms inducing the identity on $\mathcal{U}$. It is equivalent to the *opposite* category of $\mathrm{Coh}(\mathcal{O}_\mathcal{U} \otimes_{\mathcal{O}_K} K)$. For every morphism $\phi \colon \mathcal{U} \to \mathcal{U}'$ in $\mathbf{FSchemes}_A$ we have an inverse image functor $\phi^* \colon \mathbf{Sheaves}_K|_{\mathcal{U}'} \to \mathbf{Sheaves}_K|_\mathcal{U}$ such that $(\alpha \circ \beta)^* \cong \beta^* \circ \alpha^*$ for every two morphisms $\alpha \colon \mathcal{U}' \to \mathcal{U}''$ and $\beta \colon \mathcal{U} \to \mathcal{U}'$. In particular, $\mathbf{Sheaves}_K \longrightarrow \mathbf{FSchemes}_A$ is a *fibred category in groupoids* in the sense of [SGA1, Ex. VI, Def. 6.1 & Rmk]; cf. [SGA1, Ex. VI, Example 11.b].

We also have an integral variant of the category $\mathbf{Sheaves}_K$, the category **Sheaves** whose objects are pairs $(\mathcal{U}, \mathcal{F})$ where $\mathcal{U}$ is an object of $\mathbf{FSchemes}_A$ and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_\mathcal{U}$-modules on $\mathcal{U}$. Given two objects $\big(\mathcal{U}, \mathcal{F}\big)$ and $\big(\mathcal{U}', \mathcal{F}'\big)$ a morphism $\big(\mathcal{U}, \mathcal{F}\big) \longrightarrow \big(\mathcal{U}', \mathcal{F}'\big)$ is a pair $\big(\phi, \phi^+\big)$ where $\phi \colon \mathcal{U} \longrightarrow \mathcal{U}'$ is a morphism of formal schemes over S and $\phi^+ \colon \phi_K^*(\mathcal{F}') \longrightarrow \mathcal{F}$ is a morphism of $\mathcal{O}_\mathcal{U}$-modules over $\mathcal{U}$. Then also $\mathbf{Sheaves} \longrightarrow \mathbf{FSchemes}_A$ is a *fibred category.*

c) Let F denote a totally real number field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$ and different ideal $\mathcal{D}_F$. Let $(\mathfrak{c}_1, \mathfrak{c}_1^+), \ldots, (\mathfrak{c}_h, \mathfrak{c}_h^+)$ be representatives of the strict class group of F. Fix an integer $N \geq 4$ and assume that p is unramified in F and does not divide N. We define the category $\mathbf{Hilb}(\mu_N)_A^w$ to be the category whose objects are $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ where

• $\mathcal{U}$ is an object of $\mathbf{FSchemes}_A$.

• $G \longrightarrow \mathcal{U}$ is a formal abelian scheme of relative dimension g. We assume that $G \longrightarrow \mathcal{U}$ is relatively algebrizable by which we mean that there exists an affine open covering $\{\mathcal{U}_i = \mathrm{Spf}(R_i)\}_{i\in I}$ of $\mathcal{U}$, with R_i a normal domain, such that $G \times_{\mathcal{U}} \mathcal{U}_i$ is algebrizable to an abelian scheme G_i over $\mathrm{Spec}(R_i)$.

• $\iota\colon \mathcal{O}_F \to \mathrm{End}_{\mathcal{U}}(G)$ is a ring homomorphism having the following property: if $e\colon \mathcal{U} \longrightarrow G$ is the zero section of G and we let $\Omega_{G/\mathcal{U}} := e^*(\Omega^1_{G/\mathcal{U}})$, then $\Omega_{G/\mathcal{U}}$ is an invertible $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{U}}$-module.

• if $\mathcal{P}_G \subset \mathrm{Hom}_{\mathcal{O}_F}\big(G, G^\vee\big)$ is the sheaf of $\mathcal{O}_F$-modules for the finite étale topology of $\mathcal{U}$, defined as the symmetric $\mathcal{O}_F$-linear homomorphisms $G \to G^\vee$, and if $\mathcal{P}_G^+ \subset \mathcal{P}_G$ is the subset of polarizations, then λ is an isomorphism $\lambda\colon \big(\mathcal{P}_G, \mathcal{P}_G^+\big) \cong (\mathfrak{c}_t, \mathfrak{c}_t^+)$, for some t, of invertible $\mathcal{O}_F$-modules with a notion of positivity for the finite étale topology of $\mathcal{U}$; [R, Def. 1.19].

• we have a closed immersion $\Psi_N\colon \mu_N \otimes \mathcal{D}_F^{-1} \to G[N]$ of group schemes compatible with the $\mathcal{O}_F$-actions over $\mathcal{U}$.

• $Y \in \mathrm{H}^0\big(\mathcal{U}, \omega_{G/\mathcal{U}}^{1-p} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K\big)$ is such that $Y h\big(G/\mathcal{U}, \iota, \lambda, \psi_N\big) = p^w$. Here, $\omega_{G/\mathcal{U}} := \wedge^g \Omega_{G/\mathcal{U}}$ and $h\big(G/\mathcal{U}, \iota, \lambda, \psi_N\big) \in \mathrm{H}^0\big(\mathcal{U}, \omega_{G/\mathcal{U}}^{p-1} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K\big)$ is the Hasse invariant; see [AGo, Def. 7.12].

Remark 3.1. If $(G/\mathcal{U}, \iota, \lambda, \psi_N, Y)$ is as above, then $G_K \to \mathcal{U}_K$ has a canonical subgroup $C_G \subset G_K[p]$. See 2.1.

The morphisms of the category $\mathbf{Hilb}(\mu_N)_A^w$ are defined as follows. Let $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ and $\big(G'/\mathcal{U}', \iota', \lambda', \psi_N', Y'\big)$ be objects, then a morphism $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big) \longrightarrow \big(G'/\mathcal{U}', \iota', \lambda', \psi_N', Y'\big)$ is a pair $\big(\phi, \alpha\big)$ such that

• $\phi\colon \mathcal{U} \longrightarrow \mathcal{U}'$ is a morphism of formal schemes.

• $\alpha\colon G \longrightarrow G'_{\mathcal{U}} := G' \times_{\mathcal{U}'} \mathcal{U}$ is an $\mathcal{O}_F$-linear morphism of group schemes defined over $\mathcal{U}$ such that $\psi_N \circ \alpha = \psi_N'$ and $\alpha^*(Y') = Y$ and α induces an isomorphism $C_G \cong C_{G'} \times_{\mathcal{U}'_K} \mathcal{U}_K$.

We have a functor $\mathbf{Hilb}(\mu_N)_A^w \longrightarrow \mathbf{FSchemes}_A$ defined by $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big) \longrightarrow \mathcal{U}$ and $\big(\phi, \alpha\big) \longrightarrow \phi$. Given an object $\mathcal{U}$ of $\mathbf{FSchemes}_A$ we let $\mathbf{Hilb}(\mu_N)_A^w|_{\mathcal{U}}$ be the subcategory of $\mathbf{FSchemes}_A$ consisting of objects $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ over $\mathcal{U}$ and of morphisms mapping to the identity of $\mathcal{U}$. For a morphism $\phi\colon \mathcal{U} \to \mathcal{U}'$ in $\mathbf{FSchemes}_A$ we have an inverse image functor $\phi^*\colon \mathbf{Hilb}(\mu_N)_A^w|_{\mathcal{U}'} \to \mathbf{Hilb}(\mu_N)_A^w|_{\mathcal{U}}$ defined by the fibre product with respect to ϕ. Moreover, $(\alpha \circ \beta)^* \cong \beta^* \circ \alpha^*$ for every two morphisms $\alpha\colon \mathcal{U}' \to \mathcal{U}''$ and $\beta\colon \mathcal{U} \to \mathcal{U}'$. In particular, also $\mathbf{Hilb}(\mu_N)_A^w \longrightarrow \mathbf{FSchemes}_A$ is a *fibred category.*

The category of $\mathbf{Hilb}(\mu_N)_A^w$ has a *special* object which enjoys a certain universal property as follows (see lemma 3.2). Let $\mathfrak{M}(A, \mu_N)$ be the fine moduli space classifying abelian schemes over A with real multiplication by $\mathcal{O}_F$ and μ_N-level structure. It is a smooth and quasi-projective scheme over A; see [R, §1]. Let $\widehat{\mathfrak{M}}(A, \mu_N)$ denote the formal completion of $\mathfrak{M}(A, \mu_N)$ along its special fiber and $\mathfrak{B}(w)$ denote the normalization of the blow-up of $\widehat{\mathfrak{M}}(A, \mu_N)$ along the ideal generated by $\big(p^w, \tilde h(G,\iota,\lambda,\psi_N)\big)$ where $\big(G/\widehat{\mathfrak{M}}(A,\mu_N),\iota,\lambda,\psi_N\big)$ is the pull back of the universal abelian scheme with its $\mathcal{O}_F$-multiplication, polarization and level structure on $\mathfrak{M}(A,\mu_N)$ and $\tilde h\omega^{p-1}$ is a local lift of the Hasse invariant $h(G,\iota,\lambda,\psi_N)$ with ω a local generator of $\omega_{G/\widehat{\mathfrak{M}}(A,\mu_N)}$.

We denote by $\mathfrak{M}(A,\mu_N)(w)$ the p-adic completion of the open sub-scheme of $\mathfrak{B}(w)$ which is the complement of the section at ∞ in the exceptional divisor of $\mathfrak{B}(w)$. It is a p-adic formal scheme which is an object of $\mathbf{FSchemes}_A$ defined at the beginning of this section. The rigid analytic generic fiber $\mathfrak{M}(A,\mu_N)(w)_K$ is the strict neighborhood of the ordinary locus of $\mathfrak{M}(A,\mu_N)_K^{\mathrm{rig}}$ of width p^w.

Let $\big(G^{\mathrm{univ}},\iota^{\mathrm{univ}},\lambda^{\mathrm{univ}},\psi_N^{\mathrm{univ}}\big)$ be the pull back under the canonical map $\mathfrak{M}(A,\mu_N)(w)\longrightarrow \widehat{\mathfrak{M}}(A,\mu_N)$ of the p-adic completion of the universal object over $\mathfrak{M}(A,\mu_N)$. We also define $Y^{\mathrm{univ}} = p^w/h\Big(G^{\mathrm{univ}}/\mathfrak{M}(A,\mu_N)(w),\iota^{\mathrm{univ}},\lambda^{\mathrm{univ}},\psi_N^{\mathrm{univ}}\Big)$. Then

$$\underline{G}^{\mathrm{univ}} := \big(G^{\mathrm{univ}},\iota^{\mathrm{univ}},\lambda^{\mathrm{univ}},\psi_N^{\mathrm{univ}},Y^{\mathrm{univ}}\big)$$

is an object of $\mathbf{Hilb}(\mu_N)_A^w$ and we have

Lemma 3.2. *Let $\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ be an object of $\mathbf{Hilb}(\mu_N)_A^w$. Then there exists a unique morphism in $\mathbf{FSchemes}_A$, $\phi\colon \mathcal{U}\longrightarrow \mathfrak{M}(A,\mu_N)(w)$ and a morphism $(\phi,\alpha)\colon \big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)\longrightarrow \big(G^{\mathrm{univ}}/\mathfrak{M}(A,\mu_N)(w),\iota^{\mathrm{univ}},\lambda^{\mathrm{univ}},\psi_N^{\mathrm{univ}},Y^{\mathrm{univ}}\big)$ in $\mathbf{Hilb}(\mu_N)_A^w$ such that $\alpha\colon G\longrightarrow G^{\mathrm{univ}}\times_{\mathfrak{M}(A,\mu_N)(w)} \mathcal{U}$ is an isomorphism compatible even with the polarizations data. Moreover α is unique among all such isomorphisms compatible with the $\mathcal{O}_F$-multiplications, the level structures and the polarizations data.*

Remark 3.3. The compatibility between polarizations data alluded to in lemma 3.2 means that $\lambda\colon \big(\mathcal{P}_G,\mathcal{P}_G^+\big)\cong(\mathfrak{c}_t,\mathfrak{c}_t^+)$ coincides through α with the isomorphism of sheaves on the étale topology of $\mathfrak{M}(A,\mu_N)(w)$, $\lambda^{\mathrm{univ}}\colon \big(\mathcal{P}_{G^{\mathrm{univ}}},\mathcal{P}^+_{G^{\mathrm{univ}}}\big)\cong(\mathfrak{c}_t,\mathfrak{c}_t^+)$

More precisely, the isomorphism defined by α is $\big(\mathcal{P}_{G^{\mathrm{univ}}_{\mathcal{U}}},\mathcal{P}^+_{G^{\mathrm{univ}}_{\mathcal{U}}}\big)\cong\big(\mathcal{P}_G,\mathcal{P}_G^+\big)$ which sends $f\colon G^{\mathrm{univ}}_{\mathcal{U}}\to G^{\mathrm{univ},\vee}_{\mathcal{U}}$ to $\alpha^\vee\circ f\circ\alpha$.

We now prove Lemma 3.2.

Proof. Let $\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ be an object of $\mathbf{Hilb}(\mu_N)_A^w$. Then there exists a unique morphism of formal schemes $\varphi:\mathcal{U}\longrightarrow\widehat{\mathfrak{M}}(A,\mu_N)$ such that $G\cong G^0\times_{\widehat{\mathfrak{M}}(A,\mu_N)}\mathcal{U}$ over $\mathcal{U}$, where G^0 is the universal object over $\widehat{\mathfrak{M}}(A,\mu_N)$. This isomorphism is compatible with the $\mathcal{O}_F$-multiplications, level structures and polarizations data. Moreover, because of the existence of the section Y, φ factors uniquely through the above mentioned blow-up of $\widehat{\mathfrak{M}}(A,\mu_N)$ and because $\mathcal{U}$ is a normal p-adic formal scheme, it factors uniquely through the p-adic completion of the normalization $\mathfrak{B}(w)$ of the blow-up, in other words φ factors uniquely through a morphism $\phi:\mathcal{U}\longrightarrow\mathfrak{M}(A,\mu_N)(w)$. Thus we obtain an isomorphism $\alpha: G\cong G^{\mathrm{univ}}\times_{\mathfrak{M}(A,\mu_N)(w)}\mathcal{U}$ over $\mathcal{U}$ such that $(\phi,\alpha):\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)\longrightarrow\big(G^{\mathrm{univ}}/\mathfrak{M}(A,\mu_N)(w),\iota^{\mathrm{univ}},\lambda^{\mathrm{univ}},\psi_N^{\mathrm{univ}},Y^{\mathrm{univ}}\big)$ is a morphism in $\mathbf{Hilb}(\mu_N)_A^w$. Moreover, α also preserves the polarizations data.

We claim that α is unique with these properties: if $\alpha': G\cong G^{\mathrm{univ}}\times_{\mathfrak{M}(A,\mu_N)(w)}\mathcal{U}$ is another isomorphism preserving the $\mathcal{O}_F$-action, level structures, and polarizations then $\alpha^(-1)\circ\alpha': G\cong G$ is an isomorphism over $\mathcal{U}$ which preserves all structures (including the polarizations data) so it is the identity by Lemma 1.23 in [R].

□

Definition 3.4. An *overconvergent modular sheaf of tame level* N (with degree of overconvergence w) is a functor $\mathcal{F}\colon \mathbf{Hilb}(\mu_N)_A^w \longrightarrow \mathbf{Sheaves}_K$ over $\mathbf{FSchemes}_A$.

Remark 3.5. *Let $k \in \mathbb{Z}$ and let us define the functor $\underline{\omega}^k\colon \mathbf{Hilb}(\mu_N)_A^w \longrightarrow \mathbf{Sheaves}_K$ by:*

$$\underline{\omega}^k\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big) := \omega_{G/\mathcal{U}}^{\otimes k} \otimes_{\mathcal{O}_K} K.$$

We leave it as an exercise to the reader that $\underline{\omega}^k$ is an overconvergent modular sheaf. The key point is that for every morphism $\underline{G} := \big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big) \longrightarrow \big(G'/\mathcal{U}', \iota', \lambda', \psi'_N, Y'\big) =: \underline{G}'$ in $\mathbf{Hilb}(\mu_N)_A^w$ over $\phi\colon \mathcal{U} \to \mathcal{U}'$ we have $\phi^*\big(\omega_{G'/\mathcal{U}'}^{\otimes k}\big) \otimes_{\mathcal{O}_K} K \cong \omega_{G' \times_{\mathcal{U}'} \mathcal{U}/\mathcal{U}}^{\otimes k} \otimes_{\mathcal{O}_K} K$.

Definition 3.6. If $\mathcal{F}$ is an overconvergent modular sheaf and $\big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ is an object of $\mathbf{Hilb}(\mu_N)_A^w$, we denote by $\mathcal{F}_{(G/\mathcal{U}, \iota, \lambda, \psi_N, Y)}$ the respective coherent sheaf on $\mathcal{U}$. We denote by

$$M(\mathcal{F}, N, w) := \mathrm{H}^0\left(\mathfrak{M}(A, \mu_N)(w), \mathcal{F}_{\underline{G}^{\mathrm{univ}}}\right)$$

and call these *weakly holomorphic overconvergent $\mathcal{F}$-valued modular forms.*

Remark 3.7. The sheaf $\mathcal{F}_{\underline{G}^{\mathrm{univ}}}$ defines a coherent sheaf on the rigid analytic fiber $\mathfrak{M}(A, \mu_N)(w)_K$ and we could have defined equivalently $M(\mathcal{F}, N, w)$ as $M(\mathcal{F}, N, w) := \mathrm{H}^0\left(\mathfrak{M}(A, \mu_N)(w)_K, \mathcal{F}_{\underline{G}^{\mathrm{univ}}}\right)$.

Remark 3.8. Because $\underline{G}^{\mathrm{univ}}$ is a final object in $\mathbf{Hilb}(\mu_N)_A^w$, to give a section $f \in M(\mathcal{F}, N, w)$ is equivalent to give a rule which assigns to every isomorphism class of an object $(G/\mathcal{U}, \iota, \lambda, Y)$ of $\mathbf{Hilb}(\mu_N)_A^w$ a section $f(G/\mathcal{U}, \iota, \lambda, Y) \in \mathrm{H}^0\big(\mathcal{U}, \mathcal{F}_{(G/\mathcal{U}, \iota, \lambda, Y)}\big)$ which is functorial in $(G/\mathcal{U}, \iota, \lambda, Y)$ and commutes with base change.

3.2 Hecke operators

The main property of the definition in section 3.1 is that given an overconvergent modular sheaf $\mathcal{F}$ of tame level N there is an operator U_p and a Hecke operator $T_{\mathfrak{q}}$ for every prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$ not dividing pN on the A_K-module of weakly holomorphic overconvergent $\mathcal{F}$-valued modular forms. Let $\underline{G}^{\mathrm{univ}} := \big(G^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, \psi_N^{\mathrm{univ}}, Y\big)$ be the universal abelian formal scheme over $\mathfrak{M}\big(\mathcal{O}_K, \mu_N, \big)(w)$ and let $\mathfrak{M}\big(\mathcal{O}_K, \mu_N, \big)(w)_K$ be the rigid analytic fiber of $\mathfrak{M}\big(\mathcal{O}_K, \mu_N\big)(w)$.

Lemma 3.9. *The canonical subgroup C of the universal object is isomorphic to the constant group scheme $\mathcal{O}_F/p\mathcal{O}_F$ over a finite and étale covering of $\mathfrak{M}(\mathcal{O}_K, \mu_N)(w)_K$.*

Proof. We first consider the finite and étale cover Z' where C_K becomes constant. We have to prove that finite étale locally over Z' such group is isomorphic to $\mathcal{O}_F/p\mathcal{O}_F$. It suffices to prove it for every point of Z'. Let $K \subset L$ be a finite extension, let $\mathbb{F}$ be its residue field and consider an L-valued point x of Z'.

The pull-back A_x of A to the ring of integers of L is an abelian scheme. Then, $A_{x,K}$ admits a canonical subgroup $C_{x,K}$ which is a constant group. In particular, it is a $\mathcal{O}_F/p\mathcal{O}_F$-module of dimension p^g as $\mathbb{F}_p$-vector space. We need to prove that it is a free $\mathcal{O}_F/p\mathcal{O}_F$-module. It suffices to show that given a non-trivial element e of $\mathcal{O}_F/p\mathcal{O}_F$ then e does not annihilate $C_{x,K}$. We let $C_x \subset A_x$ be the schematic closure of $C_{x,K}$ in A_x. Its special fiber $C_{x,\mathbb{F}}$ coincides with the kernel of Frobenius on $A_{x,\mathbb{F}}$. In particular its module of invariant differentials coincides with $\omega_{A_{x,\mathbb{F}}/\mathbb{F}}$ which is a free $\mathcal{O}_F \otimes \mathbb{F}$-module so that e does not annihilate it. Then e does not annihilate $C_{x,K}$ either. □

Set $\mathfrak{q} = p$ or a prime ideal not dividing pN. For every i we fix an identification $(\mathfrak{q}\mathfrak{c}_i, \mathfrak{q}^+\mathfrak{c}_i^+) \cong (\mathfrak{c}_j, \mathfrak{c}_j^+)$ (j depends on i), where let us recall the family $\{(\mathfrak{c}_i, \mathfrak{c}_i^+)\}_{i=1,h^+}$ is a set of representatives for the strict class group of F. We define $\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)_K \to \mathfrak{M}\big(A, \mu_N\big)(w)_K$ as the finite and étale cover of rigid analytic spaces classifying $\mathcal{O}_F$-invariant subgroup schemes H_K of the universal abelian object which, finite and étale locally, are isomorphic to $\mathcal{O}_F/\mathfrak{q}$ and such that H_K does not intersect the canonical subgroup C. If $\mathfrak{q}$ does not divide p, this last condition is automatic. If $\mathfrak{q} = p$ it follows from lemma 3.9 that $H_K \oplus C \cong \big(\mathcal{O}_F/p\mathcal{O}_F\big)^2$ étale locally over $\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)_K$ so that $H_K \oplus C$ coincides with the p-torsion of the universal abelian scheme G_K^{univ}. Thanks to [AGa, Thm 10.6] there is a normal formal scheme

$$p_1 \colon \mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w) \longrightarrow \mathfrak{M}\big(A, \mu_N\big)(w),$$

with rigid analytic fiber $\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)_K$ such that H_K extends to a finite and flat subgroup scheme H^{univ} of the universal abelian object G^{univ}. We also have a unique morphism

$$p_2 \colon \mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w) \longrightarrow \mathfrak{M}\big(A, \mu_N\big)(w')$$

defined by taking the quotient $G^{',\mathrm{univ}} := G^{\mathrm{univ}}/H^{\mathrm{univ}}$ with induced action of $\mathcal{O}_F$, polarization structure, ψ_N and Y. Notice that the locus where the polarization module is $(\mathfrak{c}_i, \mathfrak{c}_i^+)$ is sent to the locus where the polarization module is $(\mathfrak{q}\mathfrak{c}_i, \mathfrak{q}^+\mathfrak{c}_i^+)$; see [GK, §7.1]. Here we take $w' = w$ if $\mathfrak{q}$ does not divide p. If $\mathfrak{q} = p$ and $w < \frac{1}{p+1}$, as $H_K \oplus C = G_K^{\mathrm{univ}}[p]$, we may take $w' = pw$ by [Fa2, Thm. 5] or [GK, Thm. 5.4.3]. In both cases, the morphism p_2 induces an isomorphism on rigid analytic fibers as G_K^{univ} is canonically isomorphic to the quotient of $G_K^{',\mathrm{univ}}$ by the subgroup scheme $G_K^{\mathrm{univ}}[\mathfrak{q}]/H_K$ and H_K coincides with the image of $G_K^{',\mathrm{univ}}[\mathfrak{q}]$. We denote by

$$\pi_{\mathfrak{q}} \colon \underline{G}^{\mathrm{univ}} \longrightarrow \underline{G}^{',\mathrm{univ}}$$

the universal isogeny with kernel H^{univ} defined over $\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)$. Therefore we have a natural morphism:

$$\mathcal{F}\big(\pi_{\mathfrak{q}}\big) \colon p_2^*\big(\mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big) \longrightarrow \mathcal{F}_{\underline{G}^{',\mathrm{univ}}} \longrightarrow \mathcal{F}_{\underline{G}^{\mathrm{univ}}} \cong p_1^*\big(\mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big).$$

We define the operator $T_{\mathfrak{q}}$ (for $\mathfrak{q}$ not dividing p) and the operator U_p (for $\mathfrak{q} = p$) as $1/\mathrm{deg} p_{1,K}$ times the composition:

$$\mathrm{H}^0\big(\mathfrak{M}(A, \mu_N)(w'), \mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big) \longrightarrow \mathrm{H}^0\Big(\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w), p_2^*\big(\mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big)\Big) \xrightarrow{\mathcal{F}(\pi_{\mathfrak{q}})}$$

$$\xrightarrow{\mathcal{F}(\pi_{\mathfrak{q}})} \mathrm{H}^0\Big(\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)_K, p_1^*\big(\mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big)\Big) \cong \mathrm{H}^0\Big(\mathfrak{M}\big(A, \mu_N\big)(w)_K, p_{1,*}\big(p_1^*\mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big)\Big) \xrightarrow{\mathrm{Tr}}$$

$$\xrightarrow{\mathrm{Tr}} \mathrm{H}^0\big(\mathfrak{M}\big(A, \mu_N, \big)(w)_K, \mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big) \cong \mathrm{H}^0\big(\mathfrak{M}\big(A, \mu_N\big)(w), \mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big).$$

As in 3.7 we implicitly identify the global sections of $p_1^*\big(\mathcal{F}_{\underline{G}^{\mathrm{univ}}}\big)$ on $\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)$ with the global sections of the associated locally free sheaf on the rigid analytic space $\mathfrak{M}\big(A, \mu_N, \mathfrak{q}\big)(w)_K$. Similar considerations apply to the last isomorphism. This allows us to define the trace of $p_{1,K}$

as p_1 induces a finite and étale morphism of rigid analytic spaces but not necessarily at the level of formal schemes.

In particular we get Hecke operators

$$T_{\mathfrak{q}}\colon M(\mathcal{F},N,w) \longrightarrow M(\mathcal{F},N,w), \qquad U_p\colon M(\mathcal{F},N,w) \to M(\mathcal{F},N,w),$$

for primes $\mathfrak{q}$ not dividing p, by pre-composing the maps above with the inclusion $M(\mathcal{F},N,w) \to M(\mathcal{F},N,w')$.

Corollary 3.10. *Given overconvergent modular sheaves $\mathcal{F}$ and $\mathcal{G}$ and a natural transformation of functors $f\colon \mathcal{F} \to \mathcal{G}$, the induced map $M(\mathcal{F},N,w) \longrightarrow M(\mathcal{G},N,w)$ is Hecke equivariant.*

Proof. This follows as the Hecke operators are defined using only the fact that $\mathcal{F}$ and $\mathcal{G}$ are functors.

□

3.3 Overconvergent modular sheaves of weight k

Let $\mathcal{W}_F$ be the rigid analytic space over $\mathbb{Q}_p$ associated to the noetherian complete algebra $\mathbb{Z}_p[[(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times]]$, called the weight space for $\mathbf{GL}_{2/F}$.

Let us recall that we fixed a K-affinoid algebra A_K with good reduction (see the beginning of section 3.1). Let $k \in \mathcal{W}_F(A_K)$ be an A_K-valued weight, in other words $k\colon \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^\times \longrightarrow A_K^\times$ is a continuous homomorphism. We denote the action of k on elements $t \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ additively, i.e. $k(t) = t^k$. As $\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^\times$ is compact the image of k is contained in $A^\times$ and we denote also by $k\colon \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^\times \longrightarrow A^\times$. The goal of this section is to attach to the data (A,F,N,w,k) an overconvergent modular sheaf $\omega_A^{\dagger,k}$ on $\mathbf{Hilb}(\mu_N)_A^w$, functorial in A and w; see 3.13.

We denote by $r := \min\{n \in \mathbb{N}, \quad | \quad n > 0,\ ||k\big(1 + p^n(\mathcal{O}_F \otimes \mathbb{Z}_p)\big)|| < p^{-1/(p-1)}\}$. Any non-negative $w \in \mathbb{Q}$ such that $w < \frac{2}{p^{r-1}}$ if $p \neq 3$ and $w < \frac{1}{3^r}$ if $p = 3$ is called *adapted* to k. We fix such a w and note that there exists a unique $\mathbb{Z}_p$-linear function $a\colon \mathcal{O}_F \otimes \mathbb{Z}_p \to A$ such that

$$t^k = \exp\big(a \log(t)\big) \quad \forall t \in 1 + p^r(\mathcal{O}_F \otimes \mathbb{Z}_p).$$

Let $e_1,\ldots,e_g$ be a $\mathbb{Z}$-basis of $\mathcal{O}_F$. By [AGo, §4] we have an isomorphism $\mathcal{W}_F \cong \amalg_{\epsilon\in\hat{\mu}} D$, where $\hat{\mu}$ is the set of characters of $(\mathcal{O}_F/p\mathcal{O}_F)^\times$ and D is the open unit polydisk of dimension g given by $\mathrm{Spm}\big(\mathbb{Z}_p[\![X_1,\ldots,X_g]\!]\big[p^{-1}\big]\big)$, centered at 0. Given a $\mathbb{C}_p$-valued point $(x_1,\ldots,x_g)$ of D, the associated character of $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ is the logarithm $1+p\big(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p\big) \longrightarrow \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ composed with the map $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \mapsto \mathbb{C}_p^\times$, $\sum a_i e_i \mapsto \prod x_i^{a_i}$.

Examples 1) For every finite extension $K \subset L$ and every $k \in \mathcal{W}_F(L)$ our construction for $A = \mathcal{O}_L$ provides *overconvergent modular sheaves of weight k*:

$$\omega^{\dagger,k}\colon \mathbf{Hilb}(\mu_N)^w_{\mathcal{O}_L} \longrightarrow \mathbf{Sheaves}_K.$$

2) For every positive $r \in \mathbb{N}$ let $\mathcal{W}_r \subset \mathcal{W}_F$ be the open subspace of weights k such that $||k\big(1 + p^r\big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)|| < p^{-1/(p-1)}$. Then the family $\{\mathcal{W}_r\}_r$ defines an open admissible covering of $\mathcal{W}_F$ by affinoids associated to formal affine schemes $\mathrm{Spf}(A_r)$. As on each $\mathcal{W}_r$ there exists a

universal character $k_r\colon (\mathcal{O}_F\otimes\mathbb{Z}_p)^\times \longrightarrow A_r^\times$ defined by $\big(t^{k_r}\big)(x)=t^x$ for all $t\in(\mathcal{O}_F\otimes\mathbb{Z}_p)^\times$ and $x\in\mathcal{W}_r$, we can apply the previous construction providing a *family of overconvergent modular sheaves*

$$\omega_{r,w}^\dagger\colon \mathbf{Hilb}(\mu_N)_{A_r}^w \longrightarrow \mathbf{Sheaves}_K.$$

For every finite extension $K\subset L$ and every $k\in\mathcal{W}_r(\mathbb{C}_p)$, the functoriality implies that the restriction of $\omega_{r,w}^\dagger$ to the inclusion $\mathbf{Hilb}(\mu_N)_L^w\subset\mathbf{Hilb}(\mu_N)_{A_r}^w$ coincides with $\omega^{\dagger,k}$.

We start explaining the construction of $\omega_A^{\dagger,k}$ with the following lemma.

Lemma 3.11. *Every object $\underline{G}:=\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ in $\mathbf{Hilb}(\mu_N)_A^w$ admits a canonical subgroup $C_r\subset G_K$ of level r. Moreover, C_r is the iterated extension of C_1, it is $\mathcal{O}_F$-invariant and finite étale locally over $\mathcal{U}_K$ the $\mathcal{O}_F$-group scheme C_r is isomorphic to $\mathcal{O}_F/p^r\mathcal{O}_F$.*

Proof. The existence of the canonical subgroup follows from [Fa2, Thm. 6]. By loc. cit. it is a free $\mathbb{Z}/p^r\mathbb{Z}$-module of rank g finite étale locally over $\mathcal{U}_K$, it is stable under the action of $\mathcal{O}_F$ and it is the successive extension of the canonical subgroup C_1 of level 1. It then follows from 3.9 that it is is a free $\mathcal{O}_F/p^r\mathcal{O}_F$-module of rank 1 finite étale locally over $\mathcal{U}_K$.

□

We first define the functor $\omega_A^{\dagger,k}$ on a full subcategory $\mathbf{Hilb}'(\mu_N)_A^w$ of $\mathbf{Hilb}(\mu_N)_A^w$. We then show how $\omega_A^{\dagger,k}$ extends uniquely to $\mathbf{Hilb}(\mu_N)_A^w$.

The category $\mathbf{Hilb}'(\mu_N)_A^w$. Given an object $\underline{G}:=\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ in $\mathbf{Hilb}(\mu_N)_A^w$ with $\mathcal{U}=\mathrm{Spf}(R)$, let $\theta_K\colon\mathcal{U}_K'\longrightarrow\mathcal{U}_K$ be the finite and étale morphism representing the functor

$$\underline{\mathrm{Isom}}_{\mathcal{O}_F}\big(\mathcal{O}_F/p^r\mathcal{O}_F,C_r\big),$$

i.e., all the trivializations of C_r as $\mathcal{O}_F/p^r\mathcal{O}_F$-group scheme. By the Lemma it is Galois with group $\mathcal{G}\cong\big(\mathcal{O}_F/p^r\mathcal{O}_F\big)^\times$. Let now $\theta\colon\mathcal{U}'\longrightarrow\mathcal{U}$ denote the normalization of $\mathcal{U}$ in $\mathcal{U}_K'$ and set $\underline{G}':=\big(G'/\mathcal{U}',\iota',\lambda',\psi_N',Y'\big)$ be given by the pull–back of $\underline{G}$ via θ. It defines an object of $\mathbf{Hilb}(\mu_N)_A^w$.

We define $\mathbf{Hilb}'(\mu_N)_A^w$ as the full subcategory of $\mathbf{Hilb}(\mu_N)_A^w$ consisting of objects $\underline{G}:=\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ such that $\mathcal{U}=\mathrm{Spf}(R)$ is affine, G is algebrizable and, with the notations above setting $\mathcal{U}':=\mathrm{Spf}\big(R'\big)$, the ring R' satisfies the local assumptions of §2.

The functor $\omega_A^{\dagger,k}$ on objects. There is an $\mathcal{O}_{\mathcal{U}'}$-submodule $\mathcal{F}_{\underline{G}}$ of $\omega_{\widetilde{G}'/R'}$ given by $F_0(\widetilde{G}'/R')$ of 2.7. It is stable under the action of $\mathcal{O}_F$ on $\widetilde{G}'$ by the functoriality of F_0 proven in 2.8. Moreover we have an isomorphism of $\mathcal{O}_F\otimes\mathcal{O}_{\mathcal{U}'}$-modules provided by the map dlog:

$$(C_r)^\vee\otimes\mathcal{O}_{\mathcal{U}'}/p^{(1-v)r}\mathcal{O}_{\mathcal{U}'}\cong\mathcal{F}_{\underline{G}}/p^{(1-v)r}\mathcal{F}_{\underline{G}}. \qquad (1)$$

Define $\mathcal{F}_{\underline{G}}'$ as the inverse image in $\mathcal{F}_{\underline{G}}$ of $(C_r)^\vee-(C_r)^\vee[p^{r-1}]$. Then $\mathcal{F}_{\underline{G}}'$ is a torsor under the sheaf of groups $\mathcal{S}_{\mathcal{U}'}:=\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^\times\cdot\big(1+p^{(1-v)r}(\mathcal{O}_F\otimes\mathcal{O}_{\mathcal{U}'})\big)$ and both $S_{\mathcal{U}'}$ and $\mathcal{F}_{\underline{G}}'$ are endowed with compatible actions of $\mathcal{G}$, lifting the action on $\mathcal{U}'$. If x is a local section of $1+p^{(1-v)r}\big(\mathcal{O}_F\otimes\mathcal{O}_{\mathcal{U}'}\big)$ then we write $x^a:=\exp(a\log(x))$ where let us recall $a:\mathcal{O}_F\otimes\mathbb{Z}_p\longrightarrow A$ was a $\mathbb{Z}_p$-linear map such that $t^k=\exp(a\log(t))$ for all $t\in1+p^r(\mathcal{O}_F\otimes\mathbb{Z}_p)$. Let us then remark that with the definition above x^a is also a section of $1+p^{(1-v)r}\big(\mathcal{O}_F\otimes\mathcal{O}_{\mathcal{U}'}\big)$. We now define a twisted action of $\mathcal{S}_{\mathcal{U}'}$ on $\mathcal{O}_{\mathcal{U}'}$ as follows: let $s:=c\cdot x$ be a local section of $\mathcal{S}_{\mathcal{U}'}:=\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^\times\cdot\big(1+p^{(1-v)r}(\mathcal{O}_F\otimes\mathcal{O}_{\mathcal{U}'})\big)$ and y a local section of $\mathcal{O}_{\mathcal{U}'}$. We set

$$s*y:=x^a\cdot c^k\cdot y.$$

We denote $\mathcal{O}_{\mathcal{U}'}^{(k)}$ the sheaf $\mathcal{O}_{\mathcal{U}'}$ with this action of $\mathcal{S}_{\mathcal{U}'}$. We define the coherent $\mathcal{O}_{\mathcal{U}}\otimes_{\mathcal{O}_K}K$-module

$$\omega_{\underline{G}}^{\dagger,k}:=\Big(\theta_{K,*}\big(\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}},\mathcal{O}_{\mathcal{U}'}^{(-k)})\big)\otimes_{\mathcal{O}_K}K\Big)^{\mathcal{G}}.$$

As $\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}\big(\mathcal{F}'_{\underline{G}},\mathcal{O}_{\mathcal{U}'}^{(-k)}\big)$ is an invertible $\mathcal{O}_{\mathcal{U}'}$-module, we deduce that $\omega_{\underline{G}}^{\dagger,k}$ is an invertible $\mathcal{O}_{\mathcal{U}}\otimes_{\mathcal{O}_K}K$-module. This defines $\omega_A^{\dagger,k}$ on objects of $\mathbf{Hilb}'(\mu_N)_A^w$.

The functor $\omega_A^{\dagger,k}$ on morphisms. Consider the functor $\mathcal{S}\colon\mathbf{Hilb}'(\mu_N)_A^w\longrightarrow\mathbf{Groups}$ and the functor $\mathcal{O}'^{(-k)}\colon\mathbf{Hilb}'(\mu_N)_A^w\longrightarrow\big(\mathcal{S}-\mathbf{Sheaves}\big)$ associating to an object $\underline{G}:=\big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ the sheaf $\mathcal{S}_{\mathcal{U}'}$ and the coherent $\mathcal{O}_{\mathcal{U}'}$-module with action of $\mathcal{S}_{\mathcal{U}'}$ given by $\mathcal{O}_{\mathcal{U}'}^{(-k)}$. It follows from 2.8 and 2.10 that the association $\underline{G}\mapsto\mathcal{F}_{\underline{G}}$ defines a functor $\mathcal{F}\colon\mathbf{Hilb}'(\mu_N)_A^w\longrightarrow\big(\mathcal{S}-\mathbf{Sheaves}\big)$. Consider a morphism $\alpha\colon\underline{G}\longrightarrow\underline{H}$ in $\mathbf{Hilb}'(\mu_N)_A^w$ over a map $\mathcal{U}\to\mathcal{W}$ in $\mathbf{FSchemes}_A$. By assumption it induces an isomorphism of canonical subgroups $C_G\cong C_H\times_{\mathcal{W}_K}\mathcal{U}_K$ of level 1. Thanks to 3.11 it induces an isomorphism of the canonical subgroups of level r of G_K and $H_K\times_{\mathcal{W}_K}\mathcal{U}_K$. In particular, if we let $\mathcal{U}'_K\to\mathcal{U}_K$ and $\mathcal{W}'_K\to\mathcal{W}_K$ be $\mathcal{G}$-coverings classifying the trivializations of the the canonical subgroups of level r of G_K and H_K we have $\mathcal{U}'_K\cong\mathcal{W}'_K\times_{\mathcal{W}_K}\mathcal{U}_K$ compatibly with the action of $\mathcal{G}$. This induces a morphism $\mathcal{U}'\to\mathcal{W}'$ lifting the morphism $\mathcal{U}\to\mathcal{W}$. Due to 2.8 the map on differentials $\omega_H\to\omega_G$ induces a morphism $\alpha^*\colon\mathcal{F}_{\underline{H}}\to\mathcal{F}_{\underline{G}}$. Consider the diagram

$$\begin{array}{ccc}\alpha^*\big(\mathcal{F}_{\underline{H}}\big) & \stackrel{\alpha^*}{\longrightarrow} & \mathcal{F}_{\underline{G}}\\ \downarrow & & \downarrow\\ \alpha^*(C_H^\vee)\otimes\mathcal{O}_{\mathcal{U}'}/p^{1-v}\mathcal{O}_{\mathcal{U}'} & \stackrel{\alpha^\vee}{\longrightarrow} & C_G^\vee\otimes\mathcal{O}_{\mathcal{U}'}/p^{1-v}\mathcal{O}_{\mathcal{U}'},\end{array}$$

where the vertical arrows are induced by the isomorphisms defined in (2) and $\alpha^\vee\colon C_H^\vee\to C_G^\vee$ is induced by α so that it is an isomorphism. The diagram is commutative due to the functoriality of dlog proven in lemma 2.8. In particular α^* is an isomorphism of $\mathcal{O}_F\otimes\mathcal{O}_{\mathcal{U}'}$-modules and induces an isomorphism $\alpha^*\colon\alpha^*\big(\mathcal{F}'_{\underline{H}}\big)\otimes_{\alpha^*(S_{\mathcal{W}'})}S_{\mathcal{U}'}\cong\mathcal{F}'_{\underline{G}}$ of $S_{\mathcal{U}'}$-torsors. We thus get isomorphisms

$$\alpha^*\left(\mathfrak{H}om_{S_{\mathcal{W}'}}\big(\mathcal{F}'_{\underline{H}},\mathcal{O}_{\mathcal{W}'}\big)\right)\cong\mathfrak{H}om_{\alpha^*(S_{\mathcal{W}'})}\big(\alpha^*(\mathcal{F}'_{\underline{H}}),\mathcal{O}_{\mathcal{U}'}\big)\stackrel{\sim}{\longrightarrow}\mathfrak{H}om_{S_{\mathcal{U}'}}\big(\mathcal{F}'_{\underline{G}},\mathcal{O}_{\mathcal{U}'}\big)$$

of $\mathcal{O}_{\mathcal{U}'}$-modules with action of G. Here, the first map is induced by the natural map $\mathcal{O}_{\mathcal{W}'}\to\alpha_*\big(\mathcal{O}_{\mathcal{U}'}\big)$ and the last map is induced by inverting the isomorphism $\alpha^*\big(\mathcal{F}'_{\underline{H}}\big)\longrightarrow\mathcal{F}'_{\underline{G}}$. Inverting p and taking $\mathcal{G}$-invariants of $\theta_{K,*}$ we get the sought for isomorphism of $\mathcal{O}_{\mathcal{U}}\otimes_{\mathcal{O}_K}K$-modules

$$\omega_{\underline{G}}^{\dagger,k}\cong\alpha^*\big(\omega_{\underline{H}}^{\dagger,k}\big).$$

Remark 3.12. The right definition of the sheaves $\omega_{\underline{G}}^{\dagger,k}$ over $\mathcal{U}'$ should be as the “pushout” $\mathcal{F}'_{\underline{G}}\otimes_{S_{\mathcal{U}'}}\mathcal{O}_{\mathcal{U}'}^{(k)}\otimes K$, as the definition of $\omega_{\underline{G}}^{\dagger,k}$ on morphisms would be more direct and readily contravariant. Using local trivializations of $\mathcal{F}'_{\underline{G}}$ as $S_{\mathcal{U}'}$-torsor, one can make sense of such tensor construction as the push-out of the cocyle defining $\mathcal{F}'_{\underline{G}}$ via the map $k\colon S_{\mathcal{U}'} \to \mathcal{O}_{\mathcal{U}'} \otimes K$. The need in applications for an intrinsic definition of $\omega^{\dagger,k}_{\underline{G}}$, not based on a choice of local trivializations, lead the authors to the present definition. One should think of it as the $\mathcal{O}_{\mathcal{U}'} \otimes K$-dual of $\mathcal{F}'_{\underline{G}} \otimes_{S_{\mathcal{U}'}} \mathcal{O}^{(-k)}_{\mathcal{U}'} \otimes K$.

Recall from [SGA1, Ex. VI, Def. 5.2] that a functor between two categories over a given category is cartesian if it transforms cartesian diagrams to cartesian diagrams. We then deuce from the above that $\omega^{\dagger,k}_A$ is a cartesian functor $\mathbf{Hilb}'(\mu_N)^w_A$, namely for every morphism $\underline{G} \longrightarrow \underline{G}'$ in $\mathbf{Hilb}'(\mu_N)^w_A$ over $\phi\colon \mathcal{U} \to \mathcal{U}'$, the induced map $\phi^+\colon \phi^*\big(\omega^{\dagger,k}_{\underline{G}'}\big) \to \omega^{\dagger,k}_{\underline{G}}$ is an isomorphism of $\mathcal{O}_{\mathcal{U}} \otimes_{\mathcal{O}_K} K$-modules.

Theorem 3.13. *There exists a unique overconvergent modular sheaf*

$$\omega^{\dagger,k}_A\colon \mathbf{Hilb}(\mu_N)^w_A \longrightarrow \mathbf{Sheaves}_K,$$

which is a cartesian functor and extends the functor $\omega^{\dagger,k}_A$ on $\mathbf{Hilb}'(\mu_N)^w_A$. Furthermore,

(1) for every object $\underline{G}$ of $\mathbf{Hilb}(\mu_N)^w_A$ over a formal scheme $\mathcal{U}$ the $\mathcal{O}_{\mathcal{U}} \otimes K$-module $\omega^{\dagger,k}_{\underline{G}}$ is locally free.

(2) $\omega^{\dagger,k}_A$ is functorial in A and in w, i.e., for every morphism of formally smooth $\mathcal{O}_K$-algebras $A \to B$ and for every $w' \leq w$ satisfying the bound above, the modular sheaf $\omega^{\dagger,k}_B$ on $\mathbf{Hilb}(\mu_N)^{w'}_B$ is $\omega^{\dagger,k}_A$ on $\mathbf{Hilb}(\mu_N)^w_A$ restricted to the subcategory $\mathbf{Hilb}(\mu_N)^{w'}_B \subset \mathbf{Hilb}(\mu_N)^w_A$.

(3) Let $\mathcal{O}\colon \mathbf{Hilb}(\mu_N)^w_A \longrightarrow \mathbf{Sheaves}_K$ be the functor associating to $\underline{G} := \big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ the sheaf $\mathcal{O}_{\mathcal{U}} \otimes_{\mathcal{O}_K} K$. For weights k and k' let $k+k'$ be the weight $t^{k+k'} := t^k \cdot t^{k'}$. Then, taking w adapted to k and k' there is a natural isomorphism of functors $\omega^{\dagger,k}_A \otimes_{\mathcal{O}} \omega^{\dagger,k}_A \xrightarrow{\sim} \omega^{\dagger,k+k'}_A$, compatible with the functorialities in (2).

Proof. For every affine open $\mathcal{U} \subset \mathfrak{M}(A,\mu_N)(w)$ the pull-back of the universal object $\underline{G}^{\mathrm{univ}}_{\mathcal{U}}$ defines an object of $\mathbf{Hilb}'(\mu_N)^w_A$. In particular, $\omega^{\dagger,k}_{\underline{G}^{\mathrm{univ}}}$ is defined for the universal object over $\mathfrak{M}(A,\mu_N)(w)$. Consider an object $\underline{G}$ of $\mathbf{Hilb}(\mu_N)^w_A$ over a formal scheme $\mathcal{U}$. Possibly replacing $\mathcal{U}$ with an open affine formal sub-scheme we may assume that $\mathcal{U} = \mathrm{Spf}(R)$ is affine. We have a unique map $f\colon \mathcal{U} \to \mathfrak{M}(A,\mu_N)(w)$ such that $\underline{G}$ is isomorphic to the pull–back of the universal object via $\underline{G}^{\mathrm{univ}}$ via f. We set $\omega^{\dagger,k}_{\underline{G}} := f^*\big(\omega^{\dagger,k}_{\underline{G}^{\mathrm{univ}}}\big)$. This construction glues and provides a definition of $\omega^{\dagger,k}_{\underline{G}}$ for every object $\underline{G}$ of $\mathbf{Hilb}(\mu_N)^w_A$.

As we have already proven that the functor $\omega^{\dagger,k}_A$ on $\mathbf{Hilb}'(\mu_N)^w_A$ is cartesian, this definition agrees with the one already given for objects of $\mathbf{Hilb}'(\mu_N)^w_A$. The cartesian property implies that any functor with the properties required in the Theorem agrees on objects with our definition. This proves the uniqueness claimed in the Theorem 3.13. We are left to define $\omega^{\dagger,k}$ on morphisms and to prove that it is cartesian.

Consider an object $\underline{G}$ of $\mathbf{Hilb}(\mu_N)^w_A$ over a formal scheme $\mathcal{V}$. Possibly after replacing $\mathcal{V}$ with a covering by open formal sub-schemes we may assume that $\mathcal{V} = \mathrm{Spf}(R)$ and the morphism $f\colon \mathcal{V} \to \mathfrak{M}(A,\mu_N)(w)$, defined above, factors via an open affine sub-scheme $\mathcal{U}$ of $\mathfrak{M}(A,\mu_N)(w)$ for which $\underline{G}^{\mathrm{univ}}_{\mathcal{V}}$ is in $\mathbf{Hilb}'(\mu_N)^w_A$. Define $\theta_K\colon \mathcal{V}'_K \longrightarrow \mathcal{V}_K$ as the finite and étale morphism representing the functor $\underline{\mathrm{Isom}}_{\mathcal{O}_F}\big(\mathcal{O}_F/p^r\mathcal{O}_F, C_r\big)$ and let R' be the normalization of R in $\mathcal{V}'_K$. Recall that over $\mathcal{U}$ we have $\mathcal{F}_{\underline{G}^{\mathrm{univ}}}|_{\mathcal{U}} \subset \Omega_{\underline{G}^{\mathrm{univ}}/\mathcal{U}} \otimes_{\mathcal{O}_{\mathcal{U}}} \mathcal{O}_{\mathcal{U}'}$ and an isomorphism $(C_r)^\vee \otimes \mathcal{O}_{\mathcal{U}'}/p^{(1-v)r}\mathcal{O}_{\mathcal{U}'} \cong \mathcal{F}_{\underline{G}^{\mathrm{univ}}}/p^{(1-v)r}\mathcal{F}_{\underline{G}^{\mathrm{univ}}}$. As $\mathcal{V}'_K \cong \mathcal{V}_K \times_{\mathcal{U}_K} \mathcal{U}'_K$ the morphism $f\colon \mathcal{V} \to \mathcal{U}$ lifts to a morphism $f'\colon \mathcal{V}' \to \mathcal{U}'$ and pulling back via f' provides an invertible $\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{V}'}$-module $\mathcal{F}_{\underline{G}} \subset \Omega_{\underline{G}/\mathcal{V}} \otimes_{\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{\mathcal{V}'}$ and an isomorphism $\gamma_{\underline{G}}\colon (C_r)^\vee \otimes \mathcal{O}_{\mathcal{V}'}/p^{(1-v)(r)}\mathcal{O}_{\mathcal{V}'} \cong \mathcal{F}_{\underline{G}}/p^{(1-v)r}\mathcal{F}_{\underline{G}}$. It follows that $\omega_{\underline{G}}^{\dagger,k} \cong \Big(\theta_{K,*}\big(\mathfrak{Hom}_{\mathcal{S}_{\mathcal{V}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{V}'}^{(-k)})\big) \otimes_{\mathcal{O}_K} K\Big)^{\mathcal{G}}$ where $\mathcal{S}_{\mathcal{V}'}$, $\mathcal{F}'_{\underline{G}}$ and $\mathcal{O}_{\mathcal{V}'}^{(-k)}$ are defined as before.

By the functoriality of dlog the map $\gamma_{\underline{G}}$ is induced by the map dlog on the Tate module of G if R' is a dvr as explained in 2.7. As R' is normal and $\mathcal{F}_{\underline{G}}$ is an invertible $\mathcal{O}_{\mathcal{W}'}$-submodule of $\Omega_{\underline{G}/\mathcal{V}} \otimes_{\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{\mathcal{V}'}$, the knowledge of $\mathcal{F}_{\underline{G}}$ and $\gamma_{\underline{G}}$ in codimension one on $\mathrm{Spec}(R')$, for general R', characterizes them uniquely. This implies that they are defined for any object $\underline{G}$ of $\mathbf{Hilb}(\mu_N)_A^w$ by gluing and that they are functorial for morphisms in $\mathbf{Hilb}(\mu_N)_A^w$. Proceeding as for the definition of $\omega_A^{\dagger,k}$ on morphisms in $\mathbf{Hilb}'(\mu_N)_A^w$, this suffices to define the functor $\omega_A^{\dagger,k}$ on morphisms in $\mathbf{Hilb}(\mu_N)_A^w$ and to prove that it is cartesian.

(2) Due to uniqueness in order to prove that $\omega_A^{\dagger,k}$ is functorial in A and w we may restrict to the subcategories $\mathbf{Hilb}'(\mu_N)_A^w$ and $\mathbf{Hilb}'(\mu_N)_B^{w'}$. We remark that $\mathbf{Hilb}'(\mu_N)_B^{w'} \subset \mathbf{Hilb}'(\mu_N)_A^w$. Then, the construction implies that $\omega_B^{\dagger,k} = \omega_A^{\dagger,k}|_{\mathbf{Hilb}'(\mu_N)_B^{w'}}$ proving (1).

(3) Due to the uniqueness statement in the theorem, it suffices to construct the claimed isomorphism on the subcategory $\mathbf{Hilb}'(\mu_N)_A^w$. Take an object $\underline{G} := \big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$. The product on $\mathcal{O}_{\mathcal{U}'}$ defines a natural isomorphism of invertible $\mathcal{O}_{\mathcal{U}'}$-modules

$$\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-k)}) \otimes_{\mathcal{O}_{\mathcal{U}'}} \mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-k')}) \longrightarrow \mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-k\cdot k')}).$$

Taking $\otimes_{\mathcal{O}_K} K$ and the $\mathcal{G}$-invariants of the push-forward by θ_K, we get the isomorphism $\omega_{\underline{G}}^{\dagger,k} \otimes_{\mathcal{O}_{\mathcal{U}}} \omega_{\underline{G}}^{\dagger,k'} \cong \omega_{\underline{G}}^{\dagger,k\cdot k'}$. We leave it to the reader to verify that it is functorial in $\underline{G}$, in A and in w. □

3.4 Overconvergent modular forms of weight k

We write $\overline{\mathfrak{M}}(A,\mu_N)(w)$ for the p-adic formal scheme defining the normalization of $\mathfrak{M}(A,\mu_N)(w)$ in a smooth projective toroidal compactification $\overline{\mathfrak{M}}(A,\mu_N)$ of $\mathfrak{M}(A,\mu_N)$. The existence of $\overline{\mathfrak{M}}(A,\mu_N)$ is proven in [R, §6]. If $g = 1$, we are classifying elliptic curves and those compactifications coincide all with the strict neighborhood of the ordinary locus of width p^w in the modular curve $X_1(N) \otimes_{\mathcal{O}_K} A$. If $g \geq 2$ the definition of $\overline{\mathfrak{M}}(A,\mu_N)$ depends on the choice of a polyhedral cone decomposition.

Theorem 3.14. *Let w, A_K, r, k be as in the previous section. Then,*

i) $\omega_{\underline{G}^{\mathrm{univ}}}^{\dagger,k}$ extends uniquely to a coherent, locally free $\mathcal{O}_{\overline{\mathfrak{M}}(A,\mu_N)(w)} \otimes K$-module of rank 1 on $\overline{\mathfrak{M}}(A,\mu_N)(w)$. This extension will also be denoted by $\omega_{\underline{G}^{\mathrm{univ}}}^{\dagger,k}$.

ii) The A_K-module $\overline{M}(A,\mu_N,w) := \mathrm{H}^0\big(\overline{\mathfrak{M}}(A,\mu_N)(w), \omega_{\underline{G}^{\mathrm{univ}}}^{\dagger,k}\big)$ is independent of the toroidal compactification.

iii) Let $\mathfrak{q}$ denote p or an ideal of $\mathcal{O}_F$ not dividing pN. Let $f \in \overline{M}(A,\mu_N,w)$ and let us denote by f^o its restriction to $\mathfrak{M}(A,\mu_N)(w)$. Then $T_{\mathfrak{q}}(f^o)$ extends uniquely to an element of $\overline{M}(A,\mu_N,w)$ which will be denoted $T_{\mathfrak{q}}(f)$.

We start with the following important result.

Lemma 3.15. *There exists a unique triple of objects* $\big(\overline{G}^{\mathrm{univ}}, \overline{\iota}^{\mathrm{univ}}, \overline{Y}^{\mathrm{univ}}\big)$ *over* $\overline{\mathfrak{M}}(A,\mu_N)(w)$*, extending the triple* $(G^{\mathrm{univ}}, \iota^{\mathrm{univ}}, Y^{\mathrm{univ}})$ *over* $\mathfrak{M}(A,\mu_N)(w)$ *as follows:*

• $\overline{G}^{\mathrm{univ}} \longrightarrow \overline{\mathfrak{M}}(A,\mu_N)(w)$ *is a semiabelian scheme, locally relatively algebraizable extending the abelian scheme on* $G^{\mathrm{univ}} \longrightarrow \mathfrak{M}(A,\mu_N)(w)$*.*

• $\overline{\iota}^{\mathrm{univ}} : \mathcal{O}_F \longrightarrow \mathrm{End}_{\overline{\mathfrak{M}}(A,\mu_N)(w)}(\overline{G}^{\mathrm{univ}})$ *is a ring homomorphism compatible with the* $\mathcal{O}_F$*-multiplication on* G^{univ} *over* $\mathfrak{M}(A,\mu_N)(w)$*.*

• $\overline{Y}^{\mathrm{univ}} \in \mathrm{H}^0\big(\overline{\mathfrak{M}}(A,\mu_N)(w), \omega^{1-p}_{\overline{G}^{\mathrm{univ}}} \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K\big)$ *is such that*

$$Y \cdot h(\overline{G}^{\mathrm{univ}}, \overline{\iota}^{\mathrm{univ}}, \lambda^{\mathrm{univ}}, \psi_N^{\mathrm{univ}}) = p^w.$$

Proof. The existence of the extensions follows from [R, Thm. 6.18]. Let $\overline{G}$ and $\overline{G}'$ be two semiabelian schemes algebrizable locally over $\overline{\mathfrak{M}}(A,\mu_N)(w)$ and extending the universal abelian scheme over $\mathfrak{M}(A,\mu_N)(w)$. We consider a covering by open formal sub-schemes $\{\mathcal{U}_i = \mathrm{Spf}(R_i)\}_i$ such that R_i is an integral, noetherian domain and $\overline{G}$ and $\overline{G}'$ are algebrizable to semiabelian schemes G_i and G_i' over $U_i = \mathrm{Spec}(R_i)$ for every i. Then, there is an open dense sub-scheme $U_i^o \subset U_i$ over which G_i and G_i' are isomorphic to the pull–back of the universal abelian scheme over $\mathfrak{M}(A,\mu_N)(w)$. As the R_i's are all normal such a generic isomorphism extends uniquely to an isomorphism $G_i \cong G_i'$ thanks to [FC, Prop. I.2.7]. By uniqueness these local isomorphisms glue to a global isomorphism $\alpha\colon G \cong G'$. If $\overline{G}$ and $\overline{G}'$ are endowed with extra structures, namely polarizations and μ_N-level structures, extending the ones on the universal abelian scheme, they are preserved by α generically and, hence, they are preserved by α. This proves the uniqueness. □

Proof. (of the Theorem 3.14)

i) Let $\mathcal{U} = \mathrm{Spf}(R) \subset \overline{\mathfrak{M}}(A,\mu_N)(w)$ be an affine open and let $\underline{G} = (G/\mathcal{U}, \iota, \psi_N)$ denote the restriction of the universal semiabelian scheme, $\mathcal{O}_F$-multiplication and level μ_N-structure to $\mathcal{U}$. Let us also denote by $\mathcal{U}^o$ the open formal sub-scheme of $\mathcal{U}$ over which G is abelian, i.e. $\mathcal{U}^o = \mathcal{U} \times_{\overline{\mathfrak{M}}(A,\mu_N)(w)} \mathfrak{M}(A,\mu_N)(w)$, and let $\underline{G}^o := (G^o/\mathcal{U}^o, \iota^o, \lambda^o, \psi^o, Y^o)$ denote the restriction of $\underline{G}^{\mathrm{univ}}$ to $\mathcal{U}^o$. Then $\underline{G}^o$ is an object of $\mathbf{Hilb}\mu_N)^w_A$. By [R, Thm. 5.1] taking $\mathcal{U}$ small enough we may assume that R and G satisfy the assumptions of §2.1. In particular the pull back of G to the completion of $\mathcal{U}$ along $\mathcal{U}\backslash\mathcal{U}^o$ admits a uniformization à la Mumford by a 1-motive M and the Tate module of G^o_K is defined by the Tate module of M. Due to the real multiplication, M has purely toric semiabelian part. Thus the connected part $G[p^r]^0$ of $G[p^r]$ is finite and flat over $\mathcal{U}$ of rank p^{rg}, it is a diagonalizable group scheme and $G[p^r]^0_K$ defines the canonical subgroup C_r of G_K level p^r over $\mathcal{U}_K$. Let us define $\theta_K\colon \mathcal{U}'_K \longrightarrow \mathcal{U}_K$ to be the finite and étale morphism representing the functor

$$\underline{\mathrm{Isom}}_{\mathcal{O}_F}\big(\mathcal{O}_F/p^r\mathcal{O}_F, C_r\big),$$

i.e., all the trivializations of C_r as $\mathcal{O}_F/p^r\mathcal{O}_F$-group scheme. It is Galois with group $\mathcal{G} \cong \big(\mathcal{O}_F/p^r\mathcal{O}_F\big)^\times$. Let now $\theta\colon \mathcal{U}' \longrightarrow \mathcal{U}$ denote the normalization of $\mathcal{U}$ in $\mathcal{U}'_K$ and set $\underline{G}' := \big(G'/\mathcal{U}', \iota', \psi_N'\big)$ to be the pull–back of $\underline{G} = \big(G, \iota, \psi_N\big)$ via θ. Consider the $\mathcal{O}_{\mathcal{U}'}$-sub-module $\mathcal{F}_{\underline{G}}$ of $\omega_{G'/R'}$, given by $F_0(G'/R')$ of 2.10. Due to 2.9 it coincides with $\omega_{G'/R'}$ and in particular it is stable under the action of $\mathcal{O}_F$ on G'. Moreover we have an isomorphism of $\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{U}'}$-modules provided by the map dlog:

$$(C_r)^\vee \otimes \mathcal{O}_{\mathcal{U}'}/p^{(1-v)r}\mathcal{O}_{\mathcal{U}'} \cong \mathcal{F}_{\underline{G}}/p^{(1-v)r}\mathcal{F}_{\underline{G}}. \tag{2}$$

Define $\mathcal{F}'_{\underline{G}}$ as the inverse image in $\mathcal{F}_{\underline{G}}$ of $(C_r)^\vee - (C_r)^\vee[p^{r-1}]$. Now we repeat the construction of $\omega^{\dagger,k}_{\underline{G}}$ in the abelian case, more precisely we notice that $\mathcal{F}'_{\underline{G}}$ is a torsor under the sheaf of groups $\mathcal{S}_{\mathcal{U}'} := \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^\times \cdot \big(1 + p^{(1-v)r}(\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{U}'})\big)$ and both $\mathcal{S}_{\mathcal{U}'}$ and $\mathcal{F}'_{\underline{G}}$ are endowed with compatible actions of $\mathcal{G}$, lifting the action on $\mathcal{U}'$ given by the same expressions as in the abelian case. We denote $\mathcal{O}^{(k)}_{\mathcal{U}'}$ the sheaf $\mathcal{O}_{\mathcal{U}'}$ with that action of $\mathcal{S}_{\mathcal{U}'}$. We define the coherent $\mathcal{O}_{\mathcal{U}} \otimes_{\mathcal{O}_K} K$-module

$$\omega^{\dagger,k}_{\underline{G}} := \Big(\theta_{K,*}\big(\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}^{(-k)}_{\mathcal{U}'})\big) \otimes_{\mathcal{O}_K} K\Big)^{\mathcal{G}}.$$

As $\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}\big(\mathcal{F}'_{\underline{G}}, \mathcal{O}^{(-k)}_{\mathcal{U}'}\big)$ is an invertible $\mathcal{O}_{\mathcal{U}'}$-module, we deduce that $\omega^{\dagger,k}_{\underline{G}}$ is an invertible $\mathcal{O}_{\mathcal{U}} \otimes_{\mathcal{O}_K} K$-module. Moreover we have:

$$\omega^{\dagger,k}_{\underline{G}}|_{\mathcal{U}^o_K} = \omega^{\dagger,k}_{\underline{G}^o}.$$

By uniqueness the sheaves $\omega^{\dagger,k}_{\underline{G}}$ on $\mathcal{U}_K$ glue to give an invertible sheaf $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ on $\overline{\mathfrak{M}}(A,\mu_N)(w)$ extending $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ on $\mathfrak{M}(A,\mu_N)(w)$.

ii) We now show that the A_K-module $\mathrm{H}^0\big(\overline{\mathfrak{M}}(A,\mu_N)(w), \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\big)$ is independent of the choice of toroidal compactification $\overline{\mathfrak{M}}(A,\mu_N)(w)$. If $g = 1$ all toroidal compactifications coincide and so there is nothing to prove. For $g \geq 2$ we remark that any two smooth toroidal compactifications are dominated by a third one, still smooth, by [R, Lemme 4.2]. Thanks to 3.15 we have compatibility of the universal objects. So it suffices to prove the following claim. Consider a morphism of smooth toroidal compactifications $f\colon \overline{\mathfrak{M}}'(A,\mu_N)(w) \to \overline{\mathfrak{M}}(A,\mu_N)(w)$ and a morphism $g\colon \underline{G}'^{,\rm univ} \to \underline{G}^{\rm univ}$ over f. We have an isomorphism $f^*\big(\omega^{\dagger,k}_{\underline{G}^{\rm univ}}\big) \cong \omega^{\dagger,k}_{\underline{G}',{\rm univ}}$ by definition of overconvergent modular sheaf. By adjunction we obtain a map $\omega^{\dagger,k}_{\underline{G}^{\rm univ}} \longrightarrow f_*\big(f^*\big(\omega^{\dagger,k}_{\underline{G}^{\rm univ}}\big)\big)$. We claim that it is an isomorphism. As $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ is a locally free module $\mathcal{O}_{\overline{\mathfrak{M}}(A,\mu_N)(w)} \otimes_{\mathcal{O}_K} K$-module, it suffices to prove it for $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ replaced with $\mathcal{O}_{\mathcal{U}} \otimes_{\mathcal{O}_K} K$ for an open affine $\mathcal{U} = \mathrm{Spf}(R) \subset \overline{\mathfrak{M}}(A,\mu_N)(w)$. In this case, the claim follows from the theorem on formal functions as $f^{-1}\big(\mathcal{U}\big) \to \mathcal{U}$ is proper birational and R is normal.

iii) We now prove that he operators $T_{\mathfrak{q}}$ and U_p restricted to $\mathrm{H}^0\left(\overline{\mathfrak{M}}(A,\mu_N)(w'), \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\right)$ factor through $\mathrm{H}^0\left(\overline{\mathfrak{M}}(A,\mu_N)(w), \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\right)$. As $\omega^{\dagger,k}_{\underline{G}^{\rm univ}}$ is a locally free sheaf on $\overline{\mathfrak{M}}\big(A,\mu_N,\big)(w)$ and the latter admits an open affine covering by the formal spectrum of normal rings, it suffices to prove that the image of a section s of $\mathrm{H}^0\left(\overline{\mathfrak{M}}(A,\mu_N)(w'), \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\right)$ via $T_{\mathfrak{q}}$ or U_p, defined a priori in $\mathrm{H}^0\big(\mathfrak{M}\big(A,\mu_N,\big)(w)_K, \omega^{\dagger,k}_{\underline{G}^{\rm univ}}\big)$, extends in codimension 1 over $\overline{\mathfrak{M}}\big(A,\mu_N,\big)(w)_K$. More precisely, it suffices to prove the following.

Claim 1. Let R be a complete discrete valued field with maximal ideal I and fraction field L of characteristic 0. Consider an object $\underline{G}_L := \big(G_L/L, \iota, \lambda, \psi_N, Y\big) \in \mathbf{Hilb}(\mu_N)^w_A(L)$ such that G_L extends to a semiabelian scheme G over R with action of $\mathcal{O}_F$, having bad reduction at the closed point of R. In particular, it will have purely toric reduction and in particular the canonical subgroup of level r of G_L extends to a subgroup scheme of G, finite and flat over R, given by the connected part of $G[p^r]$. Proceeding as in the proof of (1) one gets an R-module $\omega_G^{\dagger,k} \subset \omega_{\underline{G}_L}^{\dagger,k}$. Let $L \subset L'$ be the finite and étale algebra defining the scheme theoretic fiber $p_1^{-1}\big([\underline{G}_L]\big)$ of the moduli point $[\underline{G}_L]$. In particular $G_{L'}$ admits a tautological subgroup scheme $H_{L'}$ defining the Hecke correspondence. Consider the quotient map $\pi_{L'}\colon G_{L'} \to G_{L'}/H_{L'} =: G'_{L'}$. Then, $G'_{L'}$ with induced action of $\mathcal{O}_F$, polarizations, ψ_N and Y defines an L'-valued point $\big[\underline{G}'_{L'}\big]$ of $\mathfrak{M}(A,\mu_N)(w')$. Let R' be the normalization of R in L. It is the product of finitely many discrete valuation rings. The schematic closure of $H_{L'} \subset G_{L'}$ defines a quasi-finite and flat subgroup scheme $H' \subset G \otimes_R R'$. Thus, the projection map $\pi\colon G \times_R R' \to \big(G \times_R R'\big)/H' =: G'$ is a flat and quasi-finite homomorphism of group schemes extending $\pi_{L'}$. G has purely toric reduction therefore G' also has purely toric reduction. Furthermore G' with the induced action of $\mathcal{O}_F$, the polarization, ψ_N and Y defined over L' provides a moduli interpretation for the extension of $\big[\underline{G}'_{L'}\big]$ to an R'-valued point of $\overline{\mathfrak{M}}\big(A,\mu_N,\big)(w')$ granted by the valuative criterion of properness; see [R, Prop. 5.2]. Also in this case as in the proof of (1) one gets an R'-module $\omega_{G'}^{\dagger,k} \subset \omega_{\underline{G}'_{L'}}^{\dagger,k}$. The claim we need to prove is the following.

Claim 2. The image of $\omega_{G'}^{\dagger,k}$ via the map $p_{1,*}\circ p_2^*\colon \omega_{\underline{G}'_{L'}}^{\dagger,k} \longrightarrow \omega_{\underline{G}_L}^{\dagger,k}$ is contained in $\omega_G^{\dagger,k}$.

We prove the claim. The module $\omega_G^{\dagger,k}$ (resp. $\omega_{G'}^{\dagger,k}$) is constructed from the torsor $\mathcal{F}'_G$ (resp. $\mathcal{F}'_{G'}$) under $S_R := \big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^\times\cdot\big(1+p^{(1-v)r}(\mathcal{O}_F\otimes R)\big)$ (resp. under $S_{R'}$) defined by the inverse image in ω_G (resp. $\omega_{G'}$) of $\operatorname{dlog}\big((C_r)^\vee-(C_r)^\vee[p^{r-1}]\big)$. The map $d\pi\colon \omega_{G'}\cong\omega_G\otimes_R R'$ is an isomorphism. As π induces an isomorphism between the canonical subgroups of level p^r of $G\otimes_R R'$ and of G', it induces an isomorphism of S'_R-torsors $\mathcal{F}'_{G'}\xrightarrow{\sim}\mathcal{F}_G\times^{S_R}S'_R$. Arguing as in the proof of the functoriality of $\omega_A^{\dagger,k}$ in §3.3 we get an isomorphism of R'-modules

$$d\pi^k\colon \omega_{G'}^{\dagger,k}\xrightarrow{\sim}\omega_G^{\dagger,k}\otimes_R R'.$$

The map $p_{1,*}\circ p_2^*$ is defined as the base change $\otimes_R L$ of the composite of $d\pi^k$ and the trace $\omega_G^{\dagger,k}\otimes_R R'\to\omega_G^{\dagger,k}$. The latter is defined by the trace from R' to R so that $p_{1,*}\circ p_2^*\big(\omega_{G'}^{\dagger,k}\big)$ lies in the submodule $\omega_G^{\dagger,k}$ of $\omega_{\underline{G}_L}^{\dagger,k}$ as claimed. □

Let w, A_K, r, k be as in theorem 3.14.

Definition 3.16. We call $\overline{M}(k,\mu_N,w) := \mathrm{H}^0\big(\overline{\mathfrak{M}}(A,\mu_N)(w),\omega_{\underline{G}^{\mathrm{univ}}}^{\dagger,k}\big)$ the space of *holomorphic overconvergent Hilbert modular forms* of weight k, tame level μ_N and degree of overconvergence w. We denote by

$$T_{\mathfrak{q}}\colon \overline{M}(k,\mu_N,w)\longrightarrow\overline{M}(k,\mu_N,w),\qquad U_p\colon \overline{M}(k,\mu_N,w)\to\overline{M}(k,\mu_N,w),$$

for primes $\mathfrak{q}$ not dividing p, the induced Hecke operators.

Assume now that we can take $r=1$, i.e., that $||k\big(1+p\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)\big)||<p^{-1/(p-1)}$. Given an object $\underline{G} := \big(G/\mathcal{U},\iota,\lambda,\psi_N,Y\big)$ in $\mathbf{Hilb}'(\mu_N)_A^w$, define

$$\Omega_{\underline{G}}^{\dagger,k} := \theta_*\big(\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}},\mathcal{O}_{\mathcal{U}'}^{(-k)})\big)^{\mathcal{G}}.$$

As $\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}\big(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-k)}\big)$ is an invertible $\mathcal{O}_{\mathcal{U}'}$-module and the group $\mathcal{G} \cong (\mathcal{O}_F/p\mathcal{O}_F)^*$ is of order prime to p, we conclude that $\Omega^{\dagger,k}_{\underline{G}}$ is a coherent $\mathcal{O}_{\mathcal{U}}$-module and it is a direct summand in $\mathfrak{Hom}_{\mathcal{S}_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-k)})\big)$. Arguing as in 3.13 the functor $\Omega^{\dagger,k}$ extends uniquely to a cartesian functor $\Omega^{\dagger,k}_A\colon \mathbf{Hilb}(\mu_N)^w_A \longrightarrow \mathbf{FSheaves}$ called the *integral overconvergent modular sheaf* of weight k. As in 3.13 one proves that

Corollary 3.17. *If* $||k\big(1+p^n\mathcal{O}_F\big)|| < p^{-1/(p-1)}$*, the functor* $\Omega^{\dagger,k}_A\colon \mathbf{Hilb}(\mu_N)^w_A \longrightarrow \mathbf{Sheaves}$ *is cartesian. Furthermore,*

(1) $\Omega^{\dagger,k}_A$ *is functorial in* A *and in* w*;*

(2) for weights k *and* k' *there is a natural isomorphism of functors* $\Omega^{\dagger,k}_A \otimes_{\mathcal{O}} \Omega^{\dagger,k}_A \xrightarrow{\sim} \Omega^{\dagger,k\cdot k'}_A$*;*

(3) we have an isomorphism of functors $\omega^{\dagger,k}_A \cong \Omega^{\dagger,k}_A \otimes_{\mathcal{O}_K} K$*.*

4 Properties of $\omega^{\dagger,k}_A$

We gather several properties of the overconvergent modular sheaf $\omega^{\dagger,k}_A$ defined in theorem 3.13.

4.1 The q-expansion maps

First of all we show how the general theory of overconvergent modular sheaves provides naturally q-expansion maps on the A_K-module $\overline{M}\big(k,\mu_N,w\big)$ of holomorphic overconvergent Hilbert modular forms of weight k defined in 3.16.

A *cusp* of $\overline{\mathfrak{M}}(A,\mu_N)(w)$ is defined by (i) a fractional ideal $\mathfrak{A}$ of $\mathcal{O}_F$, (ii) a notion of positivity on $\mathfrak{A}^{-1}$, (iii) an exact sequence of $\mathcal{O}_F$-modules $0 \to \mathcal{D}_F^{-1} \to H \to \mathfrak{A} \to 0$, (iv) a direct summand $\mathcal{D}_F^{-1}/N\mathcal{D}_F^{-1} \subset H/NH$ as $\mathcal{O}_F$-modules. We write $M := \mathfrak{A} = \mathfrak{A}^2\mathfrak{A}^{-1}$ and $M_N := \frac{1}{N}M$ with the notion of positivity induced by $\mathfrak{A}^{-1}$. See [Ch, §3.1]. We explain how to define the q-expansion maps at the given cusp:

$$q\text{-exp}_{\mathfrak{A}}\colon \overline{M}\big(k,\mu_N,w\big) \longrightarrow A[\![q^\alpha]\!]^{U_N^2}_{\alpha\in M_N^+\cup\{0\}} \otimes_{\mathcal{O}_K} K,$$

where $A[\![q^\alpha]\!]^{U_N^2}_{\alpha\in M_N^+\cup\{0\}}$ is the subring of $A[\![q^\alpha]\!]_{\alpha\in M_N^+\cup\{0\}}$ on which the group of squares of the units $U_N := \{x\in\mathcal{O}_F^\times | x\equiv 1 \mod pN\}$, acting on M_N, acts trivially.

Given a cusp one can construct a Tate object $\underline{\mathbf{Tate}}_{\mathfrak{A}}$ over the formal completion $\mathcal{U}_{\mathfrak{A}} = \mathrm{Spf}\big(R_{\mathfrak{A}}\big)$ of suitable affine open formal subscheme $\mathcal{W}_{\mathfrak{A}}$ of $\overline{\mathfrak{M}}(A,\mu_N)(w)$ at the given cusp. By construction $R_{\mathfrak{A}}$ is a subring of $A[\![q^\alpha]\!]_{\alpha\in M_N^+\cup\{0\}}$. The Tate object is the pull–back of the universal object $\underline{G}^{\mathrm{univ}}$ defined in 3.15. Set $\omega^{\dagger,k}\big(\underline{\mathbf{Tate}}_{\mathfrak{A}}\big)$ as the pull–back of the sheaf $\omega^{\dagger,k}\big(\underline{G}\big)^{\mathrm{univ}}$ defined in 3.14. We describe it explicitly. The Tate object $\underline{\mathbf{Tate}}_{\mathfrak{A}}$ admits a uniformization à la Mumford by a 1-motive $[\mathfrak{A} \to \mathbb{G}_m \otimes \mathcal{D}_F^{-1}]$. In particular, $\Omega_{\underline{\mathbf{Tate}}_{\mathfrak{A}}/\mathcal{U}_{\mathfrak{A}}} \cong \Omega_{\mathbb{G}_m\otimes\mathcal{D}_F^{-1}/\mathcal{U}_{\mathfrak{A}}}$ admits a canonical basis element ω_{can} as $\mathcal{O}_F\otimes R_{\mathfrak{A}}$-module provided by the standard invariant differential on $\mathbb{G}_m$. See [R, §4] and [Ch, §3.6]. The subgroup scheme $\mu_p \subset \mathbb{G}_m$ defines the canonical subgroup $\psi_p\colon \mu_p\otimes\mathcal{D}_F^{-1} \longrightarrow \mathbf{Tate}_{\mathfrak{A}}$. Via these identifications the map dlog for $\mathbf{Tate}_{\mathfrak{A}}$ is defined by the map dlog for $\mathbb{G}_m\otimes\mathcal{D}_F^{-1}$, see §2.1, so that the image of the canonical generator of the Cartier dual of $\mu_p\otimes\mathcal{D}_F^{-1}$ is ω_{can} modulo p^{1-v}. Write $\mathcal{U}'_{\mathfrak{A}} := \mathrm{Spf}\big(R'_{\mathfrak{A}}\big) \to \mathcal{U}_{\mathfrak{A}}$ for the finite étale extension classifying all trivializations of the canonical subgroup $\psi'_p\colon \mu_p\otimes\mathcal{D}_F^{-1}$ of $\mathbf{Tate}_{\mathfrak{A}}$. The group of automorphisms of this extension is the group previously denoted $\mathcal{G}=(\mathcal{O}_F/p\mathcal{O}_F)^*$ in §3.3 and $\mathcal{U}'_{\mathfrak{A}}=\coprod_{\mathcal{G}}\mathcal{U}_{\mathfrak{A}}$ as we have ψ_p over $\mathcal{U}_{\mathfrak{A}}$. An element $\alpha\in\mathcal{G}$ acts on $R'_{\mathfrak{A}}$ as a diamond operator, sending $\psi'_p\mapsto\psi'_p\circ 1\otimes\alpha$ so that it acts by pull–back on $\mathcal{F}'_{\underline{\mathbf{Tate}}_{\mathfrak{A}}}$ sending $\omega_{\mathrm{can}}\mapsto\omega_{\mathrm{can}}|\langle\alpha\rangle=\tau(\alpha)^{-1}\omega_{\mathrm{can}}$ where τ is the Teichmüller character $\mathcal{G}\to\big(\mathcal{O}_F\otimes\mathbb{Z}_p)^*$; see [AGo, Prop. 7.6]. Write $S:=\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^{\times}\cdot\big(1+p^{(1-v)r}(\mathcal{O}_F\otimes R'_{\mathfrak{A}})\big)$. Consider the $R'_{\mathfrak{A}}$-module $\mathfrak{Hom}_S(\mathcal{F}'_{\underline{\mathbf{Tate}}_{\mathfrak{A}}},R'^{(-k)}_{\mathfrak{A}})$. By construction, see the proof of 3.14(i), the sections of $\omega^{\dagger,k}\big(\underline{\mathbf{Tate}}_{\mathfrak{A}}\big)$ over $\mathcal{U}_{\mathfrak{A}}$ coincide with $\mathfrak{Hom}_S(\mathcal{F}'_{\underline{\mathbf{Tate}}_{\mathfrak{A}}},R'^{(-k)}_{\mathfrak{A}})^{\mathcal{G}}[p^{-1}]$. It is a free $R_{\mathfrak{A}}[p^{-1}]$-module of rank 1, generated by the element ω^k_{can} sending $s\cdot\omega_{\mathrm{can}}\mapsto(-k)(s)e_k$ for every $s\in S$ where $e_k=\big(-k(\alpha)\big)_{\alpha\in\mathcal{G}}\in R'_{\mathfrak{A}}$ generates the eigenspace of $R'_{\mathfrak{A}}$ on which $\mathcal{G}$ acts via the character $-k$. Hence, for every $f\in\overline{M}\big(k,\mu_N,w\big)$ there exists a unique element q-$\exp_{\mathfrak{A}}(f)$ of $R_{\mathfrak{A}}\otimes_{\mathcal{O}_K}K$ such that q-$\exp_{\mathfrak{A}}(f)\cdot\omega^k_{\mathrm{can}}=f|_{\mathcal{U}_{\mathfrak{A}}}$ as sections of $\mathfrak{Hom}_S(\mathcal{F}_{\underline{\mathbf{Tate}}'_{\mathfrak{A}}},R'^{(-k)}_{\mathfrak{A}})^{\mathcal{G}}$. Note that as f is a section of $\omega^{\dagger,k}_A$ over $\overline{\mathfrak{M}}(A,\mu_N)(w)$ by 3.6 the image of q-$\exp_{\mathfrak{A}}(f)$ in $A[\![q^\alpha]\!]_{\alpha\in M_N^+\cup\{0\}}\otimes_{\mathcal{O}_K}K$ is contained in the subring $A[\![q^\alpha]\!]^{U_N^2}_{\alpha\in M_N^+\cup\{0\}}\otimes_{\mathcal{O}_K}K$. This provides the sought for q-expansion map.

4.2 Comparison with Katz' ordinary modular forms

For $w=0$, the formal scheme $\mathfrak{M}(A,\mu_N)(0)$ is the open formal subscheme of $\mathfrak{M}(A,\mu_N)$ defined by the ordinary locus. It is affine with algebra of functions denoted in the sequel by R. Let $\mathfrak{M}\big(A,\mu_{Np^\infty}\big)$ be the formal affine scheme defined by the Igusa tower over $\mathfrak{M}(A,\mu_N)(0)$, classifying ordinary abelian schemes G with real multiplication by $\mathcal{O}_F$, polarization and a μ_{Np^∞}-level structure $\Psi_{Np^\infty}\colon\mu_{Np^\infty}\mathcal{D}_F^{-1}\hookrightarrow G$. The natural map

$$r\colon\mathfrak{M}\big(A,\mu_{Np^\infty}\big)\longrightarrow\mathfrak{M}\big(A,\mu_N\big)(0)$$

is Galois with group $\mathcal{I}:=(\mathcal{O}_F\otimes\mathbb{Z}_p)^*$. Following [K2, §1.9], see also [AGo, Def. 11.4], we have:

Definition 4.1. The R_K-module of *ordinary p-adic modular forms à la Katz of level μ_N and weight k* is the space $M(A,\mu_N,k)^{\mathrm{ord}}$ of eigenfunctions in $H^0\left(\mathfrak{M}\big(A,\mu_{Np^\infty}\big),\mathcal{O}_{\mathfrak{M}\left(A,\mu_{Np^\infty}\right)}\otimes_{\mathcal{O}_K}K\right)[k]$ i.e. the set of sections on which $\mathcal{I}$ acts via the character k.

For every cusp as in §4.1 the connected part of the p-divisible group of the Tate object $\underline{\mathbf{Tate}}_{\mathfrak{A}}$ is canonically isomorphic to $\mu_{p^\infty}\otimes\mathcal{D}_F^{-1}$ providing a unique morphism $f\colon\mathrm{Spec}\widehat{A(\!(q^\alpha)\!)}_{\alpha\in M_N^+\cup\{0\}}\longrightarrow\mathfrak{M}\big(A,\mu_{Np^\infty}\big)$. Here ^ denotes p-adic completion. The pull–back of functions via f defines a q-expansion map

$$q\text{-}\exp_{\mathfrak{A}}\colon M(A,\mu_N,k)^{\mathrm{ord}}\longrightarrow\widehat{A(\!(q^\alpha)\!)}_{\alpha\in M_N^+\cup\{0\}}\otimes_{\mathcal{O}_K}K;$$

see [K2, (1.9.8)].

On the other hand, using 3.17 and proceeding as in §4.1, we also have an R_K-module $\mathrm{H}^0\big(\mathfrak{M}\big(A,\mu_N\big)(0),\omega^{\dagger,k}_A|_{\underline{G}^{\mathrm{univ}}}\big)$ and a q-expansion map

$$q\text{-}\exp_{\mathfrak{A}}\colon\mathrm{H}^0\big(\mathfrak{M}\big(A,\mu_N\big)(0),\omega^{\dagger,k}_A|_{\underline{G}^{\mathrm{univ}}}\big)\longrightarrow\widehat{A(\!(q^\alpha)\!)}_{\alpha\in M_N^+\cup\{0\}}\otimes_{\mathcal{O}_K}K.$$

The main result of this section is a statement comparing the two spaces:

Proposition 4.2. *We have a natural isomorphism of R_K-modules*

$$\Phi_k^{\rm ord}\colon \mathrm{H}^0\big(\mathfrak{M}\big(A,\mu_N\big)(0),\omega_A^{\dagger,k}|_{\underline{G}^{\rm univ}}\big) \stackrel{\sim}{\longrightarrow} M(A,\mu_N,k),$$

which is compatible with q-expansions.

Proof. Let $R_K\subset \overline{R}_K$ be the Galois extension, with group $\mathcal{H}$, over which the p-adic Tate module of T of the dual of the universal abelian scheme $G^{\rm univ}$ over $\mathrm{Spec}(R)$ is trivial. Note that we have an exact sequence

$$0\longrightarrow T^{\rm co}\longrightarrow T\to T^{\rm et}\longrightarrow 0$$

of $\mathcal{H}$-modules, where $T^{\rm co}$ is the Tate module of the connected part of $G^{{\rm univ},\vee}[p^\infty]$ and $T^{\rm et}\cong \mathbb{Z}_p(\phi)$, for $\phi\colon \mathcal{H}\longrightarrow \big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^\times$ an étale character, is the Tate module of the maximal étale quotient of T. They are both invertible $\mathcal{O}_F\otimes\mathbb{Z}_p$-modules. By construction ϕ factors via $\mathcal{I}$. Let $\overline{R}$ be the normalization of R in $\overline{R}_K$. We denote by $\Gamma\subset\mathcal{H}$ the Galois group of $R'_K\subset\overline{R}_K$ where R' is the algebra underlying the formal scheme $\mathfrak{M}\big(A,\mu_{Np}\big)(0)$. The quotient $\mathcal{H}/\Gamma$ is the group previously denoted by $\mathcal{G}\cong\big(\mathcal{O}_F/p\mathcal{O}_F\big)^*$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega^\vee_{G^{\rm univ}/R}\otimes_R\widehat{\overline{R}}(1) & \longrightarrow & T\otimes\widehat{\overline{R}} & \stackrel{\rm dlog}{\longrightarrow} & \omega_{G^{\rm univ}/R}\otimes_R\widehat{\overline{R}} & \longrightarrow & 0\\
 & & \downarrow & & \| & & \uparrow\alpha & & \\
0 & \longrightarrow & T^{\rm co}\otimes\widehat{\overline{R}} & \longrightarrow & T\otimes\widehat{\overline{R}} & \longrightarrow & T^{\rm et}\otimes\widehat{\overline{R}} & \longrightarrow & 0
\end{array}$$

The morphism α is defined by dlog and is an isomorphism so that $F^0=\omega_{G^{\rm univ}/R}\otimes_R\widehat{\overline{R}}$ in this case. Modulo p the morphism α induces the isomorphism $(\mathcal{O}_F/p\mathcal{O}_F)\otimes R'\cong G^{{\rm univ},\vee}[p]^{\rm et}\otimes R'\cong\omega_{G^{\rm univ}/R}\otimes_R R'/pR'$. Let G' (resp. F') be the inverse image of $(\mathcal{O}_F/p\mathcal{O}_F)^*$ under the map $\omega_{G^{{\rm univ},\vee}/R}\otimes\widehat{\overline{R}}\longrightarrow\big(\mathcal{O}_F/p\mathcal{O}_F\big)\otimes\overline{R}/p\overline{R}$ (resp. $\omega_{G^{\rm univ}/R'}\longrightarrow(G^{\rm univ}[p]/C)\otimes R'$). Then, G' (resp. F') is a torsor under $S^{\rm ord}=\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^\times(1+p\mathcal{O}_F\otimes\mathbb{Z}_p)\widehat{\overline{R}}\big)$ (resp. $S=\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^\times\big(1+p(\mathcal{O}_F\otimes\mathbb{Z}_p)R'\big)$) and the inclusion $F'\subset G'$ induces the isomorphism $G'\cong F'\times^S S^{\rm ord}$, where the latter is the push-out torsor. Let $\widehat{\overline{R}}^{(-k)}$ (resp $R'^{(-k)}$) be $\widehat{\overline{R}}$ (resp. R') with the action by $S^{\rm ord}$ (resp. S) twisted by the character $-k$. Then, $\big(S^{\rm ord}\big)^\Gamma=S$ and $\big(\widehat{\overline{R}}^{(-k)}\big)^\Gamma=R'^{(-k)}$. Then,

$$\mathrm{Hom}_S\big(F',R'^{(-k)}\big)=\left(\mathrm{Hom}_S\left(F',\widehat{\overline{R}}^{(-k)}\right)\right)^\Gamma=\mathrm{Hom}_{S^{\rm ord},\Gamma}\left(G',\widehat{\overline{R}}^{(-k)}\right).$$

An element $\sigma\in\Gamma$ acts on $g\in\mathrm{Hom}_{S^{\rm ord}}\big(G',\widehat{\overline{R}}^{(-s)}\big)$ by $(\sigma g)(x):=\sigma\big(g\big(\sigma^{-1}(x)\big)\big)$. Note that for $s\in S^{\rm ord}$ we have $(\sigma g)(sx)=s(\sigma g)(x)$ so that the action is well defined. The map

$$\rho\colon \mathrm{Hom}_S\left(F',\widehat{\overline{R}}^{(-k)}\right)\longrightarrow\mathrm{Hom}_{S^{\rm ord}}\left(G',\widehat{\overline{R}}^{(-k)}\right)$$

is defined by sending $f\mapsto g$ where for $y\in G'$ we let $g(y):=sf(x)$ for $s\in S^{\rm ord}$ and $x\in F'$ such that $y=xs$. Since f is S-invariant, $g(y)$ does not depend on the choice of s and x such that $xs=y$ i.e. g is a well defined and $S^{\rm ord}$-equivariant. Moreover, ρ is Γ-equivariant. Indeed, if $\sigma\in\Gamma$ then $(\sigma g)(y)=s(\sigma g)(x)=s\sigma\big(g(\sigma^{-1}(x))\big)=s\sigma\big(f(x)\big)$ since $\sigma^{-1}(x)=x$. Because

$\rho\big(\sigma(f)\big)(y) = s\sigma\big(f(x)\big)$ we have $\sigma\big(\rho(f)\big) = \rho\big(\sigma(f)\big)$. Moreover, ρ is an isomorphism whose inverse is defined by $g \mapsto g|_{F'}$.

The isomorphism $\alpha\colon T^{\mathrm{et}} \otimes \widehat{\widehat{R}} = \widehat{\widehat{R}}(\phi) \longrightarrow \omega_{G^{\mathrm{univ}}/R} \otimes_R \widehat{\widehat{R}}$ gives $\alpha^{-1}(G') \cong S^{\mathrm{ord}}$, where the Γ-action is defined by: if $\sigma \in \Gamma$, $y \in S^{\mathrm{ord}}$ then $\sigma * y = \sigma(y)\phi(\sigma) \in S^{\mathrm{ord}}$. Therefore α induces an isomorphism

$$\mathrm{Hom}_S\big(F', R'^{(-k)}\big) \cong \mathrm{Hom}_{S^{\mathrm{ord}},\Gamma}\left(S^{\mathrm{ord}}, \widehat{\widehat{R}}^{(-k)}\right) = \left(\mathrm{Hom}_{S^{\mathrm{ord}}}\left(S^{\mathrm{ord}}, \widehat{\widehat{R}}^{(-k)}\right)\right)^{\Gamma}.$$

Let us observe that $\mathrm{Hom}_{S^{\mathrm{ord}}}(S^{\mathrm{ord}}, \widehat{\widehat{R}}^{(-k)}) \cong \widehat{\widehat{R}}$, as $\widehat{\widehat{R}}$-modules, and the Γ-action is given as follows. Let $\sigma \in \Gamma$ and $g\colon S^{\mathrm{ord}} \longrightarrow \widehat{\widehat{R}}^{(-k)}$ be an S^{ord}-morphism. Then $(\sigma g)(u) := \sigma(g(\sigma^{-1}(u))$, in particular $(\sigma g)(1) = \sigma(g(\sigma^{-1}(1))) = \sigma(g(\phi(\sigma)^{-1}\cdot 1)) = k\big(\phi(\sigma)\big)g(1)$.

In other words, $\mathrm{Hom}_{S^{\mathrm{ord}}}(S^{\mathrm{ord}}, \widehat{\widehat{R}}^{(-k)}) \cong \widehat{\widehat{R}}(k \circ \phi)$ as Γ-modules and therefore α induces an isomorphism of R'-modules $\mathrm{Hom}_S\big(F', R'^{(-k)}\big) \cong \big(\widehat{\widehat{R}}(k\circ\phi)\big)^{\Gamma}$, which is compatible with the residual action of $\mathcal{G}$ on both sides. Now inverting p and passing to the invariants with respect to $\mathcal{G} = \big(\mathcal{O}_F/p\mathcal{O}_F\big)^*$ we get the claimed isomorphism

$$\Phi_k^{\mathrm{ord}}\colon \big(\mathrm{Hom}_S\big(F', R'^{(-k)}\big) \otimes_{\mathcal{O}_K} K\big)^{\mathcal{G}} \xrightarrow{\ \sim\ } \big(\widehat{\widehat{R}}(k\circ\phi) \otimes_{\mathcal{O}_K} K\big)^{\mathcal{H}},$$

where the first space is $\mathrm{H}^0\big(\mathfrak{M}\big(A, \mu_N,\big)(0), \omega_A^{\dagger,k}|_{\underline{G}^{\mathrm{univ}}}\big)$ and the second one is $M(A, \mu_N, k)$. We are left to check the compatibility with q-expansions. We remark that $T^{\mathrm{et}}\big(\mathbf{Tate}_{\mathfrak{A}}^{\vee}\big) = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as it is the Cartier dual of $T^{\mathrm{co}}\big(\mathbf{Tate}_{\mathfrak{A}}\big)$. The pull–back of α via $f\colon \mathrm{Spec}\widehat{A(\!(q^\alpha)\!)}_{\alpha\in M_N^+\cup\{0\}} \longrightarrow \mathfrak{M}\big(A, \mu_{Np^\infty}\big)$ arises then from the isomorphism of $R_{\mathfrak{A}}$-modules

$$R_{\mathfrak{A}} \cong T^{\mathrm{et}}\big(\mathbf{Tate}_{\mathfrak{A}}^{\vee}\big) \otimes R_{\mathfrak{A}} \xrightarrow{\mathrm{dlog}} \Omega_{\mathbb{G}_m\otimes\mathcal{D}_F^{-1}/\mathcal{U}_{\mathfrak{A}}} \cong \Omega_{\underline{\mathbf{Tate}}_{\mathfrak{A}}/\mathcal{U}_{\mathfrak{A}}},$$

coming from the map dlog on $\mathbb{G}_m$, sending $a \mapsto a\omega_{\mathrm{can}}$. Using the notation of 4.1, this induces isomorphisms of $R_{\mathfrak{A}}$-modules

$$\beta\colon \mathfrak{Hom}_S(\mathcal{F}'_{\underline{\mathbf{Tate}}_{\mathfrak{A}}}, R_{\mathfrak{A}}'^{(-k)})^{\mathcal{G}} \xrightarrow{\ \sim\ } \big(R_{\mathfrak{A}}'^{(-k)}\big)^{\mathcal{G}} \cong R_{\mathfrak{A}},$$

sending $\omega_{\mathrm{can}}^k \mapsto e_k \mapsto 1$. The map on the right is induced by the first projection $R'_{\mathfrak{A}} = \prod_{g\in\mathcal{G}} R_{\mathfrak{A}} \to R_{\mathfrak{A}}$ and sends $e_k \mapsto 1$. The pull–back of β to $\widehat{A(\!(q^\alpha)\!)}_{\alpha\in M_N^+\cup\{0\}}$ is the pull–back of Φ_k^{ord} via f. In particular, given $g \in \mathrm{H}^0\big(\mathfrak{M}\big(A, \mu_N,\big)(0), \Omega_A^{\dagger,k}|_{\underline{G}^{\mathrm{univ}}}\big)$ we have $q\text{-}\mathrm{exp}_{\mathfrak{A}}(g)\cdot\omega_{\mathrm{can}}^k = g|_{\mathcal{U}_{\mathfrak{A}}}$. Thus $q\text{-}\mathrm{exp}_{\mathfrak{A}}\big(\Phi_k^{\mathrm{ord}}(g)\big) = f^*\big(\Phi_k^{\mathrm{ord}}(g)\big) = q\text{-}\mathrm{exp}_{\mathfrak{A}}(g)\cdot\beta(\omega_{\mathrm{can}}^k) = q\text{-}\mathrm{exp}_{\mathfrak{A}}(g)$ as claimed.

□

4.3 Comparison with classical Hilbert modular forms

We conclude this section with a result comparing our construction with the spaces of classical Hilbert modular forms. Let K be a finite extension of $\mathbb{Q}_p$ containing a Galois closure of F. Let $\sigma_1, \ldots, \sigma_g$ be the field homomorphisms $F \to K$. Each σ_i defines a so called *universal* character

$\chi_i = (\sigma_i \otimes 1)\colon (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \to \mathcal{O}_K^\times$, see [AGo, Def. 4.1]. Then, inside $\mathcal{W}(K)$ one has the so called *classical* weights which are of the form $k := \chi \cdot \varepsilon$ where $\chi := \prod_{i=1}^{g} \chi_i^{k_i}$ with $(k_1, \ldots, k_g) \in \mathbb{Z}^g$ and $\varepsilon\colon \big(\mathcal{O}_F/p\mathcal{O}_F\big)^* \to \mathcal{O}_K^*$ is a character. We say that χ, and hence k, is *non-negative* if $(k_1, \ldots, k_g) \in \mathbb{N}^g$. For any such we write $|\chi| := \sum_i k_i$.

Let $k = \chi \cdot \varepsilon$ be a classical weight. Let $\underline{G} = \big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ be an object of $\mathbf{Hilb}(\mu_N)^w_{\mathcal{O}_K}$. In particular, $\Omega_{G/\mathcal{U}}$ is an invertible $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{U}}$-module. Let $\mathcal{U}_K^o$ the open dense subspace where G is an abelian scheme and let $\theta_K\colon \mathcal{U}_K^{o,'} \to \mathcal{U}_K^o$ be the finite étale covering classifying $\mathcal{O}_F$-invariant subgroup schemes $\Psi_p\colon \mu_p \mathcal{D}_F^{-1} \hookrightarrow G_K$. It is Galois with group $\mathcal{G} := \big(\mathcal{O}_F/p\mathcal{O}_F\big)^*$. Let $\theta\colon \mathcal{U}' \to \mathcal{U}$ be the normalization of $\mathcal{U}$ in $\mathcal{U}_K'$. Let us point out that this definition makes sense. Let $\mathcal{V} = \mathrm{Spf}(R) \subset \mathcal{U}$ be an affine open, then $\mathcal{V}_K = \mathrm{Spm}(R_K)$ and let $\mathcal{V}_K' \subset \mathcal{U}_K'$ be the inverse image of $\mathcal{V}_K$ under θ_K. This morphism is finite and étale therefore $\mathcal{V}_K'$ is an affinoid, $\mathcal{V}_K' = \mathrm{Spm}(S_K)$, with $R_K \longrightarrow S_K$ a finite and étale K-algebra homomorphism. Let S be the normalization of R in S_K and set $\mathcal{V}' = \mathrm{Spf}(S)$. Then S is a p-adically complete, separated and normal R-algebra and for varying $\mathcal{V}$'s, the various $\{\mathcal{V}'\}$'s constructed above glue to give a formal scheme $\mathcal{U}'$ with a unique morphism to $\mathcal{U}$. Define the $\mathcal{O}_{\mathcal{U}'}$-module $\omega^\chi_{G/\mathcal{U}'}$ pushing-out $\theta^*\big(\Omega_{G/\mathcal{U}}\big)$ by χ. If we put $T_{\mathcal{U}'} := \big(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{U}'}\big)^*$ and we let $\omega^\times$ be the $T_{\mathcal{U}'}$-torsor defined by $\theta^*\big(\Omega_{G/\mathcal{U}}\big)$, then $\omega^\chi_{G/\mathcal{U}'}$ is the sheaf $\mathfrak{H}om_{T_{\mathcal{U}'}}\big(\omega^\times, \mathcal{O}_{\mathcal{U}'}^{(\chi^{-1})}\big)$ where $\mathcal{O}_{\mathcal{U}'}^{(\chi^{-1})}$ is the sheaf $\mathcal{O}_{\mathcal{U}'}$ with action by $T_{\mathcal{U}'}$ twisted by χ^{-1} (cf. 3.12). Recall that $k \in \mathcal{W}(K)$ is given by $\chi \cdot \varepsilon\colon (\mathcal{O}_F \otimes \mathbb{Z}_p)^* \to \mathcal{O}_K^*$. The elements of $\mathcal{G}$ act on $\mathcal{U}'$ and on $\omega^\chi_{G/\mathcal{U}'}$. We write $\omega^k_{\underline{G}} := \theta_*\big(\omega^\chi_{G/\mathcal{U}'}\big)^{[\overline{k}]}$ as the eigenspace of elements of $\theta_*\big(\omega^\chi\big)$ on which $\mathcal{G}$ acts via the character $\overline{k}\colon (\mathcal{O}_F/p\mathcal{O}_F)^* \to \mathcal{O}_K^*$ defined by k. It is a $\mathcal{O}_{\mathcal{U}}$-module and ω^k defines a functor

$$\omega^k\colon \mathbf{Hilb}(\mu_N)^w_{\mathcal{O}_K} \longrightarrow \mathbf{Sheaves}, \quad \underline{G} \mapsto \theta_*\big(\omega^\chi_{G/\mathcal{U}'}\big)^\varepsilon.$$

The example 3.5 coincides with this definition for the so called *parallel weights*, i.e., weights of the form $\chi := (k, \ldots, k)$ and $\varepsilon \equiv 1$. In fact ω^k can be defined with the rule above also for the universal object G^{univ} over any toroidal compactification $\overline{\mathfrak{M}}(\mathcal{O}_K, \mu_N)$ of the moduli space $\mathfrak{M}(\mathcal{O}_K, \mu_N)$ and the global sections $\mathrm{H}^0\big(\overline{\mathfrak{M}}(\mathcal{O}_K, \mu_N), \omega^k_{G^{\mathrm{univ}}}\big)$ are the so called *classical Hilbert modular forms of weight χ and nebentypus ε.*

Proposition 4.3. *For every classical non-negative weight k there is a natural transformation of functors $\varphi_k\colon \Omega^{\dagger,k}_{\mathcal{O}_K} \longrightarrow \omega^k$ such that for every object $\underline{G}$ of $\mathbf{Hilb}(\mu_N)^w_{\mathcal{O}_K}$ the map $\varphi_k\big(\underline{G}\big)$ is an injective morphism of invertible modules with cokernel annihilated by $p^{|\chi|v}$.*

The induced functor $\varphi_{k,K}\colon \omega^{\dagger,k}_{\mathcal{O}_K} \longrightarrow \omega^k \otimes_{\mathcal{O}_K} K$ is an equivalence and it induces an isomorphism, compatible with Hecke operators,

$$\mathrm{H}^0\big(\overline{\mathfrak{M}}(\mathcal{O}_K, \mu_N), \omega^k_{G^{\mathrm{univ}}}\big) \longrightarrow M\big(\omega^{\dagger,k}_{\mathcal{O}_K}, N, w\big).$$

Proof. We use the notations above. Let $\underline{G} = \big(G/\mathcal{U}, \iota, \lambda, \psi_N, Y\big)$ be an object of $\mathbf{Hilb}(\mu_N)^w_{\mathcal{O}_K}$. Write

$$S_{\mathcal{U}'} := \big(\mathcal{O}_F \otimes \mathbb{Z}_p\big)^\times \big(1 + p^{1-v}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{U}'})\big) \subset T_{\mathcal{U}'}.$$

We remark that χ extends to a morphism of multiplicative monoids $\chi\colon \mathcal{O}_F \otimes \mathcal{O}_{\mathcal{U}'} \to \mathcal{O}_{\mathcal{U}'}$. The inclusion $\mathcal{F}_{\underline{G}} \subset \theta^*\big(\Omega_{G/\mathcal{U}}\big)$, see the explanation before 3.13, is compatible via the action of $S_{\mathcal{U}'}$ and $T_{\mathcal{U}'}$ via the natural inclusion $S_{\mathcal{U}'} \subset T_{\mathcal{U}'}$ and with the action of the Galois group $\mathcal{G}$. It provides an injective morphism of $\mathcal{O}_{\mathcal{U}'}$-modules

$$\varphi_\chi\big(\underline{G}\big)\colon \mathfrak{H}om_{S_{\mathcal{U}'}}\big(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-\chi)}\big) \longrightarrow \mathfrak{H}om_{T_{\mathcal{U}'}}\big(\theta^*\big(\Omega_{G/\mathcal{U}}\big), \mathcal{O}_{\mathcal{U}'}^{(-\chi)}\big) \cong \omega^\chi_{G/\mathcal{U}'}$$

as follows. Assume that over an open $\mathcal{W} \subset \mathcal{U}'$ the torsor $\mathcal{F}'_{\underline{G}}$ is generated by an element f and $\omega^\times_{G/\mathcal{U}'}$ is generated by an element e and write $f = ae$ with a a non-zero section of $\mathcal{O}_F \otimes \mathcal{O}_\mathcal{W}$, invertible after inverting p. Then, over $\mathcal{W}$, the map $\varphi_\chi(\underline{G})$ sends a local section $t\colon \mathcal{F}'_{\underline{G}}|_\mathcal{W} \to \mathcal{O}_\mathcal{W}^{(-\chi)}$ of $\mathfrak{H}om_{S_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_\mathcal{W}^{(-\chi)})$ to $\varphi_\chi(\underline{G})(t)\colon \omega^\times_{G/\mathcal{U}'}|_\mathcal{W} \to \mathcal{O}_\mathcal{W}^{(-\chi)}$ given by $\alpha e \mapsto \chi^{-1}(\alpha)\chi(a)t(f)$. One verifies that this morphism is well defined, i.e., it does not depend on local generators of $\mathcal{F}'_{\underline{G}}$ and $\omega^\times_{G/\mathcal{U}'}$ and that it is injective as $\chi(a)$ is invertible after inverting p. In particular, these morphisms glue for varying $\mathcal{W}$'s to a global injective morphism of $\mathcal{O}_{\mathcal{U}'}$-modules and the latter is equivariant with respect to the actions of $\mathcal{G}$. Note that

$$\Omega^{\dagger,k}_{\mathcal{O}_K}(\underline{G}) = \theta_*\left(\mathfrak{H}om_{S_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-k)})\right)^{\mathcal{G}} = \theta_*\left(\mathfrak{H}om_{S_{\mathcal{U}'}}(\mathcal{F}'_{\underline{G}}, \mathcal{O}_{\mathcal{U}'}^{(-\chi)})\right)^{[\overline{k}]},$$

the eigenspace on which $\mathcal{G}$ acts via $\overline{k}$. Thus, the map $\varphi_k(\underline{G})$ is defined by $\theta_*(\varphi_\chi(\underline{G}))$ by taking on both sides the eigenspaces on which $\mathcal{G}$ acts via $\overline{k}$. It is an injective morphism of invertible $\mathcal{U}$-modules as $\varphi_\chi(\underline{G})$ is an injective morphism of invertible $\mathcal{O}_{\mathcal{U}'}$-modules and $\mathcal{G}$ is a group of order prime to p. We leave to the reader the verification that φ_k defines a natural transformation of functors.

Since the cokernel of $\mathcal{F}_{\underline{G}} \subset \theta^*(\Omega_{G/\mathcal{U}})$ is annihilated by p^v, the cokernel of $\varphi_\chi(\underline{G})$ is annihilated by $\chi(p^v) = p^{|\chi|v}$. We conclude that $\varphi_{k,K} = \varphi_k \otimes_{\mathcal{O}_K} K$ is an isomorphism of functors. As $\omega^{\dagger,k}_{\mathcal{O}_K}$ and $\omega^k \otimes_{\mathcal{O}_K} K$ are both overconvergent modular sheaves, the fact that $\varphi_{k,K}$ induces an Hecke equivariant map follows from 3.10.

□

5 Overconvergent modular sheaves in the elliptic case

In this section we review what has been done so far in the case that $g = 1$, i.e., we are studying elliptic modular forms. The main purpose of this section is to show that the overconvergent modular forms defined in this article coincide, in the elliptic case, with the ones previously defined by R. Coleman in [C1].

We fix in this section the following notation: Let $N \geq 4$ be an integer, $p > 2$ a prime, K and w be as in section §1, $F = \mathbb{Q}$, $\mathcal{W}(K) := \mathcal{W}_\mathbb{Q}(K) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, K^\times)$. We denote $\mathcal{W}^*(K)$ the subset of weights $k \in \mathcal{W}(K)$ such that there exists a (unique) pair (s, i), $s \in K$ with $v(s) > \dfrac{-p+2}{p-1}$ and $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $t^k = \langle t\rangle^s \tau(t)^i$ where let us recall $\tau : \mathbb{Z}_p^\times \longrightarrow \mu_{p-1}$ is the Teichmüler character and for $t \in \mathbb{Z}_p^\times$ we denote $\langle t\rangle := t/\tau(t)$. We call such weights *accessible* and in what follows we identify the accessible weight k with the pair (s, i) as above associated to it. We set $A_K = K$ and let us recall the Eisenstein series ([C1], section B1)

$$E(q) := 1 + \frac{2}{L_p(0, \mathbf{1})} \sum_{n\geq 1} \Big(\sum_{d|n,(p,d)=1} \tau^{-1}(d) \Big) q^n \in K[[q]].$$

In this section we will denote by $X(w)$ the rigid analytic subspace of the modular curve $X_1(N)_{/K}$ which was denoted $\overline{\mathfrak{M}}(\mathcal{O}_K, \mu_N)(w)_K$ in section 3.4 and by $\mathfrak{X}(w)$ its formal model over $\mathcal{O}_K$ which was denoted $\overline{\mathfrak{M}}(\mathcal{O}_K, \mu_N)(w)$ there. Let us recall that if $g = 1$ there is a unique compactification (toroidal and minimal) of the open modular curve $Y_1(N)_{/K} = \mathfrak{M}(\mathcal{O}_K, \mu_N)_K$ so there is no ambiguity. Moreover, let us recall the important fact that $X(w)$ is an affinoid over K. We also denote by $X(p)(w)$ the inverse image under the natural projection $X_1(Np)_{/K} \longrightarrow X_1(N)_{/K}$ of $X(w)$. As $X(p)(w) \longrightarrow X(w)$ is a finite Galois extensions of affinoids, of Galois group $G \cong \mathbb{F}_p^\times$, we have a natural action of this group on $X(p)(w)$ and on $H^0(X(p)(w), \mathcal{O}_{X(p)(w)})$, the latter being denoted $F|\langle a\rangle$ for $F \in H^0(X(p)(w), \mathcal{O}_{X(p)(w)})$ and $a \in \mathbb{F}_p^\times$.

We now recall the definition B4 in [C1]. If $k \in \mathcal{W}(K)^*$ is an accessible weight attached to the pair (s,i) as above and $f(q) \in K[[q]]$ is a power series, we say that $f(q)$ is the q-expansion of an overconvergent modular form of tame level N, weight k and degree of overconvergence w if the power series $f(q)/E(q)^s$ is the q-expansion of a section $F \in H^0(X(p)(w), \mathcal{O}_{X(p)(w)})$ with the additional property that for all $a \in \mathbb{F}_p^\times$ we have $F|\langle a\rangle = \tau(a)^i F$.

In this sense, of course $E(q)$ is the q-expansion of an overconvergent modular form of weight $(1,0)$. We denote by $M(N,k,w)$ the K-vector space of all "Coleman" overconvergent modular forms of tame level N, weight k and degree of overconvergence w and by $M^\dagger(N,k) := \lim_{w\to 0} M(N,k,w)$. It is shown in [C1] that we have natural Hecke operators on $M^\dagger(N,k)$.

On the other hand in 3.13 we have introduced an overconvergent modular sheaf $\omega_{\mathcal{O}_K}^{\dagger,k}$ and the space $M\big(\omega_{\mathcal{O}_K}^{\dagger,k}, N\big) := \lim\limits_{w\to 0} M\big(\omega_{\mathcal{O}_K}^{\dagger,k}, N, w\big)$ of global sections for varying w's, see 3.6, also provides a Hecke module. The two spaces coincide for classical weights $(s,i) \in \big(\mathbb{Z}, \mathbb{Z}/(p-1)\mathbb{Z}\big)$ due to 4.3. Our aim is to prove the following result:

Theorem 5.1. *For all $k \in \mathcal{W}^*(K)$ we have a natural K-linear, Hecke-equivariant isomorphisms,*

$$\Phi_k \colon M\big(\omega_{\mathcal{O}_K}^{\dagger,k}, N\big) \xrightarrow{\sim} M^\dagger(N,k)_K.$$

Moreover, if $\kappa = (s,i) \in \big(\mathbb{Z}, \mathbb{Z}/(p-1)\mathbb{Z}\big)$ then Φ_k coincides with the one defined in proposition 4.3.

In order to prove the theorem the first step is to study more closely the (integral) sheaf

$$\Omega_w^{\dagger,k} := \Omega_{\mathcal{E}^{\mathrm{univ}}/\mathfrak{X}(w)}^{\dagger,k}$$

constructed in 3.17. Here $\mathcal{E}^{\mathrm{univ}}$ is the universal semiabelian scheme over the formal scheme $\mathfrak{X}(w)$.

We review its construction. Let $\mathfrak{X}(p)(w)$ denote the normalization of $\mathfrak{X}(w)$ in $X(p)(w)$ and $\vartheta : \mathfrak{X}(p)(w) \longrightarrow \mathfrak{X}(w)$ the corresponding finite map. As ϑ is normal and $\vartheta_K : X(p)(w) \longrightarrow X(w)$ is finite Galois with Galois group $G = \mathbb{F}_p^\times$, this group acts on $\mathfrak{X}(p)(w)$.

Let $k = (s,i)$ be an accessible weight. The condition on the valuation of s assures that $\exp\big(s\log\langle t\rangle\big)$ is well defined. Let $w = w(s) \in \mathbb{Q}$ be such that $0 \le w < p/(p+1)$ and $w < (p-1)v(s) + p - 2$. Write $v := w/(p-1)$. If x is a section of the sheaf of abelian groups $1 + p^{1-v}\mathcal{O}_{\mathfrak{X}(p)(w)}$ we then write $x^s := \exp\big(s\log(x)\big)$ which converges thanks to the assumption on w. It is another section of $1 + p^{1-v}\mathcal{O}_{\mathfrak{X}(p)(w)}$. We write $\mathcal{O}_{\mathfrak{X}(p)(w)}^{(k)}$ for the sheaf $\mathcal{O}_{\mathfrak{X}(p)(w)}$ with action of $S_v := \mathbb{Z}_p^\times \cdot \big(1 + p^{1-v}\mathcal{O}_{\mathfrak{X}(p)(w)}\big)$ defined as follows. For $c \in \mathbb{Z}_p^*$ and local sections x of $1 + p^{1-v}\mathcal{O}_{\mathfrak{X}(p)(w)}$ and y of $\mathcal{O}_{\mathfrak{X}(p)(w)}$ define

$$(cx)\cdot y := x^s c^k y.$$

Since for every $u \in 1+p\mathbb{Z}_p$ we have $(ux)^s\big(u^{-1}c\big)^k = x^s c^k$, the given action is well defined. We let G act on $\mathcal{O}^{(k)}_{\mathfrak{X}(p)(N)(w)}$ and on S_v via the pull–back action on $\mathcal{O}_{\mathfrak{X}(p)(w)}$.

Let us recall the S_v-torsor $\mathcal{F}'_v$ defined in section §3.3 and denoted $\mathcal{F}'_{\mathcal{E}^{\rm univ}/\mathfrak{X}(w)}$ there. Then G acts on $\mathcal{F}'_v$ and the construction in 3.17 gives:

Lemma 5.2. *The sheaf $\Omega^{\dagger,k}_{\mathcal{E}^{\rm univ}/\mathfrak{X}(w)}$ is the sheaf Ω^k_w, defined as the $\mathcal{O}_{\mathfrak{X}(w)}$-module*

$$\Omega^\kappa_w := \Big(\vartheta_*\Big(\mathfrak{Hom}_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-k)}_{\mathfrak{X}_1(Np)(w)}\big)\Big)\Big)^G.$$

In particular, if we denote $\mathcal{M}(N,k,w) := \mathrm{H}^0\big(\mathfrak{X}(w), \Omega^k_w\big)$ we have

$$\mathcal{M}(N,k,w)_K := \mathcal{M}(N,k,w)\otimes_{\mathcal{O}_K} K \cong M\big(\omega^{\dagger,k}_{\mathcal{O}_K}, N, w\big),$$

the latter being defined in 3.6, and

$$\mathcal{M}(N,k)_K := \lim_{w\to 0}\mathcal{M}(N,k,w)_K \cong M\big(\omega^{\dagger,k}_{\mathcal{O}_K}, N\big).$$

Note that $\mathfrak{Hom}_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-k)}_{\mathfrak{X}(p)(w)}\big)$ is an invertible $\mathcal{O}_{\mathfrak{X}(p)(w)}$-module since $\mathcal{F}'_v$ is a torsor locally trivial for the Zariski topology on $\mathfrak{X}(p)(w)$. Since ϑ is finite, $\vartheta_*\Big(\mathfrak{Hom}_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-k)}_{\mathfrak{X}(p)(w)}\big)\Big)$ is a coherent and p-torsion free $\mathcal{O}_{\mathfrak{X}(w)}$-module. If k is associated to the pair (s,j), it depends only on s and not on j. The action of G on $\mathcal{F}'_v$ and on $\mathcal{O}^{(-k)}_{\mathfrak{X}(p)(w)}$ induces an action of G on $\mathfrak{Hom}_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-k)}_{\mathfrak{X}(p)(w)}\big)$ lifting the action of G on $\mathfrak{X}(p)(w)$. The action of G depends on j. Then, Ω^k_w consists of the G-invariants of $\vartheta_*\Big(\mathfrak{Hom}_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-k)}_{\mathfrak{X}(p)(w)}\big)\Big)$ so that it is a coherent $\mathcal{O}_{\mathfrak{X}(w)}$-module.

We denote by Ω^s_w the sheaf $\mathfrak{Hom}_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-s,0))}_{\mathfrak{X}(p)(w)}\big)$. It is an invertible $\mathcal{O}_{\mathfrak{X}(p)(w)}$-module, endowed with an action of G.

Coleman introduced an overconvergent modular form D_p of level $\Gamma_1(Np)$ and weight $(1,0)\in\mathcal{W}^*(\mathbb{Q}_p)$ in lemma 9.2 of [C3] (see also proposition 6.2 and corollary 6.3 of the present article). The modular form D_p is a characterized by the folllowing properties of its q-expansion: $D_p(q)^{p-1} = E_{p-1}(q)$ and $D_p(q)\big(\mathrm{mod}\ p\big)=1$. The relationship between E and D_p is given by the following lemma.

Lemma 5.3. *(1) For all $\epsilon > 1/p$ there is $w>0$ such that D_p/E and E/D_p are sections on $X(p)(w)$ and $|(D_p/E)-1||_{X(p)(w)}\le\epsilon$ and $|(D_p/E)-1|_{X(p)(w)}\le\epsilon$.*

(2) Let k be a weight associated to a pair $(s,i)\in\mathcal{W}^*(K)$. Then $D_p(q)^s/E(q)^s$ and $E(q)^s/D_p(q)^s$ are q-expansions of overconvergent modular functions of trivial character.

Proof. 2) follows from 1) which is really an adaptation of lemma B3.1 of [C1]. More precisely, as $E(q)$ and $D_p(q)$ are congruent to 1 (mod p) it follows that $|(D_p/E)|_{X_1(Np)(0)}-1|_{X_1(Np)(0)}\le |p| = 1/p$. As the family $\{X_1(Np)(w)\}_{w>0}$ is a basis of strict neighborhoods of $X_1(Np)(0)$ and as D_p/E is overconvergent, it follows that

$$|(D_p/E)|_{X_1(Np)(0)}-1|_{X_1(Np)(0)} = \lim_{w\to 0^+}|(D/E)-1|_{X_1(Np)(w)}.$$

Now (1) is clear. Since E and D_p have trivial character, claim (2) follows as well. □

Let us fix $\mathcal{V} \subset \mathfrak{X}(w)$ an affine such that the invariant differentials $\omega_{\mathcal{E}/\mathcal{V}}$ is free with generator ω. Let $\mathcal{U} = \mathrm{Spf}(R) \subset \mathfrak{X}(p)(w)$ be the inverse image of $\mathcal{V}$ via the map $\vartheta \colon \mathfrak{X}(p)(w) \to \mathfrak{X}(w)$. Consider the differential, which we'll call **standard differential**:

$$\omega^{\mathrm{std}} := D_p(\mathcal{E}/R, \psi) = D_p(\mathcal{E}/R, \omega, \psi)\omega, \text{ for every generator } \omega \in \omega_{\mathcal{E}/\mathfrak{X}(p)(w)}(\mathcal{U}). \tag{3}$$

Here, ψ is the level $\Gamma_1(Np)$-structure of the restriction of $\mathcal{E}$ to $\mathcal{U}$.

Lemma 5.4. *(1) The R-module $\mathcal{F}_{\mathcal{U}}$ is the free R-submodule of $\omega_{\mathcal{E}/\mathcal{U}}$ generated by ω^{std}. In particular, $\mathcal{F}'_v$ is the trivial $\mathbb{Z}_p^*\big(1+p^{1-v}R\big)$-torsor defined by $\mathbb{Z}_p^*\big(1+p^{1-v}R\big)\omega^{\mathrm{std}}$.*

(2) The diamond operators $a \in \mathbb{F}_p^*$ act on ω^{std} via $\omega^{\mathrm{std}}|\langle a\rangle = \tau^{-1}(a)\omega^{\mathrm{std}}$.

Proof. (1) The first claim is a consequence of Proposition 2.7 and corollary 6.3. The second follows since ω^{st} modulo p^{1-v} is the image via $d\log$ of the generator $\gamma \in C^\vee$ by Proposition 2.7 and $\mathcal{F}'_v$ is the inverse image of $\mathbb{F}_p^*\gamma = C^\vee \backslash \{0\}$ under $\mathcal{F}_{\mathcal{U}} \longrightarrow C^\vee \otimes R/p^{1-v}R$.

(2) The claim follows from Coleman's result that D_p has p-adic weight $(1,0)$. □

As $D_p(\mathcal{E}/\mathfrak{X}(p)(w), \psi)$ is a canonical global section of $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}$, it follows that $\mathcal{F}$ is a free $\mathcal{O}_{\mathfrak{X}(p)(w)}$-submodule of $\omega_{\mathcal{E}/\mathfrak{X}(p)(w)}$ and $\mathcal{F}'_v$ is the trivial S_v-torsor, generated by the standard differential ω^{std}. Consider $s \in K$ with p-adic valuation $v(s) > \frac{2-p}{p-1}$. Let $X_{s,v}$ be the global section of Ω_w^s defined as follows. For every $\mathcal{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(N)(w)$ as above and every $cu \in S_v(\mathcal{U})$ with $c \in \mathbb{Z}_p^*$ and $u \in \big(1+p^{1-v}R\big)$, define

$$X_{s,v}(cu\omega^{\mathrm{std}}) := u^{-s} \in R.$$

Due to the action of the diamond operators on ω^{std}, it follows that for every $a \in \mathbb{F}_p^*$ we have $X_{s,v}\big(\omega^{\mathrm{std}}|\langle a\rangle\big) = X_{s,v}$. We deduce:

Corollary 5.5. *The $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module Ω_w^s is a free $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module with basis element $X_{s,v}$ and $X_{s,v} \in \mathcal{M}\big(N,(s,0),p^w\big)$.*

Now we can prove the following lemma.

Lemma 5.6. *We have a decomposition*

$$\vartheta_*\left(\Omega_w^s\right) = \oplus_{j=0}^{p-2}\Omega_w^{(s,j)},$$

as $\mathcal{O}_{\mathfrak{X}(w)}$-modules, which identifies $\Omega_w^{(s,j)}$ as the direct factor of $\vartheta_*\left(\Omega_w^s\right)$ on which G acts via the j-th power τ^j of the Teichmüller character τ. In particular,

(a) $\mathrm{H}^0\left(\mathfrak{X}(p)(w), \Omega_w^s\right) = \oplus_{j=0}^{p-2}\mathcal{M}\big(N,(s,j),p^w\big)$ and similarly after inverting p.

(b) The $\mathcal{O}_{\mathfrak{X}(w)} \otimes_{\mathcal{O}_K} K$-module $\Omega_w^k \otimes_{\mathcal{O}_K} K$ is invertible and identifying it with an invertible sheaf on the rigid analytic fiber $X(w)$ we get that

$$\mathcal{M}(N,k,w)_K = \mathrm{H}^0\big(X(w), \Omega_w^k \otimes_{\mathcal{O}_K} K\big).$$

c) if p^v is a uniformizer of K, then $\Omega_w^{(s,j)}$ is a locally free $\mathcal{O}_{\mathfrak{X}(w)}$-module of rank 1 for all $0 \le j \le p-2$.

Proof. Since the order of G is $p-1$ which is invertible in $\mathcal{O}_K$ and Ω^s_w is an invertible $\mathcal{O}_{\mathfrak{X}(p)(w)}$-module, then $\vartheta_*(\Omega^s_w)$ admits a decomposition into coherent $\mathcal{O}_{\mathfrak{X}(w)}$-modules defined, locally on $\mathfrak{X}(w)$, as the eigenspaces on which G acts via τ^j for $j = 0, \ldots, p-1$. We have

$$\mathfrak{H}om_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{-(s,i)}_{\mathfrak{X}(p)(w)}\big) = \Omega^s_w[-i]$$

as $\mathcal{O}_{\mathfrak{X}(p)(w)}$-modules with G-action where $\Omega^s_w[-i]$ is Ω^s_w, as $\mathcal{O}_{\mathfrak{X}(p)(w)}$-module, with action of G twisted by τ^{-i}. Since $\Omega^{(s,i)}_w$ consists by definition of the $\mathcal{G}$-invariants of $\vartheta_*\Big(\mathfrak{H}om_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-s,-i)}_{\mathfrak{X}(p)(w)}\big)\Big)$, it is identified with the G-invariants of $\vartheta_*(\Omega^s_w[-i])$, i.e., with the $\mathcal{O}_{\mathfrak{X}(w)}$-submodule of $\vartheta_*(\Omega^s_w)$ on which G acts via τ^i. Claim (a) follows.

Now we wish to show that in the decomposition of the locally free $\mathcal{O}_{X(w)}$-module $\vartheta_*\big(\Omega^s_w \otimes_{\mathcal{O}_K} K\big) \cong \vartheta_*\big(\mathcal{O}_{X(p)(w)}\big)$ (by corollary 5.5), into eigensheaves for the action of G, $\Omega^{(s,j)}_w \otimes_{\mathcal{O}_K} K \cong \mathcal{O}^{(j)}_{X(p)(w)}$, each of them is locally free of ramk 1. For this let us recall Igusa's theorem which states that $X(p)(0)$, the ordinary locus in $X(p)(w)$, is connected, therefore $X(p)(w)$ is also connected. Thus the rank of $\mathcal{O}^{(j)}_{X(p)(0)}$ can be checked on the generic point of $X(p)(0)$, and there it follows that the rank is 1 by Kummer theory. This proves b).

c) Under the assumption that p^w is a uniformizer of K, the formal scheme $\mathfrak{X}(w)$ is regular formal scheme and as $\mathfrak{X}(p)(w)$ is a normal formal scheme of relative dimension 2 over $\mathrm{Spf}(\mathcal{O}_K)$, it is Cohen-Macauley. Therefore the finite morphism $\vartheta\colon \mathfrak{X}(p)(w) \longrightarrow \mathfrak{X}(w)$ is also flat and as Ω^s_w is a locally free $\mathcal{O}_{\mathfrak{X}(p)(w)}$-module of finite rank, $\vartheta_*\big(\Omega^s_w\big)$ is a locally free $\mathcal{O}_{\mathfrak{X}(w)}$-module of finite rank. Thus $\Omega^{(s,j)}_w$ is itself locally free. Now as $\mathfrak{X}(w)$ is connected, the rank of $\Omega^{(s,j)}_w$ is constant and it is equal to 1 as by b) above the rank of $\Omega^{(s,j)}_w \otimes_{\mathcal{O}_K} K$ is 1. □

5.1 Comparison with Coleman's definition for non-integral weights

In this section we assume that $p \geq 5$ as in [C3] and we fix an accessible weight $k = (s, j) \in \mathcal{W}^*(K)$.

We prove the main result of this section, i.e. theorem 5.1.

Proof. (of 5.1)

Let $g \in H^0\big(X(w), \omega^{\dagger,k}\big)$. By lemma 5.6 this is equivalent to the fact that there is $f \in H^0\big(X(w), \omega^{(0,j)}\big)$ such that $g = fX_{s,v}$. In other words $f \in H^0\big(X(p)(w), \mathcal{O}_{X(p)(w)}\big)$ is such that $f|\langle a\rangle = \tau^j(a)f$ for all $a \in G = \mathbb{F}_p^\times$. Therefore for $w > 0$ small enough we have $g = fu^sE^s$ with $u \in H^0\big(X(p)(w), \mathcal{O}_{X(p)(w)}\big)^G$ with the property that u is congruent to 1 modulo a large enough power of p such that u^s makes sense. This is equivalent to g being an overconvergent modular form of weight k in Coleman's sense. □

6 Appendix A: The map dlog

Let p be a prime number ≥ 3 and $0 \leq w \leq \frac{1}{p}$ be a rational number. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and containing the p-th roots of unity. We fix such a root ζ_p so that over $\mathcal{O}_K$ we have a canonical homomorphism of group schemes $\mathbf{Z}/p\mathbf{Z} \to \mu_p$, sending

$1 \mapsto \zeta_p$, which is an isomorphism over K. We normalize the induced discrete valuation on K so that p has valuation 1. Let R be a normal and flat $\mathcal{O}_K$-algebra, which is a p-adically complete and separated integral domain. Let us recall the notation introduced after section §1: if $u \in \mathbb{Q}$, we'll denote by p^u an element of $\mathbb{C}_p$ of valuation u. If $p^u \in \mathcal{O}_K$ and M is an object over R (an R-module, an R-scheme or formal scheme) then we denote by $M_u := M \otimes_R R/p^u R$. In particular $M_1 = M \otimes_R R/pR$.

Let $\pi\colon A \longrightarrow \mathcal{U} := \operatorname{Spec}(R)$ be an abelian scheme of relative dimension $g \geq 1$. Assume that the determinant ideal of the Frobenius $\varphi\colon R^1\pi_*(A_1) \longrightarrow R^1\pi_*(A_1)$ contains p^w. Then, it follows by the main theorem [AGa, Thm. 3.5] that A_K over $\mathcal{U}_K$ admits a canonical subgroup C. Let $D \subset A[p]_K^\vee \cong A^\vee[p]_K$ be the Cartier dual of $A[p]_K/C$ over $\mathcal{U}_K$. Then, we have:

Proposition 6.1. *(1) The map* $\operatorname{dlog}\colon A^\vee[p]_K \longrightarrow \omega_{A/R} \otimes_R \overline{R}/p\overline{R}$ *has* D *as kernel.*

(2) The cokernel of the $\overline{R}$*-linear extension* $A^\vee[p]_K \otimes \overline{R}/p\overline{R} \longrightarrow \omega_{A/R} \otimes_R \overline{R}/p\overline{R}$ *of* dlog *is annihilated by* $p^{\frac{w}{p-1}}$.

The proof of (1) follows from [AGa], but we need to recall some preliminaries. The proof of (2) is a combination of the techniques of (1) and the results of [Fa2].

Congruence group schemes: Assume that K contains a p-th root of 1, let R be a flat, p-adically complete and separated $\mathcal{O}_K$-algebra which is an integral domain and let $\lambda \in R$ such that $\lambda^{p-1} \in pR$.

Let $G_\lambda = \operatorname{Spec}(A_\lambda)$, with $A_\lambda = R[T]/\big(P_\lambda(T)\big)$, be the finite and flat group scheme over R as in [AGa, Def. 5.1]. Here, $P_\lambda(T) := \frac{(1+\lambda T)^p - 1}{\lambda^p}$ and the group scheme structure is given as follows. The co-multiplication is $T \mapsto T \otimes 1 + 1 \otimes T + \lambda T \otimes T$, the co-unit $T \mapsto 0$, the co-inverse by $T \mapsto -T(1+\lambda T)^{-1}$. This makes sense since $1 + \lambda T$ is a unit in A_λ.

Homomorphisms between congruence group schemes: If μ is an element of R dividing λ the $\mathbb{F}_p$-vector space $\operatorname{Hom}_R(G_\lambda, G_\mu)$ is of dimension 1 generated by the map $\eta_{\lambda,\mu}\colon G_\lambda \to G_\mu$ sending $Z \mapsto 1 + \lambda\mu^{-1}T$; see [AGa, §5.3]. If $\nu \in R$ divides μ one has $\eta_{\nu,\mu} \circ \eta_{\lambda,\mu} = \eta_{\lambda,\nu}$. In particular, if μ is a unit then there is a canonical isomorphism $G_\mu \cong \mu_p$ by [AGa, Ex. 5.2(b)] and we put $\eta_\lambda := \eta_{\lambda,\mu}$. It follows that $\operatorname{Hom}_R(G_\lambda, G_\mu) = 0$ if μ does not divide λ.

Relation to Oort-Tate theory: In terms of Oort–Tate theory G_λ corresponds to the group scheme $G_{(a,c)} = \operatorname{Spec}\big(R[Y]/(Y^p - aY)\big)$ where $ac = p$ and $c = c(\lambda) = \lambda^{p-1}(1-p)^{p-1}w_{p-1}^{-1}$. There is a canonical isomorphism $G_{(a,c)} \cong G_\lambda$ sending $T \mapsto \sum_{i=1}^{p-1} \lambda^{i-1}\frac{Y^i}{w_i}$ where $w_1, \ldots, w_{p-1}$ are the universal constants of Oort-Tate; see [AGa, §5.4]. We remark for later purposes that $a = p\lambda^{1-p}$ up to unit so that $a = 0$ modulo $p\lambda^{1-p}$. Recall also that $w_i \equiv i!$ modulo p for $i = 1, \ldots, p-1$ so that $T \mapsto \sum_{i=1}^{p-1} \lambda^{i-1}\frac{Y^i}{i!}$ modulo p. In particular,

$$1 + \lambda T \mapsto \sum_{i=0}^{p-1} \lambda^{i-1}\frac{Y^i}{i!}$$

and at the level of differentials we have $dT \mapsto \sum_{i=1}^{p-1} \lambda^{i-1}\frac{Y^{i-1}}{(i-1)!}dY = \beta dY$ with $\beta = \sum_{i=1}^{p-2} \lambda^{i-1}\frac{Y^i}{i!}$. In particular, $\beta = 1$ modulo λY and $(1+\lambda T)^{-1}dT \mapsto \big(1 - \lambda^{p-1}Y^{p-1}\big)dY$ modulo $\lambda^{p-1}\lambda Y^p = \lambda^p aY$ which is a multiple of λp and hence is 0 modulo p.

Differentials: Since $P_\lambda(T) := \frac{(1+\lambda T)^p - 1}{\lambda^p}$ the derivative of $P_\lambda(T)$ is $p\lambda^{1-p}(1+\lambda T)^{p-1}$ which is a up to unit. Hence, we have $\Omega_{G_\lambda/R} \cong \lambda dT/aA_\lambda dT$ with $a = 0$ modulo $p\lambda^{1-p}$. In particular,

$\Omega_{G_\lambda/R}$ is free of rank 1 as $A_\lambda/(p\lambda^{1-p})$-module so that also the module of invariant differentials $\omega_{G_\lambda/R}$ of G_λ is a free $R/(p\lambda^{1-p})$-module of rank 1. The image of the invariant differential dT/T of μ_p under the map η_λ is then $\lambda(1+\lambda T)^{-1}dT$, i.e.,

$$\mathrm{dlog}\colon G_\lambda^\vee \longrightarrow \omega_{G_\lambda}, \qquad \eta_\lambda \mapsto \lambda(1+\lambda T)^{-1}.$$

Proof. (of 6.1) To prove (1) it suffices to prove that for every S-valued point x of $A^\vee[p]$, where $S=\mathrm{Spec}(R')$ and R' is a finite normal extension of R, we have $x_K \in D(S_K)$ if and only if $x_K \in \mathrm{Ker}\big(\mathrm{dlog}\big)$. Replacing R' with the completion at its prime ideals above p, we may assume that R' is a complete dvr. Passing to a faithfully flat extension we may further replace R' with its normalization $\overline{R}'$ in an algebraic closure of the fraction field of R'. Replace S with $S:=\mathrm{Spec}(\overline{R}')$.

To prove (2) we remark that $A^\vee[p]_K\otimes \overline{R}/p\overline{R}$ and $\omega_{A/R}\otimes_R \overline{R}/p\overline{R}$ are free $\overline{R}/p\overline{R}$-modules of the same rank. Take the determinant of the $\overline{R}$-linear extension of dlog and call it $d\in \overline{R}/p\overline{R}$. Then, d annihilates the cokernel. We may assume that $d\in R'/pR'$ for some $R\subset R'$ finite and normal and $R'\subset \overline{R}$. Since R' is normal, to prove that $d=\alpha p^{\frac{gw}{p-1}}$ for some $\alpha\in R'$ it suffices to show that this holds after localizing at the prime ideals of R' over p. As before, passing to a faithfully flat extension we may further replace R' with its normalization $\overline{R}'$ in an algebraic closure of the fraction field of R'.

Thus both in case (1) and (2) we may assume that R is a complete discrete valuation ring and $S=\overline{R}$. In this case it is proven in [AGa, prop. 13.5] that the map dlog modulo p can also be defined in terms of torsors and corresponds to the following map $\mathrm{dLog}\colon A^\vee(\overline{R})=\mathrm{H}^1_{\mathrm{fppf}}(A_{\overline{R}},\mu_p)\to \omega_{A_{\overline{R}}/\overline{R}}/p\omega_{A_{\overline{R}}/\overline{R}}$. A μ_p-torsor over $A_{\overline{R}_K}$ extends to a μ_p-torsor $Y\to A_{\overline{R}}$, which is defined by giving a Zariski affine cover $\mathcal{U}_i$ of $A_{\overline{R}}$ and units u_i in $\Gamma(U_i,\mathcal{O}_{U_i})$ so that the $Y|_{U_i}$ is given by the equation $Z_i^p-u_i$. Then, $\mathrm{dLog}(Y)$ is defined locally by $u_i^{-1}du_i\in\Gamma(U_i,\Omega^i_{U_i/\overline{R}})/(p)$ and these glue to a global section of $\omega_{A_{\overline{R}}/\overline{R}}/p\omega_{\mathcal{E}_{\overline{R}}/\overline{R}}$.

(1) Let $\lambda\in\overline{R}$ be an element of valuation $\frac{1-w}{p-1}$. It follows from [AGa, Prop. 13.4] and [AGa, Prop. 12.1] that the kernel of dLog has dimension g and is isomorphic to $\mathrm{H}^1_{\mathrm{fppf}}(A_{\overline{R}},G_\lambda)$. Note that $\mathrm{H}^1_{\mathrm{fppf}}(A_{\overline{R}},G_\lambda)\cong \mathrm{Hom}_{\overline{R}}\big(G_\lambda^\vee,A^\vee_{\overline{R}}\big)$ by Cartier duality [AGa, §5.12] so that we get a map $\Psi\colon G_\lambda^{\vee,g}\longrightarrow A^\vee_{\overline{R}}$ which is a closed immersion after inverting p. Let $D\subset A^\vee_{\overline{R}}$ be the schematic closure of Ψ_K. Then, by [AGa, Def. 12.4] it is the Cartier dual of $A_{\overline{R}}/C$. This concludes the proof of (1).

(2) Consider on $E=A^\vee_{\overline{R}}/D$ an increasing filtration by g finite and flat subgroup schemes Fil^iE such that $E_i=\mathrm{Fil}^{i+1}E/\mathrm{Fil}^iE$ is of order p. Such filtration exists over K with $E_{i,K}\cong\mathbb{Z}/p\mathbb{Z}$ since $E_K\cong(\mathbb{Z}/p\mathbb{Z})^g$. One then defines $\mathrm{Fil}^1E=E_1$ to be the schematic closure of $E_{1,K}$ in E and this is a finite and flat subgroup scheme of E since R is a dvr. One lets E_{i+1} be the schematic closure of $E_{i+1,K}$ in E/E_i and one puts Fil^iE to be the inverse image of E_{i+1} via the quotient map $E\to E/E_i$. In particular, $E_i\cong G^\vee_{\lambda_i}$ for some $\lambda_i\in R$. The invariant differentials $\omega_{G_{\lambda_i}}$ of G_{λ_i} define a free rank 1 module over $\overline{R}/p\lambda_i^{1-p}\overline{R}$.

It is one of the key results of Fargues [Fa2] that D is the canonical subgroup of $A^\vee$ and that $\prod_{i=1}^g\lambda_i^{1-p}$ divides p^w. Hence, each $\omega_{G_{\lambda_i}}/(p^{1-w})$ is a free $\overline{R}/p^{1-w}\overline{R}$-module. The image of the map $\mathrm{dlog}\colon G_{\lambda_i}^\vee \otimes \overline{R} \longrightarrow \omega_{G_{\lambda_i}}$ is generated by λ_i so that its cokernel is annihilated by λ_i. Since $E^\vee \subset A_{\overline{R}}$ is a closed immersion, the module of invariant differentials $\omega_{E^\vee}$ of $E^\vee$ modulo p^{1-w} is a quotient of the module of invariant differentials of $A_{\overline{R}}$ which is a free $\overline{R}$-module of rank g. Note that $\omega_{E^\vee}/(p^{1-w}) \cong \omega_{A_{\overline{R}}}/(p^{1-w})$ admits a filtration, with graded pieces $\omega_{E_i^\vee}/(p^{1-w})$, compatible with the given filtration on E. Due to its functoriality the map $\mathrm{dlog}\colon E \otimes \overline{R} \longrightarrow \omega_{E^\vee}/(p^{1-w})$ preserves the filtrations and we conclude that $\prod_{i=1}^{g} \lambda_i$ annihilates its image. In particular, $p^{\frac{w}{p-1}}$ annihilates the cokernel of dlog as claimed. □

We now pass to the case of elliptic curves. Let N be an integer prime to p. Let $\pi\colon \mathcal{E} \to Y(N,p)$ be the universal relative elliptic curve where $Y(N,p)$ is the modular curve associated to the congruence subgroup $\Gamma_1(N) \cap \Gamma_0(p)$. Write $\mathcal{E}_1$ for the mod p reduction of $\mathcal{E}$. Let $F\colon \mathcal{E}_1 \to \mathcal{E}_1^{(p)}$ be the Frobenius isogeny and let

$$\varphi\colon \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1} \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1}$$

be the induced σ-linear morphism on cohomology. This defines a modular form H of weight $p-1$ on the mod p-reduction of $Y(N,p)$ which is the pull-back to level (N,p) of the Hasse invariant, a modular form of level 1. It coincides with the modular form E_{p-1} modulo p for $p \geq 5$. Let $U(w)$ be the formal sub-scheme of $Y(N,p)$ defined by $|H| \leq w$. We proceed as follows: locally lift H so that $|H| \leq w$ makes sense in unequal characteristic and then we show that the formal scheme does not depend on the choice of the lift. Let $Z(w)$ be the normalization of the inverse image of $U(w)$ in the p-adic formal scheme associated to $Y_1(Np)$. Its rigid analytic geometric fibre is finite and étale of degree $p-1$ over $U(w)_{\mathbf{Q}_p}$. Recall that we have a map

$$\mathrm{dlog}\colon \mathcal{E}[p] \longrightarrow \omega_{\mathcal{E}_1}$$

defined as follows. Consider a formal scheme S over $U(w)$ and an S-valued point x of $\mathcal{E}[p]$. Via the canonical isomorphism $\mathcal{E}[p] \cong \mathcal{E}[p]^\vee$ it defines an S-valued point of $\mathcal{E}[p]^\vee$, i.e., a group scheme homomorphism $f_x\colon \mathcal{E}[p]_S \to \mathbf{G}_{m,S}$. Then, the $\mathrm{dlog}(x)$ is the invariant differential on $\mathcal{E}_1$ given by the inverse image via f_x of the standard invariant differential $Z^{-1}dZ$ on $\mathbf{G}_{m,S}$. In general, a similar construction provides for every finite and locally free group scheme G over a base S a map $\mathrm{dlog}\colon G^\vee \to \omega_{G/S}$ where $\omega_{G/S}$ is the sheaf of invariant differentials of G.

Proposition 6.2. *Over $Z(w)$ we have an exact sequence*

$$0 \longrightarrow C \longrightarrow \mathcal{E}[p] \xrightarrow{\mathrm{dlog}} \omega_{\mathcal{E}_1},$$

where C is the canonical subgroup of $\mathcal{E}[p]^\vee$. Moreover,

1.) *H admits a unique $p-1$-root $H^{\frac{1}{p-1}}$ on $Z(w)_1$ which extends at the cusps and has q-expansion 1. If $p \geq 5$ then $H^{\frac{1}{p-1}}$ lifts uniquely to a $(p-1)$-th root $E_{p-1}^{\frac{1}{p-1}}$ of E_{p-1} on $Z(w)$, i.e., a weight 1 modular form whose $(p-1)$-th power is E_{p-1};*

2.) *$\mathcal{E}[p]/C$ admits a canonical $Z(w)$-section, which we denote by γ and which is a generator over $Z(w)_{\mathbb{Q}_p}$. Modulo p^{1-w} the image of γ via* dlog *is the weight 1 modular form $H^{\frac{1}{p-1}}$. There is $\alpha \in \mathcal{O}_{Z(w)_1}$ such that $\alpha \cdot H^{\frac{1}{p-1}} = p^{\frac{w}{p-1}}$.*

Proof. The first statement follows from 6.1. We simply write Z for $Z(w)$.

(1) Choose a basis e_R of $\mathrm{R}^1\pi_*\mathcal{O}_\mathcal{E}$ on an open formal sub-scheme $\mathcal{U} = \mathrm{Spf}(R) \subset Z$ and denote by $A \in R$ an element such that $F(e_R) = Ae_R$ modulo p. For $p \geq 5$ we take A so that $E_{p-1}(\mathcal{E}, e_R^\vee) = Ae_R^{\vee,p-1}$. Here we identify the element $\Omega_R := e_R^\vee$ with a generator of the module of invariant differentials $\omega_\mathcal{U}$ via the isomorphism $\big(\mathrm{R}^1\pi_*\mathcal{O}_\mathcal{E}\big)^\vee \cong \omega_\mathcal{U}$ given by Serre's duality. We may then write $\varphi - p^w \colon \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_{1-w}} \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1}$ on $\mathcal{U}$ as the map $S_1 \to S_1$ sending $X \to HX^p - p^w X$. For every ring extension $R \subset R'$ we let $\mathbf{Z}_{R'}(A)$ be the set of solutions of the equation $X \to AX^p - p^w X$ in R'. We let $\mathbf{Z}_{R',1}(A)$ and $\mathbf{Z}_{R',1-w}(A)$ be the solutions in R'_1 and R'_{1-w}. Write $\mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big)$ for the image of $\mathbf{Z}_{R',1}(A)$ in $\mathbf{Z}_{R',1-w}(A)$. It coincides with the set of solutions of $R'_{1-w} \ni X \mapsto AX^p - p^w X \in R'_1$. Then, [AGa, Lemma 9.5] asserts that if R' is normal, noetherian, p-torsion free and p-adically complete and separated the natural map

$$\mathbf{Z}_{R'}(A) \longrightarrow \mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big)$$

is a bijection. By [AGa, Thm. 8.1 & Def. 12.4] we have an isomorphism

$$\mathcal{E}[p]/C(R'_K) \cong C^\vee(R'_K) \cong \mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big).$$

Thus, the set $\mathbf{Z}_{R'}(A)$ is an $\mathbf{F}_p$-vectors space of dimension ≤ 1 and it is of dimension 1 if and only if $C^\vee(R'_K)$ is a constant group scheme over R'_K.

Since by construction the canonical subgroup exists and has a generator c over R_K and since by assumption $\mu_{p,K} \cong \mathbb{Z}/p\mathbb{Z}$, then also $C^\vee(R'_K)$ admits a canonical generator $c^\vee$. Then, $\mathbf{Z}_R(A)$ has dimension 1 as an $\mathbf{F}_p$-vector space and the image of $c^\vee$ defines a basis element. This image is of the form $p^{\frac{w}{p-1}}\delta^{-1}$ where δ is a given $(p-1)$–root of A in R. This already implies that H has a $(p-1)$-root in R_1 defined by δ and it also implies the last claim in (2).

Assume that $\mathcal{U}$ is contained in the ordinary locus of $Z(w)$. Then, the canonical subgroup is canonically isomorphic to μ_p and we can take the invariant differential of μ_p as a generator of $\omega_\mathcal{U}$ modulo p and, hence, of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1}$. With respect to this basis H is 1 and, hence, $\delta = 1$; see [AGa, Prop. 3.4]. This construction applied to the Tate curve gives the claim on the q-expansion.

Assume that $p \geq 5$. Two different local trivializations of $\omega_\mathcal{E}$ on $\mathcal{U}$ differ by a unit u. Thus, we get two different functions A and B with $B = u^{p-1}A$. In particular, multiplication by u defines a bijection from $\mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big)$ to $\mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(B)\big)$ and the root $p^{\frac{w}{p-1}}\delta_A^{-1}$ is sent to $p^{\frac{w}{p-1}}\delta_B^{-1} \cdot u$. This implies that over $Z(w)$ the modular form $p^w E_{p-1}^{-1}$, and hence also the modular form E_{p-1}, admits a globally defined $(p-1)$-root as claimed.

(2) Choose $\mathcal{U}$, e_R and δ as in the proof of (1). Let G_δ be the group scheme introduced at the beginning of this section. The canonical subgroup of $\mathcal{E}[p]$ is isomorphic to the subgroup scheme $G_{(-p\delta^{1-p},-\delta^{p-1})}$ by [C2, Thm. 2.1]. It is denoted by B_{-u} with $u = p\delta^{1-p}$ in loc. cit. using the relation between Coleman's approach and the Oort-Tate description given in the proof of [C2, Prop. 1.1]. Such group scheme is isomorphic to $G_{(a,c)}$ modulo p with a and c as at the beginning of the section. We remark that in this case $a = p^{1-w}(p^w\delta^{(1-p)})$ up to a unit so that $a \equiv 0$ modulo p^{1-w}. By loc. cit. the immersion $h\colon G_{(a,c)} \subset \mathcal{E}$ has the property that $h^*(\Omega_R) = (1 - \delta^{p-1}Y^{p-1})dY$ modulo p. The latter is equivalent to $(1 - \delta^{p-1}Y^{p-1})dY$ and hence to $(1 + \delta T)^{-1}dT$ modulo p.

By the first claim of the proposition and identifying $\mathcal{E}[p]$ with $\mathcal{E}[p]^\vee$ via the principal polarization on $\mathcal{E}$, the map dlog factors via the map $\mathcal{E}[p]^\vee \to G_\delta^\vee$ and by functoriality of dlog it is compatible with the map dlog for G_δ. Moreover, the morphism $\eta_\delta\colon G_\delta \to \mu_p$ introduced at the beginning of this section defines a canonical section of $\mathcal{E}[p]^\vee/C \cong G_\delta^\vee$ which is a generator over $\mathbf{Q}_p$. This defines the section γ of $\mathcal{E}[p]/C$ claimed in (2). Denote by $\mathcal{U}_{1-w}$ and $\mathcal{E}_{1-w}$ the reduction of $\mathcal{U}$ and $\mathcal{E}$ modulo $p^{1-w}\mathcal{O}_K$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}[p]^\vee & \stackrel{\text{dlog}}{\longrightarrow} & \omega_{\mathcal{E}_{1-w}/\mathcal{U}_{1-w}} \\ \downarrow & & \downarrow \\ G_\delta^\vee & \stackrel{\text{dlog}}{\longrightarrow} & \omega_{G_\delta/\mathcal{U}_{1-w}}. \end{array}$$

Since $G_\delta \subset \mathcal{E}[p]$ is a closed immersion, the natural map $\omega_{\mathcal{E}_{1-w}/\mathcal{U}_{1-w}} \to \omega_{G_\delta/\mathcal{U}_{1-w}}$ is surjective. Since $\omega_{G_\delta/\mathcal{U}_{1-w}}$ is free as $\mathcal{O}_{\mathcal{U}_{1-w}}$-module, such map is an isomorphism. The map dlog: $G_\delta^\vee \to \omega_{G_\delta/\mathcal{U}_{1-w}}$ sends η_δ to the pull–back of the invariant differential of μ_p in $\omega_{G_\delta/\mathcal{U}_1}$ which is $\delta(1+\delta T)^{-1}dT$, i.e., $\delta\Omega_R$ via the isomorphism $\omega_{\mathcal{E}_{1-w}/\mathcal{U}_{1-w}} \cong \omega_{G_\delta/\mathcal{U}_{1-w}}$. Hence, $\text{dlog}(\eta_\delta)$ is equal to $H^{\frac{1}{p-1}}$ modulo p^{1-w} concluding the proof of (2). □

In [C3, Lemma 9.2] and for $p \geq 5$ Coleman introduces a weight $(1,-1) \in \mathcal{W}^*(\mathbb{Q}_p)$ overconvergent modular form D_p of level $\Gamma_1(Np)$ whose $(p-1)$-power is E_{p-1} and, from the proof of the Lemma, it has q-expansion 1 modulo p. These two properties characterize such a modular form. We deduce:

Corollary 6.3. *The overconvergent modular form defined by* $E_{p-1}^{\frac{1}{p-1}}$ *over* $Z(w)_{\mathbb{Q}_p}$ *is the weight* $(1,0)$ *overconvergent modular form* D_p *of level* $\Gamma_1(Np)$ *introduced by Coleman.*

In particular, our approach can be seen as a refinement of [C1, Lemma 9.2] providing a formal model for D_p.

7 Appendix B: V. Pilloni's overconvergent modular forms

In this appendix we use the notations of chapter 5. As mentioned in the introduction, in this section we compare the constructions of elliptic overconvergent modular forms in [P] and the one in this article and prove that they produce the same objects.

Let us briefly present the construction in [P]. We fix $k \in \mathcal{W}(K)$ a weight; then there is a $w > 0$ and a rigid analytic space $T^\times$ (depending on w), with a morphism $\beta : T^\times \longrightarrow X(w)$ which is a Galois cover with Galois group $\mathbb{Z}_p^\times$. Then the overconvergent modular sheaf ω^k on $X(w)$ is defined to be the sub-sheaf of $\beta_*\big(\mathcal{O}_{T^\times}\big)$ on which the Galois group $\mathbb{Z}_p^\times$ acts via the character $-k$ and the overconvergent modular forms of weight k are global sections of this sheaf.

We'll now be more precise. We choose a p-th root of 1 in K and assume for simplicity that the weight $k = (s,j) \in \mathcal{W}^*(K)$ is accessible. Let $\mathcal{T}^\times$ denote the formal scheme whose points over a formal scheme $S \longrightarrow \text{Spf}(\mathcal{O}_K)$ are isomorphism classes of triples (x,γ,ω) where:

• $x \in \mathfrak{X}(w)(S)$ corresponding to a pair $(\mathcal{E}_x/S, \psi_x)$ consisting of an elliptic curve $\mathcal{E}_x \longrightarrow S$ and a $\Gamma_1(N)$-level structre on it

• γ denotes a generator of the dual of the canonical subgroup $C_x^\vee$ of $\mathcal{E}_x$ corresponding to the p-the root of 1 chosen

and

• ω is an invariant 1-differential form on $\mathcal{E}_x$ over S such that $\omega(\bmod p^{1-v}) = \text{dlog}(\gamma)$.

We have natural morphisms $\mathcal{T}^\times \xrightarrow{\alpha} \mathfrak{X}(p)(w) \xrightarrow{\vartheta} \mathfrak{X}(w)$ and let $\beta := \vartheta \circ \alpha$. We denote as usual by $T^\times$, $X(p)(w)$ and $X(w)$ the rigid analytic generic fibers of these formal schemes and by α, ϑ, β the restrictions of the morphisms with the same names to the generic fibers. Then $\beta : T^\times \longrightarrow X(w)$ is Galois with Galois group $\mathbb{Z}_p^\times$, where the action is defined by: if $a \in \mathbb{Z}_p^\times$, $(x, \gamma, \omega) \in T^\times$, we set $a * (x, \gamma, \omega) := (x, a\gamma, a\omega)$. We define

$$\omega^\kappa := \beta_*\big(\mathcal{O}_{T^\times}\big)^{(-\kappa)},$$

where the exponent $(-\kappa)$ indicates the sub-sheaf of sections on which $\mathbb{Z}_p^\times$ acts via $-\kappa$, more precisely if s is a section of $\mathcal{O}_{\mathcal{T}^\times}$, $a \in \mathbb{Z}_p^\times$ and (x, γ, ω), s is a section of ω^κ if and only if $s(x, a\gamma, a\omega) = \kappa^{-1}(a)s(x, \gamma, \omega)$.

We define $M^P(N, k, w) := \mathrm{H}^0\big(X(w), \omega^k\big)$, and the main purpose of the appendix is to prove

Lemma 7.1. *There is a natural isomorphism, compatible with the actions of the Hecke operators:*

$$\mathcal{M}(N, \kappa, w)_K \cong M^P(N, k, w),$$

where the first module was studied in section §5 of this article.

Proof. We use the notations of section §5. Let us first remark that by the very definition of $T^\times$, the torsor sheaf $\alpha^*(\mathcal{F}')$ has a canonical trivialisation, i.e. $\alpha^*(\mathcal{F}') = S_{v,T^\times} \cdot \omega$, where $S_{v,T^\times} = \mathbb{Z}_p^*(1 + p^{1-v}\mathcal{O}_{T^\times})$ and ω is the universal differential form. Therefore (again using the notations of section §5) we have

$$\mathfrak{Hom}_{S_{v,T^\times}}\big(\alpha^*(\mathcal{F}'), \mathcal{O}_{T^\times}^{(-k)}\big) \cong \mathcal{O}_{T^\times}^{(-k)} = \alpha^*(\Omega_w^s).$$

It follows that we have, on the one hand

$$\beta_*\big(\mathfrak{Hom}_{S_{v,T^\times}}(\alpha^*(\mathcal{F}'), \mathcal{O}_{T^\times}^{(-k)}\big)^{\mathbb{Z}_p^\times} = \beta_*\big(\mathcal{O}_{T^\times}\big)^{(-k)} = \omega^k.$$

On the other hand we have

$$\beta_*\big(\mathfrak{Hom}_{S_{v,T^\times}}(\alpha^*(\mathcal{F}'), \mathcal{O}_{T^\times}^{(-k)})\big)^{\mathbb{Z}_p^\times} = \beta_*\big(\alpha^*(\Omega_w^s)\big)^{\mathbb{Z}_p^\times} = \vartheta_*\Big(\alpha_*\big(\alpha^*(\Omega_w^s)\big)^{1+p\mathbb{Z}_p}\Big)^G = \vartheta_*(\Omega_w^s)^G = \Omega_w^\kappa.$$

This gives the desired isomorphism and we leave it to the reader to prove that this isomorphism commutes with the action of Hecke operators.

□

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