Homework 3; due Friday 11/06/10

Local Fields

June 4, 2010

All rings in the problem set are commutative with identity.

1) Let A be a ring. An element $x \in A$ is called nilpotent if $x^n = 0$ for some $n \geq 1$. Let $\mathfrak{N}_A$ be the set of all nilpotent elements of A

i) Show that $\mathfrak{N}_A$ is an ideal of A.

ii) Show that $\mathfrak{N}_A$ is the intersection of all the prime ideals of A

iii) If $x \in \mathfrak{N}_A$ then $1 + x$ is a unit in A, i.e. $1 + x \in A^\times$.

2) Let A be a ring and $B := A[X]$ the polynomial ring in one variable with coefficents in A. Let $f = a_0 + a_1 X + ... + a_n X^n \in A[X]$. Show that:

i) f is a unit in $A[X]$ if and only if $a_0 \in A^\times$ and $a_1, a_2, ..., a_n \in \mathfrak{N}_A$.

ii) f is nilpotent if and only if $a_0, a_1, ..., a_n \in \mathfrak{N}_A$.

iii) f is a zero divisor in $A[X]$ if and only if there is $a \in A$, $a \neq 0$ such that $af = 0$.

iv) Let $J_{A[X]}$ be the intersection of all the maximal ideals of $A[X]$. Show that $J_{A[X]} = \mathfrak{N}_{A[X]}$.

3) Let A be a ring and $A[[X]]$ the ring of power series with coefficeints in A. Let $f = \sum_{n=0}^{\infty} a_n X^n \in A[[X]]$.

i) f is a unit in $A[[X]]$ if and only if $a_0 \in A^\times$.

ii) If f is nilpotent then $a_n \in \mathfrak{N}_A$ for all $n \geq 0$. Show that the converse if true if A is Noetherian.

iii) $f \in J_{A[[X]]}$ if and only if $a_0 \in J_A$.

iv) If $\mathfrak{m}$ is a maximal ideal of $A[[X]]$, then $\mathfrak{n} := \mathfrak{m} \cap A$ is a maximal ideal of A and $\mathfrak{m} = \big(\mathfrak{n}, X\big)$.

v) If $\mathfrak{p}$ is a prime ideal of A, then there is a prime ideal $\mathfrak{P}$ of $A[[X]]$ such that $\mathfrak{p} = \mathfrak{P} \cap A$.

4) Let A be a ring and let $X := \mathrm{Spec}(A)$ be the set of prime ideals of A with the Zariski topology, i.e. a subset Y of X is closed if Y is the set $V(\mathfrak{a})$ of all prime ideals of A containing a certain ideal $\mathfrak{a}$ of A. If $f \in A$ denote by X_f denote the complement of $V\big(fA\big)$ in X.

i) Show that the family of sets $\{X_f\}_{f \in A}$ is a basis of open sets for the Zariski topology on X.

ii) $X_f \cap X_g = X_{fg}$.

iii) $X_f = \phi$ if and only if $f \in A^\times$.

iv) X is quasi-compact, i.e. every covering by open sets of X has a finite sub-covering. In fact every X_f is quasi-compact.

v) Let $x \in X$ be a point. Show that the set $\{x\}$ is closed in X if and only if x is a maximal ideal.