Number Theory 2, Homework 1

April 26, 2010

The homework is due Monday, May 3, 2010 in class. You may **NOT** consult with your classmates while working on this assignement.

1) Consider $x, y, z \in \mathbb{Q}_p$. Show that if x, y, z are not colinear then the triangle with verices x, y, z is isoceles.

2) Suppose $p > 2$ is a prime integer. Show that the group $\mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$ is finite and has 4 elements.

3) Let (K, v) be a complete discrete valued field and $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \in K$ for all $n \in \mathbb{N}$.

a) Show that $\{a_n\}_n$ is a Cauchy sequence if and only if $\lim_{n\to\infty} v(a_{n+1} - a_n) = \infty$.

b) Show that the series $\sum_{n=0}^{\infty} a_n$ is convergent in K if and only if $\lim_{n\to\infty} v(a_n) = \infty$.

c) Decide if the following sequences converge in $\mathbb{Q}_p$ and if they do find their limit: i) $a_n = n!$, ii) $a_n = n$, iii) $a_n = p^n$, iv) $a_n = (1+p)^{p^n}$.

d) Define the norm $|\ |_p : \mathbb{Q}_p \longrightarrow \mathbb{R}_>$ by the formula: $|x|_p := p^{-v_p(x)}$. Show that if $\{a_n\}_n$ is a sequence of elements in $\mathbb{Q}_p$ such that the series $\sum_{n=0}^{\infty} |a_n|_p$

converges in $\mathbb{R}$, then the series $\sum_{n=0}^{\infty} a_n$ converges in $\mathbb{Q}_p$.

4) Let $f(X) = \sum_{n=0}^{\infty} a_n X^n = a_0 + a_1 X + a_2 X^2 + ... \in \mathbb{Q}_p[[X]]$ such that: $f(X) \neq 0$ and $\lim_{n\to\infty} a_n = 0$.

a) Show that for all $\alpha \in \mathbb{Z}_p$, the series $f(\alpha)$ converges in $\mathbb{Q}_p$.

b) Show that there is an $N \in \mathbb{N}$ such that: $v(a_N) = \inf_{n\in\mathbb{N}} v(a_n)$ and $v(a_n) > v(a_N)$ for all $n > N$.

c) Suppose that N found at b) above is 0. Show that for all $\alpha \in \mathbb{Z}_p$, $f(\alpha) \neq 0$.

d) Suppose this time that N found at b) is 1 and that $\alpha \in \mathbb{Z}_p$ is such that $f(\alpha) = 0$.

i) Show that there is a series $g(X) = b_0 + b_1 X + b_2 X^2 + ... \in \mathbb{Q}_p[[X]]$ such that: $f(X) = (X - \alpha)g(X)$ and $\lim_{n\to\infty} b_n = 0$.

ii) Calculate N of b) above for $g(X)$ and deduce that $f(X)$ has at most one zero in Z_p.

e) In general, show that $f(X)$ as in the statement has at most N zeroes in Z_p.

f) Let $f(X), g(X) \in \mathbb{Q}_p[[X]]$ be two power series which converge on $\mathbb{Z}_p$. Show that if $f(\alpha) = g(\alpha)$ for infinitely many elements $\alpha \in \mathbb{Z}_p$ then $f(X) = g(X)$.

5) Use 2-adic analysis in $\mathbb{Q}_2$ to show that for all $M > 0$, $M \in \mathbb{Z}$ there is $n \in \mathbb{N}$ such that 2^M divides $2 + 2^2/2 + 2^3/3 + 2^4/4 + ... + 2^n/n$.

6) Use exercise 4) to count the number of zeroes of the series $\log(1 - X) = \sum_{n=1}^{\infty} X^n/n$ in $\mathbb{Q}_p$.

7) Let $p > 2$ be an integer.

i) Show that if $x \in 1 + p\mathbb{Z}_p$ is such that $x^p = 1$ then $x = 1$.

ii) Show that the only p-th root of 1 in $\mathbb{Q}_p$ is 1.