Homework 2; due Monday 24 May

Local Fields

February 11, 2000

1) (Krasner's lemma) Let (K,v) be a complete valued field and E a finite Galois extension of K. This means that $v: K^\times \longrightarrow \mathbb{Q}$ is the valuation of K and let $w: E^\times \longrightarrow \mathbb{Q}$ be the valuation of E such that $w|_{K^\times} = v$.

Let $x \in E$ and denote $x_1 = x, x_2, ..., x_n$ the conjugates of x over K. Let $y \in E$ be such that $w(y-x) > w(y-x_i)$ for all $i = 2, ..., n$. Show that $x \in K(y)$.

2) Let (K,v) be a complete valued field and $f(X) \in K[X]$ an irreducible and separable polynomial of degree n and let L be the degree n extension of K defined by $f(X)$. If $h(X) \in K[X]$ is a polynomial of degree n such that $v(f(X)-h(X))$ is "large enough" then $h(X)$ is irreducible and the extension of K defined by $h(X)$ is isomorphic (over K) with L.

Here if $g(X) = a_n X^n + ... + a_0 \in K[X], a_n \neq 0$ we define $v(g(X)) := \min\{v(a_i) \mid 0 \leq i \leq n\}$.

3) Let now $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ be its completion with respect to the unique valuation v on $\overline{\mathbb{Q}}_p$. Then v extends uniquely to a valuation on $\mathbb{C}_p$.

a) Let $\mathbb{Q}_p \subset K \subset \overline{\mathbb{Q}}_p$ be a subfield with the valuation $v|_{K^\times}$. Let $\widehat{K}$ be the completion of K with respect to this valuation.

i) Show that $\widehat{K}$ is a subfield of $\mathbb{C}_p$.

ii) Use Krasner's lemma above to show that $\widehat{K} \cap \overline{\mathbb{Q}}_p = K$.

b) Consider the following:

Theorem 0.1 (J. Ax)**.** *Let $\mathbb{Q}_p \subset K \subset \overline{\mathbb{Q}}_p$ be a subfield and $a \in \overline{\mathbb{Q}}_p$. Then there exists $\alpha \in K$ such that*

$$v(a-\alpha) + c_0 \geq \inf_{\sigma \in G_K} \{v(\sigma(a) - a)\},$$

where $c_0 = p/(p-1)$.

Let now $\mathbb{Q}_p \subset L \subset \mathbb{C}_p$ be a complete subfield and let $K = L \cap \overline{\mathbb{Q}}_p$. Use Ax's theorem above to prove that $L = \widehat{K}$ as follows:

i) Show that $\widehat{K} \subset L$.

ii) Let $z \in \mathcal{O}_L$ and $\epsilon > 0$ a (large) real number. Let $\alpha \in \overline{\mathbb{Q}}_p$ be an element such that $v(z-\alpha) > \epsilon$ (Why does such an element exist?) and let $\alpha_1 = \alpha, \alpha_2, ..., \alpha_n$ be the conjugates of α over K. Show that $\alpha_1 - z, \alpha_2 - z, ..., \alpha_n - z$ are the conjugates of $\alpha - z$ over L.

iii) Use Ax's theorem above to show that there is $x_\epsilon \in K$ such that $v(z - x_\epsilon) \geq \epsilon - c_0$. Deduce that $z \in \widehat{K}$ and that $L = \widehat{K}$.

4) Let us use the notations of Problem 3) above and consider $\mathbb{Q}_p \subset K \subset \overline{\mathbb{Q}}_p$ a subfield and let $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$. Show that $\mathbb{C}_p^{G_K} = \widehat{K}$ as follows:

i) Let $L := \mathbb{C}_p^{G_K}$. Show that L is a complete subfield of $\mathbb{C}_p$ and that $\widehat{K} \subset L$.

ii) Show that $L \cap \overline{\mathbb{Q}}_p = \widehat{K} \cap \overline{\mathbb{Q}}_p = K$.

iii) Use Problem 3) above to conclude that $\widehat{K} = L$.

5) In the notations of Problem 3) let $\mathbb{Q}_p \subset K \subset L \subset \overline{\mathbb{Q}}_p$ be subfields such that $[L : \mathbb{Q}_p] < \infty$.

a) Let $T : L \longrightarrow \mathrm{Hom}_K(L, K)$ denote the map $T(x)(y) := \mathrm{Tr}_{L/K}(xy)$ for $x, y \in L$. Show that T is an isomorphism.

b) Let $\mathcal{D}_{L/K} \subset \mathcal{O}_L$ denote the different ideal of L over K. Show that the restriction of T defines an $\mathcal{O}_K$-linear map:

$$T : \mathcal{D}_{L/K}^{-1} \longrightarrow \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K),$$

which is an isomorphism.

Suppose that we know the statement:

Let $R \subset S$ be commutative rings such that S is a finite projective R-module. Then, if M, N are R-modules we have a canonical isomorphism of S-modules

$$\mathrm{Hom}_R(M, N) \otimes_R S \cong \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S).$$

c) i) Use the statement above to deduce that $T \otimes_K \mathrm{Id}_L : L \otimes_K L \longrightarrow \mathrm{Hom}_L(L \otimes_K L, L)$ is an L-linear isomorphism.

ii) Let $m : L \otimes_K L \longrightarrow L$ be the map $m(x \otimes y) = xy$ and let $e \in L \otimes_K L$ be the element such that $(T \otimes_K \mathrm{Id}_L)(e) = m$. Show that $e^2 = e$ (e is an idempotent) and is the unique element of $L \otimes_K L$ satisfying: if we write $e = \sum_{i=1}^n a_i \otimes b_i$, $a_i, b_i \in L$, then for all $x, y \in L$

$$(*) \quad xy = \sum_{i=1}^n y b_i \mathrm{Tr}_{L/K}(x a_i).$$

iii) Show that the exact annihilator in $\mathcal{O}_L$ of the $\mathcal{O}_L$-module

$$\mathrm{Coker}\big((T \otimes_{\mathcal{O}_K} \mathrm{Id}_{\mathcal{O}_L}) : \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \longrightarrow \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L, \mathcal{O}_L)\big)$$

is exactly $\mathcal{D}_{L/K}$.

d) Deduce from 5) c) that if α is a generator of the ideal $\mathcal{D}_{L/K}$ then $(1 \otimes \alpha)e \in \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L$.

e) If we write $(1 \otimes \alpha)e = \sum_{i=1}^n a_i \otimes b_i$, with $a_i, b_i \in \mathcal{O}_L$ then we have $\alpha = \sum_{i=1}^n b_i \mathrm{Tr}_{L/K}(a_i)$.

f) Apply $\mathrm{N}_{L/K}$ to 5) f) and deduce that there is an element $\beta \in \mathcal{O}_K$ such that $\beta \in T(\mathcal{O}_L)$ and $v(\beta) = v(\alpha)[L : K]$.