ISM
Final Examination – MAST 699M
Algebraic Geometry

Time: Return the exam no later than Thursday, December 9, 5:00 PM
Directions: May use any documentation but do not discuss the exam with anybody.
Your name and ID:

Instructions: *There are 6 problems in this test. Solve any 5 of them. Write the proofs carefully. If you use other exercises in the book give solutions for those exercises also.*

1. Let X, Y be varieties, $\varphi : X \longrightarrow Y$ a function. Suppose that $X = \cup_{i=1}^{n} U_i$, $Y = \cup_{i=1}^{n} V_i$, where U_i, V_i are open sub-varieties of X respectively Y for all $1 \leq i \leq n$. Suppose that $\varphi(U_i) \subset V_i$ for all i.

a) Show: φ is a morphism if and only if $\varphi|_{U_i} : U_i \longrightarrow V_i$ is a morphism for all $1 \leq i \leq n$.

b) Show that if each U_i, V_i is an affine variety then φ is a morphism if and only if $\widetilde{\varphi}(\Gamma(V_i)) \subset \Gamma(U_i)$, for all $1 \leq i \leq n$.

2. a) Let $V_1, V_2 \subset \mathbb{P}^3$ be the given by: $V_1 = V(XY - ZW)$ and $V_2 = V(X^2 - YW)$. Show that V_1 and V_2 are projective varieties but that $V_1 \cap V_2$ is not a variety.

b) Let $C = V(X^2 - YZ)$ and $L = V(Y)$ be algebraic sets in $\mathbb{P}^2$. Show that $C \cap L$ is a projective variety but that $I(C \cap L) \neq I(C) + I(L)$ as ideals in $k[X, Y, Z]$.

3. Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ be the map: $\varphi([x, y]) = [x^2, xy, y^2]$.

a) Show that φ is a well defined and it is a morphism.

b) Show that $\varphi(\mathbb{P}^1) = V(XZ - Y^2) \subset \mathbb{P}^2$.

c) Show that φ defines an isomorphism between $\mathbb{P}^1$ and $V(XZ - Y^2)$.

d) Show that $\widetilde{\varphi}$ does not define an isomorphism between $\Gamma_h(\mathbb{P}^1)$ and $\Gamma_h(V(XZ - Y^2))$.‘

4 Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ be the map $\varphi([x, y]) = [x^3, x^2y, xy^2, y^3]$.

a) Show that φ is well defined and it is a morphism.

b) Show that $V := \varphi(\mathbb{P}^1)$ is a projective sub-variety of $\mathbb{P}^3$. Find $I(V)$.

c) Let $P = [0, 0, 1, 0] \in \mathbb{P}^3$ and define $\psi : V \longrightarrow \mathbb{P}^2$ by

$\psi(Q)$ =the intersection of the projective line containing P, Q with $V(Z) = \mathbb{P}^2$.

i) If $Q = [a, b, c, d]$ find $\psi(Q)$ as function of a, b, c, d.

ii) Show that $\psi : V \longrightarrow \mathbb{P}^2$ is a morphism of projective varieties and that $\psi(V)$ is a projective sub-variety of $\mathbb{P}^2$.

Note A line in $\mathbb{P}^3$ is a projective variety $L = V(F, G)$, where $F, G \in k[X, Y, Z, W]$ are homogeneous of degree one such that $F \neq aG$ for any $a \in k$.

5 a) Find the intersection points of the following curves: $F = (X^2 + Y^2)Z + X^3 + Y^3$ and $G = X^3 + Y^3 - 2XYZ$.

b) Find the intersection numbers of the intersection points at a)

6 Prove that an irreducible cubic curve, i.e. a curve $C = V(F) \subset \mathbb{P}^2$ where $F \in k[X, Y, Z]$ is a homogeneous, irreducible polynomial of degree 3, is either non-singular or has at most one double point.