Homework 1; due Wednesday 13 May

Number Theory 2

May 4, 2009

In this homework let K be a perfect field and $\overline{K}$ a fixed algebraic closure of K. We let $G_K := \mathrm{Gal}(\overline{K}/K)$ with the topology defined in class.

1) For every finite, Galois extension L of K contained in $\overline{K}$ let $G(L/K) := \mathrm{Gal}(L/K)$ and let $\rho_{L/K} : G_K \longrightarrow G(L/K)$ be the map: $\rho_{L/K}(\sigma) := \sigma|_L$. Show that $\rho_{L/K}$ is a continuous, surjective group homomorphism. Here the topology on $G(L/K)$ is the discrete one.

2) Denote by I the set of finite, Galois extensions of K contained in $\overline{K}$, and think of I as an ordered set with order relation given by the inclusion of fields.

a) Show that $(I, \subset)$ is a directed set, i.e. for every $L_1, L_2 \in I$ there is $L_3 \in I$ such that $L_i \subset L_3$ for $i = 1, 2$.

b) Let $L_1, L_2 \in I$ be such that $L_1 \subset L_2$. Define $\rho_{2,1} : G(L_2/K) \longrightarrow G(L_1/K)$ by $\rho_{2,1}(\sigma) := \sigma|_{L_1}$. Show that $\big(G(L/K), \rho_{2,1}\big)_{L\in I}$ is a projective system of finite groups.

c) Show that the family of maps $(\rho_{L/K})_{L\in I}$ define a continuous isomorphism of topological groups

$$\rho : G_K \longrightarrow \lim_{\leftarrow, L} G(L/K).$$

3) Show that $\lim\limits_{\leftarrow, L} G(L/K)$ is a closed subgroup of $\prod_{L\in I} G(L/K)$ to deduce that G_K is a compact group.

4) Let M be an extension of K contained in $\overline{K}$.

a) Show that $G_M := \mathrm{Gal}(\overline{K}/M)$ is a closed subgroup of G_K, hence compact.

b) Show that G_M is an open subgroup of G_K if and only if M/K is finite.

c) Show that $M = \overline{K}^{G_M} := \{x \in \overline{K} \quad | \quad \sigma(x) = x \text{ for every } \sigma \in G_M\}$.

5) Let $H \subset G_K$ and let $M := \overline{K}^H := \{x \in \overline{K} \quad | \quad \sigma(x) = x \text{ for every } \sigma \in H\}$.

i) Show that M is a field.

ii) Show that $G_M = H$.

iii) Suppose that H is a closed norma subgroup of G_K. Show that $M := \overline{K}^H$ is a Galois extension of K and $\mathrm{Gal}(M/K) \cong G_K/G_M$, as topological groups.

6) What happens if we do not assume K perfect?