On overconvergent modular forms

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1 Introduction

Let $p \geq 5$ be a prime integer, K a finite extension of $\mathbb{Q}_p$ and let N be a positive integer. We fix once for all an algebraic closure $\overline{K}$ of K and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{K}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We denote by $\mathbb{C}_p$ the completion of $\overline{K}$.
We suppose that N is divisible by $p^f - 1$, where p^f is the number of elements of the residue field of K and that a primitive Np-th root of 1 in $\overline{K}$ is contained in K. We denote by v the p-adic valuation of $\mathbb{C}_p$ normalized such that $v(p) = 1$.
We consider the modular curve $X := X_1(Np)$ over K classifying generalized elliptic curves with $\Gamma_1(Np)$-level structure, we let $\mathcal{E} \longrightarrow X$ denote the universal semiabelian scheme over $X_1(Np)$ and $e : X \longrightarrow \mathcal{E}$ its identity section. We let $\omega_{\mathcal{E}/X} := e^*(\Omega^1_{\mathcal{E}/X})$; it is a line bundle over X and it can be identified with the sheaf of invariant sections of $\Omega^1_{\mathcal{E}/X}$.

Let $\mathcal{W}$ denote the weight space for $\mathbf{GL}_{2/\mathbb{Q}}$, i.e. the rigid analytic space over $\mathbb{Q}_p$ whose K-points are $\mathcal{W}(K) := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, K^\times)$. We embed $\mathbb{Z}$ in $\mathcal{W}(\mathbb{Q}_p)$ by sending $k \in \mathbb{Z}$ to the character which maps $a \in \mathbb{Z}_p^\times$ to a^k. If $U \subset \mathcal{W}$ is an affinoid sub-domain we denote by $A(U)$ its affinoid algebra.

H. Hida in [H] and R. Coleman in [C1] proved the following magnificent theorem. Let f be a modular eigenform of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight $k_0 + 2 \geq 2$. We suppose that all Hecke eigenvalues of f are in K, in particular we denote by a_p the U_p-eigenvalue.

Theorem 1.1 ([H],[C1]). *In the notations above suppose that $a_p^2 \neq p^{k_0+1}$ and $v(a_p) < k_0 + 1$. Then there exists an affinoid sub-domain $U \subset \mathcal{W}$ defined over $\mathbb{Q}_p$, containing k_0 and a power series*

$$\mathbb{F} := \sum_{n=0}^{\infty} A_n q^n \in A(U)[[q]],$$

such that:

- *for every $k \in U(K) \cap \mathbb{Z}$, $k > k_0$, $\mathbb{F}(k) = \sum_{n=0}^{\infty} A_n(k) q^n \in K[[q]]$ is the q-expansion of a (classical) modular eigenform of level $\Gamma_1(N)$ and weight $k+2$,*

  and
- *$\mathbb{F}(k_0)$ is the q-expansion of f.*

The power series $\mathbb{F}$ above is called a p-adic eigenfamily of modular forms deforming f. Let us remark that for every $\kappa \in U(K)$ we have a specialization $\mathbb{F}(\kappa) = \sum_{n=0}^{\infty} A_n(\kappa) q^n$ and these can be thought of as constituents of the p-adic family $\mathbb{F}$. It is shown in [C1] that these power series have remarkable properties in particular they are q-expansions of so called "overconvergent modular eigenforms of weight $\kappa + 2$". The definition of these objects in [C1] depends if κ is an integer or not, as follows.

a) Suppose $\kappa \in \mathbb{Z} \cap U(K)$.

Let $w \in \mathbb{Q}$ be such that $0 \leq w < p/(p+1)$ and let us suppose that there is an element in K whose valuation is w. Such an element will be denoted p^w. Let us recall that the normalized Eisenstein series E_{p-1} of level 1 and weight $p-1$ is a lift to characteristic 0 of the Hasse invariant and can be seen as a modular form on $X_1(N)$. We denote $X_1(N)(w) := \{x \in X_1(N) \quad | \text{ such that } |E_{p-1}(x)| \geq p^{-w}\}$. We have the following tower of modular curves: $X_1(Np) \longrightarrow X(N,p) \longrightarrow X_1(N)$, where $X(N,p)$ classifies generalized elliptic curves with $\Gamma_0(p) \cap \Gamma_1(N)$-level structure. Let us first remark that the natural (forgetful) map $X(N,p) \longrightarrow X_1(N)$ has a canonical section over $X_1(N)(w)$ whose image is the connected component of $X(N,p)(w)$ containing the cusp ∞. This section is defined by sending a point $(\mathcal{E}, \psi_N) \in X_1(N)(w)$, where $\mathcal{E}$ is an elliptic curve and ψ_N is a $\Gamma_1(N)$-level structure, to the point $(\mathcal{E}, \psi_N, C) \in X(N,p)$. Here $C \subset \mathcal{E}[p]$ is the canonical subgroup. We define $X_1(Np)(w) := X_1(Np) \times_{X(N,p)} X_1(N)(w)$, which we see as an affinoid sub-domain of $X_1(Np)$.

We define an overconvergent modular form of weight k and level $\Gamma_1(Np)$ to be an element of

$$M^\dagger(\Gamma_1(Np), k)_K := \lim_{\rightarrow, w} H^0(X_1(Np)(w), \omega_{\mathcal{E}/X}^{\otimes(k)}).$$

These forms have q-expansions at the cusps in $X_1(Np)(w)$ and natural actions of the Hecke operators T_ℓ for $\ell \neq p$ and of U_p.

b) Suppose $\kappa \in U(K) - \mathbb{Z}$.

In this situation Coleman did not define $\omega^{\kappa}_{\mathcal{E}/X}$ or even $\big(\omega_{\mathcal{E}/X}|_{X_1(Np)(w)}\big)^{\kappa}$ if $w > 0$ and so the definition of orverconvergent modular forms of weight κ is not geometric and uses a trick. More precisely, for any $\kappa \neq 1$ in $\mathcal{W}(K)$ such that $\zeta^*(\kappa) \neq 0$ we define the series

$$E_\kappa(q) := 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n\geq 1} \Big(\sum_{d|n,(d,p)=1} \kappa(d) d^{-1} \Big) q^n.$$

Here

$$\zeta^*(\kappa) := \frac{1}{\kappa(c) - 1} \int_{\mathbb{Z}_p^\times} \kappa(a) a^{-1} d(E_{1,c})(a)$$

where $c \in \mathbb{Z}_p^\times$ is any element such that $\kappa(c) \neq 1$ and $E_{1,c}$ is the Bernoulli measure defined in [C1], section B1.

If $\kappa = 1$, we define the series ([C1], section B1)

$$E(q) := 1 + \frac{2}{L_p(0, \mathbf{1})} \sum_{n\geq 1} \Big(\sum_{d|n,(p,d)=1} \tau^{-1}(d) \Big) q^n,$$

where $\tau : \mathbb{Z}_p^\times \longrightarrow \mu_{p-1} \subset \mathbb{Q}_p^\times$ is reduction modulo p composed with the Teichmüler character.

It is proved in [C1] that the series $E_\kappa(q)$ are in fact q-expansions at the cusp ∞ of sections (denoted E_κ) of $H^0\Big(X_1(Np)(0), \big(\omega|_{X_1(Np)(0)}\big)^\kappa\Big)$ where $\big(\omega|_{X_1(Np)(0)}\big)^\kappa$ was defined in [K1] and is recalled in section §3 of this article. If $\kappa = k$ is an integer then E_k is in fact an overconvergent modular form of weight k as defined at a) above.

Let $f(q) \in K[[q]]$ be a power series and suppose that it extends to a section denoted f of $H^0\Big(X_1(Np)(0), \big(\omega|_{X_1(Np)(0)}\big)^\kappa\Big)$ (i.e. f is a p-adic modular form of weight κ). We say that f is overconvergent. with degree of overconvergence $w > 0$ if the function $f/E_\kappa \in H^0(X_1(Np)(0), \mathcal{O}_{X_1(Np)})$ extends to a section $H^0(X_1(Np)(w), \mathcal{O}_{X_1(Np)})$. The set of overconvergent modular forms of weight κ and degree of overconvergence w is denoted $M(\Gamma_1(Np), \kappa, w)_K$. It is proved in [C1] that these forms have q-expansions at all the cusps in $X_1(Np)(w)$ and actions of Hecke operators T_ℓ for $(\ell, pN) = 1$ and U_p.

Remark 1.2. A priori it is not clear that the definition above of overconvergent modular forms of non-integral weight makes any sense. However, the eigenforms defined by Coleman have Galois representations attached to them, and these have the prescribed properties in particular they fit into p-adic analytic families of Galois representations. This suggests that the definition is right.

Remark 1.3. As the definition of overconvergent elliptic modular forms of non-integral weight makes heavy use of the family of Eisenstein series $\{E_\kappa\}_{\kappa\in\mathcal{W}(K)}$ introduced above and as such families are sparse in general for other groups (for example the Hilbert modular Eisenstein series all have parallel weight) overconvergent modular forms of non-integral weight have not yet been defined for a general algebraic group.

The main goal of this article is to remedy the problem outlined in remark 1.3 as follows.

I) Let $\kappa \in \mathcal{W}(K)$. Then there is $w \in \mathbb{Q}$ depending on κ with $0 < w < p/(p+1)$ and a locally free sheaf $\Omega^\kappa_{w,K}$ on $X_1(Np)(w)$ such that if we denote

$$\mathcal{M}(\Gamma_1(Np), \kappa, w)_K := H^0(X_1(Np)(w), \Omega^\kappa_{w,K}),$$

then we have:

• There exist natural actions of the Hecke operators T_ℓ, U_p $((\ell, Np) = 1)$ on $\mathcal{M}(\Gamma_1(Np), \kappa, w)_K$ and the elements of $\mathcal{M}(\Gamma_1(Np), \kappa, w)_K$ have Fourier expansions at the cusps in $X_1(Np)(w)$.

and

• We have natural K-linear isomorphisms which are Hecke-equivariant

$$\mathcal{M}(\Gamma_1(Np), \kappa, w)_K \cong M^\dagger(\Gamma_1(Np), \kappa, w)_K.$$

Moreover the sheaves $\Omega^\kappa_{w,K}$ can be put into p-adic analytic families in the following sense. Let us identify $\mathcal{W}$ as rigid analytic space with $\amalg_{\epsilon \in \widehat{\mu}_{p-1}} D$, where D is the open unit disk in $\mathbb{C}_p$ centered at 0 and $\widehat{\mu}_{p-1}$ is the group of characters of the group $\mu_{p-1}(C_p)$. If $r > 0$ we denote by $\mathcal{W}_r := \{z \in \mathcal{W}(\mathbb{C}_p) \quad | \quad |z| \leq p^{-r}\}$. Then $\{\mathcal{W}_r\}_{r>0}$ is a family of affinoids of $\mathcal{W}$ which defines an admissible covering of it. For each $r \in \mathbb{Q}$, $r > 0$ we show that there is $0 < w \leq p/(p+1)$ depending on r and a locally free sheaf $\Omega_{r,w}$ on $\mathcal{W}_r \times X_1(Np)(w)$ such that if $\kappa \in \mathcal{W}_r(K)$ and if we denote by $f_\kappa : X_1(Np)(w) \longrightarrow \mathcal{W}_r \times X_1(Np)(w)$ the morphism $f_\kappa(z) = (\kappa, z)$, we have a natural (specialization) morphism of sheaves $\psi_\kappa : (f_\kappa)^*(\Omega_{r,w}) \longrightarrow \Omega^\kappa_{w,K}$. We have

• Let $r_1, r_2 > 0$ be rational numbers and let $0 < w_1, w_2 \leq p/(p+1)$ be the associated degrees of overconvergence. Then the restrictions of the sheaves Ω_{r_1,w_1} and Ω_{r_2,w_2} on $\big(\mathcal{W}_{r_1} \cap \mathcal{W}_{r_2}\big) \times \big(X_1(Np)(w_1) \cap X_1(Np)(w_2)\big)$ coincide.

• If $\mathbb{F}$ is p-adic analytic family of modular forms over an affinoid subdomain $U \subset \mathcal{W}$ as in theorem 1.1, then first there is an $r > 0$ such that $U \subset \mathcal{W}_r$. Than, $\mathbb{F}$ is the q-expansion at the cusp ∞ of a section $\mathbf{G} \in H^0(U \times X_1(Np)(w), \Omega_{r,w})$, for $w > 0$ as above

and

• If $\kappa \in U(K)$ then the specialization $\mathbb{F}(\kappa)$ described in theorem 1.1 is the q-expansion of the section $\psi_{\kappa, U \times X_1(Np)(w)}(\mathbf{G})$.

II) We construct sheaves $\Omega^\kappa_{w,K}$ and $\Omega_{r,w}$ for Hilbert modular varieties for $\mathbf{GL}_{2/F}$, where F is a totally real number field such that p is unramified in F. In this situation the definition of overconvergent modular forms is a new definition.

We'd like to point out that Vincent Pilloni independently has constructed in [P], using a slightly different method, sheaves like our $\Omega^\kappa_{w,K}$ and therefore realized geometrically the overconvergent elliptic modular forms of non-integral weight defined in [C1].

Notations Throughout this article we'll use the following notations: if $u \in \mathbb{Q}$, we'll denote by p^u an element of $\mathbb{C}_p$ of valuation u. If $p^u \in \mathcal{O}_K$ and M is an object over R (an R-module, an R-scheme or formal scheme) then we denote by $M_u := M \otimes_R R/p^u R$. In particular $M_1 = M \otimes_R R/pR$.

2 The Hodge-Tate sequence

We fix N, p and w where N is a positive integer, $p \geq 3$ is a prime integer and w is a rational number such that $0 \leq w \leq 1/p$. We denote by K a finite extension of $\mathbb{Q}_p$ containing an element of valuation w. Let k be its residue field.

Our standard local assumptions will be the following. Let us denote by R an $\mathcal{O}_K$-algebra which is an integral domain, p-adically complete and separated and such that there is a formally étale morphism of $\mathcal{O}_K$-algebras: $\mathcal{O}_K\{T_1,...,T_d\}/(T_1...T_j-\pi^a)\longrightarrow R$, where $1\le j\le d$, π is a uniformizer of K and $a\ge 0$ is an integer. We let $\pi\colon A\longrightarrow \mathcal{U}:=\mathrm{Spec}(R)$ be an abelian scheme of relative dimension $g\ge 1$. We denote by $\omega_{A/R}:=\pi_*(\Omega^1_{A/R})$ and will assume that $\omega_{A/R}$ is a free $\mathcal{O}_{\mathcal{U}}$-module of rank g.

Let us now consider $\varphi_A\colon R^1\pi_*(A_1)\longrightarrow R^1\pi_*(A_1)$ the Frobenius morphism, let $\det(\varphi)$ denote the ideal of R/pR generated by the determinant of φ_A in a basis of $R^1\pi_*(A_1)$. *We assume* that there exists $0\le w<1/p$ such that $p^w\in\det(\varphi_A)$ and $p^w\in\det(\varphi_{A^\vee})$. By [AG] it follows that there exists a canonical subgroup $C\subset A[p]$ of A defined over $\mathcal{U}_K$. We let $D\subset A[p]^\vee\cong A^\vee[p]$ be the Cartier dual of $A[p]/C$ over $\mathcal{U}_K$. We fix a geometric generic point $\eta=\mathrm{Spec}(\mathbb{K})$ of $\mathcal{U}$ where $\mathbb{K}$ is an algebraic closure of the fraction field of R which contains $\overline{K}$. We denote by $\Delta:=\pi_1(\mathcal{U}_{\overline{K}},\eta)$ and by $\mathcal{G}:=\pi_1(\mathcal{U}_K,\eta)$. Let $T:=T_p(A_\eta)$, where A_η is the fiber of A at η. $\mathcal{G}$ acts continuously on T.

We denote by $\overline{R}$ the inductive limit of all R-algebras $S\subset\mathbb{K}$ which are normal and such that $R_K\subset S_K$ is finite and étale and by $\widehat{\overline{R}}$ the p-adic completion of $\overline{R}$. The group $\mathcal{G}$ is then the Galois group of $\overline{R}_K$ over $R\overline{K}$ and as such acts continuously on $\widehat{\overline{R}}$. Let us remark that we have an isomorphism as $\mathcal{G}$-modules $T\cong\lim\limits_{\leftarrow,n}A[p^n](\overline{R})$.

We'd now like to recall a classical construction which will be essential for the rest of this article, namely the map dlog. Let G be a finite and locally free group scheme over $\mathcal{U}$ annihilated by p^m, let $G^\vee$ denote its Cartier dual and we denote by $\omega_{G^\vee/R}$ the R-module of global invariant differentials on $G^\vee$. We fix an affine, noetherian, normal scheme, p-torsion free $S\to\mathcal{U}$ and define the map

$$\mathrm{dlog}_{G,S}\colon G(S_K)\longrightarrow \omega_{G^\vee/R}\otimes_R\mathcal{O}_S/p^m\mathcal{O}_S$$

as follows. Let x be an S_K-point of G. Since S is normal, affine and p-torsion free and G is finite and flat over $\mathcal{U}$, it extends uniquely to an S-valued point, abusively denoted by x, of G. Such point defines a group scheme homomorphism over S, $f_x\colon G^\vee\longrightarrow\mathbf{G}_{\mathrm{m}}$ and we set $\mathrm{dlog}(x)=f_x^*(dT/T)$ where dT/T is the standard invariant differential of $\mathbf{G}_{\mathrm{m}/S}$.

Lemma 2.1. *The map* $\mathrm{dlog}_{G,S}$ *is functorial with respect to* $\mathcal{U},G$ *and* S*, more precisely:*

a) Let $\mathcal{U}'\longrightarrow\mathcal{U}$ *be a morphism of schemes, let* $G\longrightarrow\mathcal{U}$ *be a finite locally free group scheme and denote by* $G'\longrightarrow\mathcal{U}'$ *the base change of* G *to* $\mathcal{U}'$ *and let* $S\longrightarrow\mathcal{U}'$ *be a morphism with* S *normal, noetherian, affine and flat over* $\mathcal{O}_K$*. Then the natural diagram commutes*

$$\begin{array}{ccc} G'(S) & \stackrel{\mathrm{dlog}_{G',S}}{\longrightarrow} & \omega_{(G')^\vee/R'}\otimes_{R'}\mathcal{O}_S/p^m\mathcal{O}_S \\ \downarrow & & \downarrow \\ G(S) & \stackrel{\mathrm{dlog}_{G,S}}{\longrightarrow} & \omega_{G^\vee/R}\otimes_R\mathcal{O}_S/p^m\mathcal{O}_S \end{array}$$

b) Let G *and* G' *be group schemes, finite and locally free over* $\mathcal{U}=Spec(R)$ *and* $G'\longrightarrow G$ *a homomorphism of group schemes over* $\mathcal{U}$*. As before we fix a morphism* $S\longrightarrow\mathcal{U}$ *with* S *normal, noetherian, affine and flat over* $\mathcal{O}_K$*. Then, we have a natural commutative diagram*

$$\begin{array}{ccc} G'(S) & \stackrel{\mathrm{dlog}_{G',S}}{\longrightarrow} & \omega_{(G')^\vee/R}\otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \\ \downarrow & & \downarrow \\ G(S) & \stackrel{\mathrm{dlog}_{G,S}}{\longrightarrow} & \omega_{G^\vee/R}\otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \end{array}$$

c) Finally let us suppose that we have a morphism of normal, noetherian, affine schemes $S' \longrightarrow S$ over $\mathcal{U}$, which are flat over $\mathcal{O}_K$. Then we have a natural commutative diagram

$$\begin{array}{ccc} G(S) & \stackrel{\mathrm{dlog}_{G,S}}{\longrightarrow} & \omega_{(G)^\vee/R}\otimes_R \mathcal{O}_S/p^m\mathcal{O}_S \\ \downarrow & & \downarrow \\ G(S') & \stackrel{\mathrm{dlog}_{G,S'}}{\longrightarrow} & \omega_{G^\vee/R}\otimes_R \mathcal{O}_{S'}/p^m\mathcal{O}_{S'} \end{array}$$

Proof. The proof is standard and we leave it to the reader. □

Applying the construction above to the group schemes $A^\vee[p^n]\cong (A[p^n])^\vee$ for $n\geq 1$ over the tower of normal R-algebras S whose union is $\overline{R}$ (see above) we obtain compatible $\mathcal{G}$-equivariant maps (for varrying n)

$$\mathrm{dlog}_n\colon A^\vee[p^n](\overline{R}_K)\longrightarrow \omega_{A[p^n]/R}\otimes_R \overline{R}/p^n\overline{R}\cong \omega_{A/R}\otimes_R \overline{R}/p^n\overline{R}.$$

By taking the projective limit of these maps we get the morphism of $\mathcal{G}$-modules

$$\mathrm{dlog}_{A^\vee}\colon T_p(A_\eta^\vee)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\longrightarrow \omega_{A/R}\otimes_R\widehat{\overline{R}}.$$

We also have the analogous map for A itself i.e., a map $\mathrm{dlog}_A\colon T_p(A_\eta)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\longrightarrow \omega_{A^\vee/R}\otimes_R\widehat{\overline{R}}$. The Weil pairing identifies $T_p(A_\eta)$ with the $\mathcal{G}$-module $T_p(A_\eta^\vee)^\vee(1)$ so that the $\widehat{\overline{R}}$-dual of dlog_A provides a map $a\colon \omega_{A^\vee/R}^{-1}\otimes_R\widehat{\overline{R}}(1)\longrightarrow T_p(A_\eta^\vee)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}$. We thus obtain the following sequence of $\widehat{\overline{R}}$-modules compatible with the semilinear action of $\mathcal{G}$:

$$0\longrightarrow \omega_{A^\vee/R}^{-1}\otimes_R\widehat{\overline{R}}(1)\stackrel{a}{\longrightarrow} T_p(A_\eta^\vee)\otimes_{\mathbb{Z}_p}\widehat{\overline{R}}\stackrel{\mathrm{dlog}}{\longrightarrow}\omega_{A/R}\otimes_R\widehat{\overline{R}}\longrightarrow 0.$$

Since $\mathrm{H}^0\big(\mathcal{G},\widehat{\overline{R}}(-1)\big)=0$ we have that $\mathrm{dlog}\circ a=0$ i.e. this sequence is in fact a complex. We have

Theorem 2.2 ([F],[Fa]). *The homology of the complex above, which we call the Hodge-Tate sequence attached to A, is annihilated by a power of p.*

Remark 2.3. It follows from [Bri] that p is not a zero divisor in $\widehat{\overline{R}}$, therefore the morphism a above is injective.

From now on whenever we write D, $A^\vee[p]$ and $A^\vee[p]/D$ we mean the $\mathcal{G}$-representations $D(\overline{R}_K)$, $A^\vee[p](\overline{R}_K)$ and $\big(A^\vee[p]/D\big)(\overline{R}_K)$ respectively. Let us denote by $F^0:=\mathrm{Im}(\mathrm{dlog})$ and $F^1=\mathrm{Ker}(\mathrm{dlog})$. They are $\widehat{\overline{R}}$-modules and because dlog is $\mathcal{G}$-equivariant it follows that F^0 and F^1 have natural continuous actions of $\mathcal{G}$.

Proposition 2.4. *a) The $\widehat{\overline{R}}$-modules F^0 and F^1 are free of rank g and we have a commutative diagram with exact rows and vertical isomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1/p^{1-v}F^1 & \longrightarrow & T_p(A_\eta^\vee)\otimes_{\mathbb{Z}_p}\overline{R}/p^{1-v}\overline{R} & \longrightarrow & F^0/p^{1-v}F^0 & \longrightarrow & 0 \\ & & \downarrow\cong & & || & & \downarrow\cong & & \\ 0 & \longrightarrow & D\otimes\overline{R}/p^{1-v}\overline{R} & \longrightarrow & A^\vee[p]\otimes\overline{R}/p^{1-v}\overline{R} & \longrightarrow & (A^\vee[p]/D)\otimes\overline{R}/p^{1-v}\overline{R} & \longrightarrow & 0 \end{array}$$

b) The cohomology of the Hodge-Tate sequence above is annihilated by p^v.

Proof. We divide the proof in three steps.

Step 1: F^0 is a free $\widehat{\overline{R}}$-module of rank g and p^v annihilates Coker(dlog). It follows from 5.1 that the mod p reduction of dlog factors via a map

$$\alpha\colon \big(A^\vee[p]/D\big)\otimes_{\mathbb{F}_p}\overline{R}/p\overline{R}\longrightarrow \omega_{A/R}\otimes_R\overline{R}/p\overline{R}.$$

Choose elements $\{\tilde{f}_1,\ldots,\tilde{f}_g\}$ of $T_p(A_\eta^\vee)/pT_p(A_\eta^\vee)$ which provide a basis of $A^\vee[p]/D$ over $\mathbb{F}_p$. We also fix a basis $\{\omega_1,\omega_2,\ldots,\omega_g\}$ of $\omega_{A/R}$. If $t_1,t_2,\ldots,t_g$ are elements of a group we denote by $\underline{t}$ the column vector with those coefficients and by $\tilde{\underline{t}}$ the reduction of $\underline{t}$ modulo p. Let us denote by $\tilde{\underline{\delta}}\in M_{g\times g}(\overline{R}/p\overline{R})$ the matrix with the property that $\tilde{\alpha}(\tilde{\underline{f}})=\tilde{\underline{\delta}}\cdot\tilde{\underline{\omega}}$. Let now denote by $\underline{\delta}\in M_{g\times g}(\widehat{\overline{R}})$ any matrix such that the image of $\underline{\delta}$ under the natural projection $M_{g\times g}(\widehat{\overline{R}})\longrightarrow M_{g\times g}(\overline{R}/p\overline{R})$ is $\tilde{\underline{\delta}}$. Let $G^0\subset\omega_{A/R}\otimes_R\widehat{\overline{R}}$ be the $\widehat{\overline{R}}$-module generated by the vectors $\underline{\delta}\cdot\underline{\omega}$. Thus Step 1 follows if we prove the following:

Lemma 2.5. *(1) There exists a matrix $\underline{s}\in M_{g\times g}(\widehat{\overline{R}})$ such that $\underline{\delta}\cdot\underline{s}=\underline{s}\cdot\underline{\delta}=p^v\mathrm{Id}$.*

(2) The $\widehat{\overline{R}}$-module G^0 is free of rank g and it contains $p^v\omega_{A/R}\otimes_R\widehat{\overline{R}}$.

(3) The $\widehat{\overline{R}}$-module G^0 coincides with F^0.

In particular, F^1 is a finite and projective $\widehat{\overline{R}}$-module of rank g.

Proof. The last statement follows from the others.

(1) We use proposition 5.1: as $p^v\mathrm{Coker(dlog)}=0(\bmod p)$ and $\underline{\delta}\cdot\underline{\omega}$ generates the image of dlog modulo p, it follows that there are matices A and $B\in M_{g\times g}(\widehat{\overline{R}})$ such that

$$p^v\underline{\omega}=\mathrm{dlog}(d)=pA\cdot\underline{\omega}+B\underline{\delta}\cdot\underline{\omega}.$$

Therefore, we have $p^v\big(\mathrm{Id}-p^{1-v}A\big)=B\underline{\delta}$ and as $\mathrm{Id}-p^{1-v}A$ is invertible we obtain that $\underline{s}\cdot\underline{\delta}=p^v\mathrm{Id}$. Let us now recall that p is a non-zero divisor in $\widehat{\overline{R}}$, therefore the natural morphism $M_{g\times g}(\widehat{\overline{R}})\longrightarrow M_{g\times g}(\widehat{\overline{R}}[1/p])$ is injective and in the target module the matrices $\underline{\delta},p^{-v}\underline{s}$ are inverse one to the other. Therefore we obtain the relation $\underline{\delta}\cdot\underline{s}=p^v\mathrm{Id}$ first in $M_{g\times g}(\widehat{\overline{R}}[1/p])$ and then even in $M_{g\times g}(\widehat{\overline{R}})$.

(2) By construction G^0 is generated by the g vectors $\underline{\delta}\cdot\underline{\omega}$. Let $\underline{a}\in\big(\widehat{\overline{R}}\big)^g$ be a row vector such that $\underline{a}\cdot\underline{\delta}\cdot\underline{\omega}=0$. Since $\underline{\delta}$ is invertible after inverting p and $\underline{\omega}$ is a basis of $\omega_{A/R}$, we have that $\underline{a}=0$ in $\widehat{\overline{R}}[p^{-1}]^g$. Since p is not a zero divisor in $\widehat{\overline{R}}$ and $\underline{\delta}$ is invertible after inverting p, we conclude that $\underline{a}=0$. Moroever, we have $p^v\underline{\omega}=\underline{s}\cdot\underline{\delta}\cdot\underline{\omega}$. Hence, the last claim follows.

(3) For every $d \in T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}$ there exists A and $B \in M_{g\times g}(\widehat{\overline{R}})$ such that $\mathrm{dlog}(d) = A\underline{\delta} \cdot \underline{\omega} + pB\underline{\omega}$. Since $p^v \omega_{A/R} \otimes_R \widehat{\overline{R}}$ is contained in G^0, we conclude that $F^0 \subset G^0$. Similarly, since $p^v \mathrm{Coker}(\mathrm{dlog}) = 0$ we have that $p^v \omega_{A/R} \otimes_R \widehat{\overline{R}} \subset F^0$. The vectors $\underline{\delta} \cdot \underline{\omega}$ are contained in $F^0 + p\omega_{A/R} \otimes_R \widehat{\overline{R}}$ which is contained in F^0. The conclusion follows.

$\square$

Step 2: We prove that we have a commutative diagram

$$\begin{array}{ccccccccc} & & \omega_{A/R}^{-1} \otimes_R \overline{R}/p\overline{R} & \stackrel{\tilde{a}}{\longrightarrow} & T_p(A_\eta^\vee) \otimes \overline{R}/p\overline{R} & \stackrel{\widetilde{\mathrm{dlog}}}{\longrightarrow} & \omega_{A/R} \otimes \overline{R}/p\overline{R} & & \\ & & \tilde{\beta} \downarrow & & || & & \uparrow \tilde{\alpha} & & \\ 0 & \longrightarrow & D \otimes \overline{R}/p\overline{R} & \longrightarrow & A^\vee[p] \otimes \overline{R}/p\overline{R} & \longrightarrow & \big(A^\vee[p]/D\big) \otimes \overline{R}/p\overline{R} & \longrightarrow & 0 \end{array}$$

Let H^0 and H^1 be the image and respectively the kernel of the map $\mathrm{dlog}\colon T_p(A_\eta) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \longrightarrow \omega_{A^\vee/R} \otimes_R \widehat{\overline{R}}$. Since we assumed that also $A^\vee$ admits a canonical subgroup, then we know from Step 1 that H^0 is a free $\widehat{\overline{R}}$-module of rank g and H^1 is a finite and projective $\widehat{\overline{R}}$-module of rank g. It follows from [F, §3, lemma 2] that H^1 and F^1 are orthogonal with respect to the perfect pairing $\big(T_p(A_\eta) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}\big) \times \big(T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}\big) \longrightarrow \widehat{\overline{R}}(1)$ defined by extending $\widehat{\overline{R}}$-linearly the Weil pairing. In particular, via the isomorphism

$$h\colon T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \longrightarrow T_p(A_\eta^\vee)^\vee \otimes \widehat{\overline{R}}(1),$$

induced by the pairing, we have $h(F^1) \subset (H^0)^\vee(1)$. Thus, h induces a morphism $h'\colon F^0 \longrightarrow (H^1)^\vee(1)$. Since H^1 is a projective $\widehat{\overline{R}}$-module, the map $T_p(A_\eta^\vee)^\vee \otimes \widehat{\overline{R}}(1) \to (H^1)^\vee(1)$ is surjective so that h' is a surjective morphism of finite and projective $\widehat{\overline{R}}$-modules of the same rank and, hence, it must be an isomorphism. This implies that h induces an isomorphism $F^1 \cong (H^0)^\vee(1)$. Since $(H^0)^\vee/p(H^0)^\vee)(1) \subset T_p(A_\eta^\vee)^\vee \otimes \widehat{\overline{R}}/p\widehat{\overline{R}}(1)$ is identified with $D \otimes \overline{R}/p\overline{R} \subset A^\vee[p] \otimes \overline{R}/p\overline{R}$ via h, we get the claim in Step 2.

Step 3: End of proof. From Step 1 (applied to the abelian scheme A/R) we have that $p^v(\omega_{A^\vee/R} \otimes_R \widehat{\overline{R}}) \subset H^0$ and from Step 2 we have an isomorphism $F^1 \cong (H^0)^\vee(1)$. We deduce that $p^v F^1 \subset \omega_{A^\vee/R}^{-1} \otimes_R \widehat{\overline{R}}(1)$.

Consider the map

$$\gamma\colon F^1 \subset T_p(A_\eta^\vee) \otimes \overline{R} \longrightarrow A^\vee[p] \otimes \overline{R}/p\overline{R} \longrightarrow \big(A^\vee[p]/D\big) \otimes \overline{R}/p\overline{R}.$$

Note that $\omega_{A^\vee/R}^{-1} \otimes_R \overline{R}/p\overline{R}$ goes to zero in $\big(A^\vee[p]/D\big) \otimes \overline{R}/p\overline{R}$ by Step 2 and the latter is a free $\overline{R}/p\overline{R}$-module. Since $\overline{R}$ is p-torsion free, the subset of elements of $\overline{R}/p\overline{R}$ annihilated by p^v coincides with $p^{1-v}\overline{R}/p\overline{R}$. We conclude that the image of the map γ is contained in $p^{1-v}\big(A^\vee[p]/D\big) \otimes \overline{R}/p\overline{R}$. In particular, the composition $T_p(A_\eta^\vee) \otimes \overline{R} \longrightarrow A^\vee[p] \otimes \overline{R}/p^{1-v}\overline{R} \longrightarrow \big(A^\vee[p]/D\big) \otimes \overline{R}/p^{1-v}\overline{R}$ has the property that it sends F^1 is 0. Therefore it induces a map $F^0/p^{1-v}F^0 \longrightarrow \big(A^\vee[p]/D\big) \otimes \overline{R}/p^{1-v}\overline{R}$. This is a surjective morphism of free $\overline{R}/p^{1-v}\overline{R}$-modules of the same rank. Hence, it must be an isomorphism. More concretely, it is defined by sending the reduction of $\underline{\delta} \cdot \underline{\omega}$ modulo p^{1-v}, to the basis $\underline{\tilde{f}}$ of D. Since F^0 is a free $\widehat{\overline{R}}$-module, we can find a (non canonical) splitting $T_p(A_\eta^\vee) \otimes \overline{R} = F^0 \oplus F^1$. In particular, $F^1/p^{1-v}F^1$ injects in $T_p(A_\eta^\vee) \otimes \overline{R}/p^{1-v}\overline{R}$ and it must coincide with $D \otimes \overline{R}/p^{1-v}\overline{R}$. This provides the diagram in the statement of proposition 2.4.

Note that $D \otimes \overline{R}/p^{1-v}\overline{R}$ is a free $\overline{R}/p^{1-v}\overline{R}$-module of rank g. Since F^1 is a projective $\widehat{\overline{R}}$-module of rang g, any lift of a basis of $D \otimes \overline{R}/p^{1-v}\overline{R}$ to elements of F^1 provides a basis of the latter as $\widehat{\overline{R}}$-module. We conclude that also F^1 is a free $\widehat{\overline{R}}$-module of rank g as claimed.

□

Let us now suppose that $(A^\vee[p]/D)(\overline{R}) = (A^\vee[p]/D)(R)$ and therefore, using 5.2, it follows that $\underline{\tilde{\delta}} \in M_{g\times g}(R/pR)$. We have the following

Proposition 2.6. *Let $\underline{\delta}_0 \in M_{g\times g}(R)$ be any lift of $\underline{\tilde{\delta}}$ in $M_{g\times g}(R)$. Let us denote by $G_0 \subset \omega_{A/R}$ the R-submodule generated by $\underline{\delta}_0 \cdot \underline{\omega}$.*

1) Then, G_0 is a free R-module of rank g with basis $\underline{\delta}_0 \cdot \underline{\omega}$ and $G_0 \otimes_R \widehat{\overline{R}} \cong F^0$..

2) The R-module $F_0 := (F^0)^{\mathcal{G}} \subset \omega_{A/R}$ coincides with G_0. In particular, $F_0 \otimes_R \widehat{\overline{R}} \cong F^0$.

3) We have a natural isomorphism $F_0/p^{1-v}F_0 \cong (A^\vee[p]/D) \otimes R/p^{1-v}R$ whose base change via $R \to \widehat{\overline{R}}$ provides the isomorphism $F^0/p^{1-v}F^0 \cong (A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R}$ in 2.4 via the isomorphism $F_0 \otimes_R \widehat{\overline{R}} \cong F^0$.

4) Let $F_1 := (F_0)^\vee(1)$, then we have a natural isomorphism as $\widehat{\overline{R}}$-modules, $\mathcal{G}$-equivariant: $F_1 \otimes_R \widehat{\overline{R}} \cong F^1$.

Proof. (1) As $\underline{\delta}_0(\bmod p\widehat{\overline{R}}) = \underline{\tilde{\delta}}$ lemma 2.5 implies that there is an $\underline{s}_0 \in M_{g\times g}(\widehat{\overline{R}})$ such that $\underline{\delta}_0 \cdot \underline{s}_0 = \underline{s}_0 \cdot \underline{\delta}_0 = p^v\mathrm{Id}$. In $M_{g\times g}\big(\widehat{\overline{R}}[1/p]\big)$ we have $\underline{s}_0 = p^{-v}\underline{\delta}_0 \in M_{g\times g}\big(R[1/p]\big) \cap M_{g\times g}(\widehat{\overline{R}})$. But $R[1/p] \cap \widehat{\overline{R}} = R$ because R is normal. Hence, $\underline{\delta}_0 \cdot \underline{s}_0 = \underline{s}_0 \cdot \underline{\delta}_0 = p^v\mathrm{Id}$. This implies that G_0 is a free R-module. By lemma 2.5 the $\widehat{\overline{R}}$-submodule generated by $\underline{\delta}_0 \cdot \underline{\omega}$ is F^0. This concludes the proof of (1).

(2) It follows from (1) that $G_0 \subset F^0$ and that $G_0 \subset (F^0)^{\mathcal{G}}$. Let now $x \in (F^0)^{\mathcal{G}}$. Then $x = \underline{u}^t \cdot (\underline{\delta}_0 \cdot \underline{\omega})$ for some column vector $\underline{u}$ with coefficients in $\widehat{\overline{R}}$. As x and $\underline{\delta}_0 \cdot \underline{\omega}$ are $\mathcal{G}$-invariant and the elements of $\underline{\delta}_0 \cdot \underline{\omega}$ are R-linearly independent (as $\underline{\delta}_0$ is in $\mathbf{GL}_g(R[1/p])$), it follows that $\underline{u}$ is $\mathcal{G}$-invariant and so $x \in G_0$. This concludes the proves (2).

(3) By construction the map $F^0 \longrightarrow (A^\vee[p]/D) \otimes \overline{R}/p^{1-v}\overline{R}$ in 2.4 sends the basis $\underline{\delta}_0 \cdot \underline{\omega}$ to the given basis of $A^\vee[p]/D$; see Step 3 of the proof of 2.4. Extending R-linearly and reducing modulo p^{1-v} we get the claimed isomorphism $F_0/p^{1-v}F_0 \to (A^\vee[p]/D) \otimes R/p^{1-v}R$.

(4) This follows from the natural isomorphism as $\widehat{\overline{R}}$-modules which is $\mathcal{G}$-equivariant $F^1 \cong (F^0)^\vee(1)$ (see Step 2 of the proof of proposition 2.4.) □

We now claim that the free R-module F_0 defined in proposition 2.6, and which will be now denoted $F_0(A/R)$ is functorial both in A/R. More precisely, we have:

Lemma 2.7. *a) Let $R \longrightarrow R'$ be a morphism of $\mathcal{O}_K$-algebras of the type defined at the beginning of this section, let A/R be an abelian scheme such that the assumptions of 2.6 hold and let A'/R' be the base change of A to R'. Then we have a natural isomorphism of R'-modules $F_0(A/R)\otimes_R R' \cong F_0(A'/R')$ compatible with the isomorphism $\omega_{A/R}\otimes_R R' \cong \omega_{A'/R'}$.*

b) Let us assume that we have a morphism of abelian schemes $A \longrightarrow B$ over R and that both A and that the assumptions of 2.6 hold. Then we have a natural morphism of R-modules $F_0(B/R) \longrightarrow F_0(A/R)$ compatible with the morphism $\omega_{B/R} \longrightarrow \omega_{A/R}$.

Proof. We have similar statements for F^0: in case (a) we have a morphism of $\widehat{\overline{R}}'$-modules $F^0(A/R)\otimes_{\widehat{\overline{R}}} \widehat{\overline{R}}' \longrightarrow F^0(A'/R')$, compatible with the isomorphism of invariant differentials, and in case (b) we have a natural morphism of $\widehat{\overline{R}}$-modules $F^0(B/R) \longrightarrow F^0(A/R)$ compatible with the morphism on invariant differentials. These two statements follow from the functoriality of dlog; see 2.1. Taking Galois invariants we immediately get (b) and also that we have a morphism $f\colon F_0(A/R)\otimes_R R' \longrightarrow F_0(A'/R')$. Since f is an isomorphism modulo p^{1-v} by 2.6 and since it is a linear morphism of free R'-modules of the same rank, it must be an isomorphism as claimed. □

2.1 The Hodge-Tate sequence in the semiabelian reduction case

We need a generalization of proposition 2.4 to the case of semiabelian schemes in order to deal with the cusps when implementing our constructions to moduli spaces of abelian varieties. We follow [F, §3.e] providing more details.

Let $S \subset \mathcal{U}$ be a simple normal crossing divisor transversal to the special fiber of $\mathcal{U}$. We write $\mathcal{U}^o := \mathcal{U}\backslash S$ and let $R[S^{-1}]$ be the underlying ring. Assume that its p-adic ompletion $\widehat{R[S^{-1}]}$ is an integral normal domain. As before we fix a geometric generic point $\eta = \mathrm{Spec}(\mathbb{K})$ of $\widehat{\mathcal{U}}^o$. We denote by $\Delta^o := \pi_1(\mathcal{U}^o_{\overline{K}}, \eta)$ and by $\mathcal{G}^o := \pi_1(\mathcal{U}^o_K, \eta)$. We denote by $\overline{R}$ the inductive limit of all R-algebras $M \subset \mathbb{K}$ which are normal and such that $R[S^{-1}, p^{-1}] \subset M[S^{-1}, p^{-1}]$ is finite and étale. Let $\widehat{\overline{R}}$ be the p-adic completion of $\overline{R}$ with its continuous action of $\mathcal{G}$.

Let A be an abelian scheme over $\mathcal{U}^o$ and assume that there exists an ètale sheaf X over $\mathcal{U}$ of finite and free $\mathbb{Z}$-modules, a semiabelian scheme G over $\mathcal{U}$, extension of an abelian scheme B by a torus T, and a 1-motive $M := [X \to G_{\mathcal{U}^o}]$ over $\mathcal{U}^o_K$ such that $T_p(A_\eta)$ is isomorphic to the Tate module $T_p(M_\eta)$ as $\mathcal{G}^o$-module. This is the case for generalized Tate objects used to define compactifications of moduli spaces. Let $M^\vee := [Y \to H_{\mathcal{U}^o}]$ be the dual motive; here, Y is the character group of T and H is a semiabelian scheme over $\mathcal{U}$, extension of $B^\vee$ by the torus T' with character group X. Then, $T_p(A^\vee_\eta)$ is isomorphic to the p-adic Tate module of the dual motive $M^\vee$. In particular, it admits a decreasing filtration $W_{-i}T_p(A^\vee_\eta)$ for $i = 0$, 1 and 2 by $\mathcal{G}^o$-submodules such that (1) $W_{-1}T_p(A^\vee_\eta) = T_p(H^\vee_\eta)$; (2) $\mathrm{gr}_0 T_p(A^\vee_\eta) \cong Y \otimes \mathbb{Z}_p$; (3) $\mathrm{gr}_{-2}T_p(A^\vee_\eta) \cong T_p(T')$; (4) $\mathrm{gr}_{-1}(T) \cong T_p(B^\vee_\eta)$.

Consider the map $\mathrm{dlog}\colon T_p(A^\vee_\eta)\otimes\widehat{\overline{R}[S^{-1}]} \longrightarrow \omega_{A/\mathcal{U}^o}\otimes_R \widehat{\overline{R}[S^{-1}]}$. Since the p-adic completion of $\omega_{A/\mathcal{U}^o}$ can be expressed in terms of the p-divisible subgroup of A and the latter can be expressed using M we have $\omega_{A/\mathcal{U}^o}\otimes_R \widehat{\overline{R}[S^{-1}]} \cong \omega_{G/\mathcal{U}}\otimes_R \widehat{\overline{R}[S^{-1}]}$. The latter sits in an exact sequence given by the exact sequence over $\mathcal{U}$:

$$0 \longrightarrow \omega_{B/\mathcal{U}} \longrightarrow \omega_{G/\mathcal{U}} \longrightarrow \omega_{T/\mathcal{U}} \longrightarrow 0.$$

We then define a filtration on $\omega_{G/\mathcal{U}}$ by setting $W_{-2}\omega_{G/\mathcal{U}} := 0$, $W_{-1}\omega_{G/\mathcal{U}} := \omega_{B/\mathcal{U}}$ and $W_{-2}\omega_{G/\mathcal{U}} := \omega_{G/\mathcal{U}}$.

Lemma 2.8. *The map* dlog *extends uniquely to a morphism*

$$\mathrm{dlog}\colon T_p(A_\eta^\vee) \otimes \widehat{\overline{R}} \longrightarrow \omega_{G/\mathcal{U}} \otimes_R \widehat{\overline{R}},$$

which is trivial on $\mathrm{gr}_{-2}T_p(A_\eta^\vee)$*, induces an isomorphism on* $\mathrm{gr}_0 T_p(A_\eta^\vee) \otimes \widehat{\overline{R}} \longrightarrow \omega_{T/\mathcal{U}} \otimes_R \widehat{\overline{R}}$ *and coincides with the map* $\mathrm{dlog}\colon T_p(B_\eta^\vee) \otimes \widehat{\overline{R}} \longrightarrow \omega_{B/\mathcal{U}} \otimes_R \widehat{\overline{R}[S^{-1}]}$ *on* B *via the identification* $\mathrm{gr}_{-1}T_p(A_\eta^\vee) \cong T_p(B_\eta^\vee)$.

Such extension is functorial in R *and in* A*. More precisley, assume that we have a morphism of abelian schemes* $f\colon A \to A'$ *over* $\mathcal{U}^o$ *such that the associated* p*-adic Tate modules are the Tate modules of* 1*-motives* $M = [X \to G]$ *and* $M' = [X' \to G']$ *as above and the map induced from* f *on Tate modules arises from a morphism of* 1*-motives* $M \to M'$*. Then, the following diagram is commutative*

$$\begin{array}{ccc} T_p((A'_\eta)^\vee) \otimes \widehat{\overline{R}} & \stackrel{\mathrm{dlog}_{(A')^\vee}}{\longrightarrow} & \omega_{G'/R} \otimes \widehat{\overline{R}} \\ \downarrow & & \downarrow \\ T_p(A_\eta^\vee) \otimes \widehat{\overline{R}} & \stackrel{\mathrm{dlog}_{A^\vee}}{\longrightarrow} & \omega_{G/R} \otimes \widehat{\overline{R}} \end{array}$$

Proof. The uniqueness follows from the fact that dlog is a map of free $\widehat{\overline{R}}$-modules and the map $\widehat{\overline{R}} \to \widehat{\overline{R}[S^{-1}]}$ is injective.

This also implies that the displayed diagram commutes since it commutes after base change to $\widehat{\overline{R}[S^{-1}]}$ by functoriality of dlog.

For every $n \in \mathbb{N}$ let $R \subset R_n \subset \overline{R}$ be a finite and normal extension such that $A_\eta^\vee[p^n] \cong M_\eta^\vee[p^n]$ is trivial as Galois module over $R_n[S^{-1}, p^{-1}]$. This allows to split the filtration on $M_\eta[p^n]$ so that $M_\eta[p^n] \cong B_\eta^\vee[p^n] \oplus T'_\eta[p^n] \oplus Y/p^nY$ as representations of the Galois group of $\overline{R}$ over R_n. Note that the Galois module $B_\eta^\vee[p^n] \oplus T'_\eta[p^n] \oplus Y/p^nY$ is associated to the group scheme $B^\vee[p^n] \oplus T'[p^n] \oplus Y/p^nY$ over R_n. By functoriality of the map dlog we deduce that dlog modulo p^n extends to all of $\mathrm{Spec}(R_n)$ and coincides with the sum of the maps dlog of these group schemes. Passing to the limit the conclusion follows. □

Applying the same argument to the dual abelian scheme and 1-motive we deduce that also in this case we have a Hodge-Tate sequence attached to A:

$$0 \longrightarrow \omega_{H^\vee/R}^{-1} \otimes_R \widehat{\overline{R}}(1) \stackrel{a}{\longrightarrow} T_p(A_\eta^\vee) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}} \stackrel{\mathrm{dlog}}{\longrightarrow} \omega_{G/R} \otimes_R \widehat{\overline{R}} \longrightarrow 0.$$

As before we let F^0 to be the image of dlog and F^1 to be the kernel. *We assume* that Frobenius $\varphi_B\colon R^1\pi_*(B_1) \longrightarrow R^1\pi_*(B_1)$ and $\varphi_{B^\vee}\colon R^1\pi_*(B_1^\vee) \longrightarrow R^1\pi_*(B_1^\vee)$ have determinant ideals containing p^w for some $0 \le w < 1/p$. Let $C \subset B[p]$ be the Galois submodule associated to the canonical subgroup and $D_B \subset B^\vee[p]$ to be the Cartier dual of $B[p]/C$. We define $D \subset A^\vee[p]$ (as Galois modules) to be the inverse image of D_B in $H[p] \to B^\vee[p]$ which we view in $A^\vee[p]$ via the inclusion $H[p] \subset A^\vee[p]$. Note that the kernel of $D \to D_B$ is $T'[p]$. Then,

Corollary 2.9. *The $\widehat{\overline{R}}$-modules F^0 and F^1 are free of rank g and we have a commutative diagram with exact rows and vertical isomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1/p^{1-v}F^1 & \longrightarrow & T_p(A_\eta^\vee)\otimes_{\mathbb{Z}_p}\overline{R}/p^{1-v}\overline{R} & \longrightarrow & F^0/p^{1-v}F^0 & \longrightarrow & 0 \\ & & \downarrow\cong & & || & & \downarrow\cong & & \\ 0 & \longrightarrow & D\otimes\overline{R}/p^{1-v}\overline{R} & \longrightarrow & A^\vee[p]\otimes\overline{R}/p^{1-v}\overline{R} & \longrightarrow & (A^\vee[p]/D)\otimes\overline{R}/p^{1-v}\overline{R} & \longrightarrow & 0 \end{array}$$

Moroever, if $A^\vee[p]/D$ is a constant group over $\mathcal{U}^o$, the R-module $F_0 := (F^0)^{\mathcal{G}} \subset \omega_{A/R}$ is free of rank g, we have $F_0\otimes_R\widehat{\overline{R}} \cong F^0$ and we have a natural isomorphism $F_0/p^{1-v}F_0 \cong (A^\vee[p]/D)\otimes R/p^{1-v}R$ whose base change via $R\to\overline{R}$ provides the isomorphism $F^0/p^{1-v}F^0 \cong A^\vee[p]/D)\otimes \overline{R}/p^{1-v}\overline{R}$.

Eventually, the construction of F_0 is functorial in $\mathcal{U}$ and A (see 2.8 for the meaning of the functoriality in A).

Proof. The statement concerning F_0 follows from the first arguing as in 2.6. The statement about the functoriality is the analogue of 2.7 and is proven as in loc. cit.. It is deduced from the functoriality of F^0 using the functoriality of the map dlog proven in 2.8.

For the first statement one argues that, using the filtration given in 2.8 and the description of dlog on such a filtration, it suffices to prove the claim for $B^\vee$ and this is the content of 2.4. Details are left to the reader. □

3 The sheaves Ω_w^κ in the elliptic case

Let N, $p\geq 5$, K be as in section 1. For a fixed $0\leq w\leq p/(p+1)$ let us recall that we defined in section 1 the K-rigid analytic spaces $X = X_1(Np)$, $X_1(N)(w)\subset X(N,p)$ and $X_1(Np)(w)\subset X$. We let $\mathfrak{X}_1(Np)$ and $\mathfrak{X}(N,p)$ denote the formal completions of $X_1(Np)$ and $X(N,p)$, seen as proper schemes over $\mathcal{O}_K$, along their special fibers and let $\mathfrak{X}_1(N)(w)$ denote the formal blow-up of $\mathfrak{X}_1(N)$ defined by the ideal of $\mathcal{O}_{\mathfrak{X}_1(N)}$ generated by $(p^w, E_{p-1}(\mathcal{E}/\mathfrak{X}_1(N),\omega))$, for a generator ω of $\omega_{\mathcal{E}/\mathfrak{X}_1(N)}$.

We let $\mathfrak{X}_1(Np)(w)$ denote the normalization of $\mathfrak{X}_1(N)(w)$ in $X_1(Np)(w)$. Let us point out that this definition makes sense. Let $\mathcal{U} = \mathrm{Spf}(R)\subset\mathfrak{X}_1(N)(w)$ be an affine open, let $\mathcal{U}_K = \mathrm{Spm}(R_K)\subset X_1(N,p)(w)$ denote its (rigid analytic) generic fiber and let $\mathcal{V}_K\subset X_1(Np)(w)$ be the inverse image of $\mathcal{U}_K$ under the morphism $X_1(Np)(w)\longrightarrow X(N,p)(w)$. This morphism is finite and étale therefore $\mathcal{V}_K$ is an affinoid, $\mathcal{V}_K = \mathrm{Spm}(S_K)$, with $R_K\longrightarrow S_K$ a finite and étale K-algebra homomorphism. Let S be the normalization of R in S_K and $\mathcal{V} = \mathrm{Spf}(S)$. Then S is a p-adically complete, separated and normal R-algebra and for varrying $\mathcal{U}$'s, the $\mathcal{V}$'s constructed above glue to give a formal scheme $\mathfrak{X}_1(Np)(w)$ with a unique morphism to $\mathfrak{X}_1(N)(w)$.

Moreover we have a natural semiabelian scheme $\mathcal{E}\longrightarrow\mathfrak{X}_1(Np)(w)$ which is the fiber product of the diagram $\mathcal{E}^{\mathrm{univ}}\longrightarrow\mathfrak{X}_1(N)\longleftarrow\mathfrak{X}_1(Np)(w)$, together with a $\Gamma_1(N)$-level structure, ψ_N. We also have a natural $\Gamma_1(p)$-level structure for $\mathcal{E}_K\longrightarrow\mathfrak{X}_1(Np)(w)_K = X_1(Np)(w)$. With these structures $\mathfrak{X}_1(Np)(w)$ is not a formal moduli scheme but it has a nice "universal" description as follows. Let R denote an $\mathcal{O}_K$-algebra, p-adically complete, separated and **normal**.

Lemma 3.1. *There is a natural bijection between the set of points $\mathfrak{X}_1(Np)(w)(R)-\{cusps\}$ and the set of isomorphism classes of quadruples $(\mathcal{E}/R,\psi_N,\psi_p,Y)$ where $\mathcal{E}\longrightarrow Spec(R)$ is an* *elliptic curve,* ψ_N *is a* $\Gamma_1(N)$ *level structure of* $\mathcal{E}/R$*,* ψ_p *is a* $\Gamma_1(p)$*-level structure of* $\mathcal{E}_K/R_K$ *i.e. a morphism of group-schemes over* R_K*:* $(\mathbb{Z}/p\mathbb{Z})_{R_K} \hookrightarrow \mathcal{E}_K$*, finally* Y *is a global section of* $\omega_{\mathcal{E}/R}^{1-p}$ *such that* $Y \cdot E_{p-1}(\mathcal{E}/R) = p^w$*.*

Proof. If $x \in \mathfrak{X}_1(Np)(w)(R) - \{\text{cusps}\}$ we may see it as a morphism of formal schemes $x : \mathrm{Spf}(R) \longrightarrow \mathfrak{X}_1(Np)(w)$ and its generic fiber is denoted $x_K : \mathrm{Spm}(R_K) \longrightarrow X_1(Np)(w)$. The first morphism gives by pull back a morphism of formal schemes $\widehat{\mathcal{E}} \longrightarrow \mathrm{Spf}(R)$ and a level $\Gamma_1(N)$-structure on it, together with a formal section $\widehat{Y}$, while the second morphism gives us the level $\Gamma_1(p)$-structure (after inverting p.) Now, as $\widehat{\mathcal{E}} \longrightarrow \mathrm{Spf}(R)$ is proper by GAGA it is algebrizable and so we obtain the family of elliptitc curves $\mathcal{E} \longrightarrow \mathrm{Spec}(R)$ together with ψ_N, Y and ψ_p after inverting p. This gives the correspondence in one direction.

Conversely, let $(\mathcal{E}/R, \psi_N, \psi_p, Y)$ be a quadruple as in the statement of the lemma. By formally completing along the special fiber we obtain a formal triple $(\widehat{\mathcal{E}} \longrightarrow \mathrm{Spf}(R), \widehat{\psi_N}, \widehat{Y})$, which provides a unique morphism of formal schemes $f : \mathrm{Spf}(R) \longrightarrow \mathfrak{X}_1(N)(w)$, whose generic fiber is $f_K : \mathrm{Spm}(R_K) \longrightarrow X_1(N)(w)$. Moreover, the quadruple $(\mathcal{E}_K/R_K, \psi_{N,K}, \psi_p, Y_K)$ gives us a morphism of K-rigid analytic spaces $g_K : \mathrm{Spm}(R_K) \longrightarrow X_1(Np)(w)$. We'd like to show that there is a unique morphism $g : \mathrm{Spf}(R) \longrightarrow \mathfrak{X}_1(Np)(w)$ whose generic fiber is g_K and which lifts f. For this we may asume that $f(\mathrm{Spf}(R))$ is contained in an affine open $\mathrm{Spf}(S)$ of $\mathfrak{X}_1(N)(w)$ (if this is not the case we cover $\mathrm{Spf}(R)$ by affine opens for which this property holds and reason in the same way for each of them.) As the morphism $X_1(Np)(w) \longrightarrow X_1(N)(w)$ is finite, the inverse image of $\mathrm{Spm}(S_K)$ in $X_1(Np)(w)$ is an affinoid $\mathrm{Spm}(T_K)$ and we denote by $T \subset T_K$ the normalization of S in T_K. Then $\mathrm{Spf}(T)$ is an open affine of $\mathfrak{X}_1(Np)(w)$ and we have the following diagram of rings and morphisms:

$$\begin{array}{ccccc} S_K & \xrightarrow{h_K} & T_K & \xrightarrow{g_K} & R_K \\ \cup & & \cup & & \cup \\ S & \xrightarrow{h} & T & & R \end{array}$$

where h_K is finite and étale and so h is integral. Moreover we have $f : S \longrightarrow R$ whose generic fiber is $g_K \circ h_K$. We'll show that $g_K(T) \subset R$. Indeed, let $t \in T$, then t is integral over S and therefore $g_K(t)$ is integral over R. But R is normal and so $g_K(t) \in R$. Therefore g is defined as the restriction of g_K to T and it has all the desired properties, in particular it defines a point $x \in \mathfrak{X}_1(Np)(w)(R)$. $\square$

Let now consider $\mathcal{U} = \mathrm{Spf}(R)$ an affine open of $\mathfrak{X}_1(Np)(w)$ then proposition 2.4 and its semi-abelian analogue, corolalry 2.9 give us a canonical free R-submodule $\mathcal{F}_{\mathcal{U}}$ of $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}(\mathcal{U})$. Namely, assume that $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}|_{\mathcal{U}}$ is a free $\mathcal{O}_{\mathcal{U}}$-module. Then, $\mathcal{F}_{\mathcal{U}} = \big(\mathrm{Im}(\mathrm{dlog})\big)^{\mathcal{G}}$ if $\mathcal{E}|_{\mathcal{U}}$ is proper or $\mathcal{F}_{\mathcal{U}} = \big(\mathrm{Im}(\mathrm{dlog})\big)^{\mathcal{G}^o}$ if $\mathcal{E}|_{\mathcal{U}}$ admits a description as a Tate curve. We denote by $\mathcal{F}$ the unique locally free $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module of rank 1 whose module of sections over a small formal affine $\mathcal{U}$ is $\mathcal{F}_{\mathcal{U}}$. Recall that we have an isomorphism of $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-modules:

$$\mathcal{F}/p^{1-v}\mathcal{F} \cong \big(\mathcal{E}[p]/C\big) \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}/p^{1-v}p\mathcal{O}_{\mathfrak{X}_1(Np)(w)}.$$

Let us denote by $\mathcal{F}'_v$ the inverse image under $\mathcal{F} \longrightarrow \big(\mathcal{E}[p]/C\big) \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}/p^{1-v}p\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$ of the constant sheaf (of sets) $\big(\mathcal{E}[p]/C\big) - \{0\}$. If we denote by S_v the sheaf of abelian groups on

$\mathfrak{X}_1(Np)(w)$ defined by

$$S_v := \mathbb{Z}_p^\times \cdot \big(1 + p^{1-v}\mathcal{O}_{\mathfrak{X}_1(Np)(w)}\big),$$

then 2.4 and 2.9 show that $\mathcal{F}'_v$ is an S_v-torsor, locally trivial for the Zariski topology on $\mathfrak{X}_1(Np)(w)$.

We denote by $\mathcal{W}$ the weight space defined in section 1 and let $\kappa \in \mathcal{W}(K)$. We will first assume that there is a pair $(\chi, s) \in \widehat{\mu}_{p-1} \times m_K$ such that $\kappa(t) := \chi([t])\langle t\rangle^s$ where $[t]$ denote the Teichmüler lift in $\mu_{p-1} \subset \mathbb{Z}_p^\times$ of $t(\bmod p)$, $\langle t\rangle := t/[t]$ and we have denoted $\langle t\rangle^s := \exp(s\log(\langle t\rangle))$ Let $w \in \mathbb{Q}$ be such that $0 \le w \le p/(p+1)$ and let us denote $v = w/(p-1)$. If $x = a \cdot b$ is a section of $S_v = \mathbb{Z}_p^\times(1 + p^{1-v}\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$, then we denote by $x^\kappa := \kappa(a) \cdot b^s$ where $b^s := \exp(s\log(b)$. Then x^κ is another section of S_v so we denote by $\mathcal{O}^{(\kappa)}_{\mathfrak{X}_1(Np)(w)}$ the sheaf $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$ with action by S_v twisted by κ.

Definition 3.2. We define the sheaf $\Omega_v^\kappa := \mathfrak{H}om_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big)$ on $\mathfrak{X}_1(Np)(w)$. We denote $\mathcal{M}(\Gamma_1(Np), \kappa, p^w) := H^0\big(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa\big)$ and $\mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K := H^0\big(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa \otimes_{\mathcal{O}_K} K\big) = H^0\big(X_1(Np)(w), \Omega_v^\kappa \otimes_{\mathcal{O}_K} K\big)$.

Since S_v-torsor $\mathcal{F}'_v$ is trivial locally on $\mathfrak{X}_1(Np)(w)$, the sheaf Ω_v^κ is a locally free $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module of rank one.

Remark 3.3. At this point we would like to recall a very useful result, namely proposition 1.7 of [O]: Suppose that $g : T \longrightarrow S$ is an admissible blow-up of formal schemes over $\mathcal{O}_K$. Then g^* and g_* induce equivalences between the category of coherent sheaves of $K \otimes_{\mathcal{O}_K} \mathcal{O}_K$-modules on S (denoted $\mathrm{Coh}(K \otimes \mathcal{O}_S)$ in [O]) and the category of $K \otimes_{\mathcal{O}_K} \mathcal{O}_T$-modules on T (i.e. $\mathrm{Coh}(K \otimes \mathcal{O}_T)$.)

Lemma 3.4. *Take $w' \ge w$ and $v' = w'/(p-1), v = w/(p-1)$. Then, via the morphism $f_{w,w'}\colon \mathfrak{X}_1(Np)(w) \to \mathfrak{X}_1(Np)(w')$ have a natural isomorphism $\rho_{v,v'}\colon f^*_{w,w'}(\Omega^\kappa_{v'}) \cong \Omega^\kappa_{v'}$ of $\mathcal{O}_{\mathfrak{X}_1(Np)(w')}$-modules such that (1) $\rho_{v,v} = \mathrm{Id}$ and (2) for $w'' \ge w' \ge w$ we have $\rho_{v,v'} \circ f^a_{w,w'}st\big(\rho_{v',v''}\big) = \rho_{v,v''}$.*

Proof. Since $v' \ge v$, one has that S_v is a subsheaf of $T_{v'} := \mathbb{Z}_p^\times \cdot \big(1 + p^{1-v'}\mathcal{O}_{\mathfrak{X}_1(Np)(w)}\big)$. Similalry, $\mathcal{F}'_v$ is a subsheaf of sets of $\mathcal{G}'_{v'} :=$, defined as the inverse image under $\mathcal{F} \longrightarrow \big(\mathcal{E}[p]/C\big) \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}/p^{1-v'}p\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$ of $\mathcal{E}[p]/C\backslash\{0\}$. Infact, the action of S_v on $\mathcal{G}'_{v'}$ induced by this inclusion is the same as the action induced by the map $S_v \subset T_{v'}$. In particular, since $\mathcal{F}'_v$ is an S_v-torsor and $\mathcal{G}'_{v'}$ is an $T_{v'}$-torsor, the natural map $\mathcal{F}'_v \subset \mathcal{G}'_{v'}$ induces an isomorphism $\mathcal{F}'_v \times^{S_v} S_{v'} \longrightarrow \mathcal{G}'_{v'}$ of $T_{v'}$-torsors. Here $\mathcal{F}'_v \times^{S_v} T_{v'}$ is the pushed-out torsor. Note that the action of S_v on $\mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}$ extends to an action of $T_{v'}$. By adjunction we then have

$$\mathfrak{H}om_{S_v}\big(\mathcal{F}'_v, \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big) \cong \mathfrak{H}om_{T_{v'}}\big(\mathcal{F}'_v \times^{S_v} T_{v'}, \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big) \cong \mathfrak{H}om_{T_{v'}}\big(\mathcal{G}'_{v'}, \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big).$$

We have a natural morphism of sheaves of groups $f^{-1}_{w,w'}(S_{v'}) \to T_{v'}$ and of sehaves of sets $f^{-1}_{w,w'}(\mathcal{F}_{v'}) \to \mathcal{G}'_{v'}$. The latter is $f^{-1}_{w,w'}(S_{v'})$-equivaraiant so that we have an isomorphism of $T_{v'}$-torsors $f^{-1}_{w,w'}(\mathcal{F}_{v'}) \times^{f^{-1}_{w,w'}(S_{v'})} T_{v'} \cong T_{v'}$. Again by adjunction we deduce that

$$\mathfrak{H}om_{T_{v'}}\big(\mathcal{G}'_{v'}, \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big) \cong \mathfrak{H}om_{f^{-1}_{w,w'}(S_{v'})}\big(f^{-1}_{w,w'}(\mathcal{F}_{v'}), \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big).$$

Since $\Omega^{\kappa}_{v'}$ is a locally free $\mathcal{O}_{\mathfrak{X}_1(Np)(v')}$-module, the natural map

$$f^*_{w,w'}(\Omega^{\kappa}_{v'}) \longrightarrow \mathfrak{Hom}_{f^{-1}_{w,w'}(S_{v'})}\big(f^{-1}_{w,w'}(\mathcal{F}_{v'}), \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(v)}\big)$$

is checked to be an isomorphism. This chain of isomorphisms provides $\rho_{v,v'}$. We leave to the reader to check the equalities (1) and (2). □

Definition 3.5. We define

$$\mathcal{M}(\Gamma_1(Np),\kappa) := \lim_{\to,w} \mathcal{M}(\Gamma_1(Np),\kappa,p^w) \text{ and } \mathcal{M}(\Gamma_1(Np),\kappa)_K := \lim_{\to,w} \mathcal{M}(\Gamma_1(Np),\kappa,p^w)_K.$$

Our goal in this chapter is to show that $\mathcal{M}(\Gamma_1(Np),\kappa)_K$ has natural actions of Hecke operators and that it is canonically and Hecke equivariantly isomorphic to the space of overconvergent modular forms of weight κ, as defined in section 1.

In order to achieve this we'll first exploit the "universality" of the family $\mathcal{E} \longrightarrow \mathfrak{X}_1(Np)(w)$ and give in the next section a different expression (à la N. Katz) of the elements of $\mathcal{M}(\Gamma_1(Np),\kappa,p^w)$ and of $\mathcal{M}(\Gamma_1(Np),\kappa,p^w)_K$.

Let us first recall the overconvergent Eisentsein series of weight 1 and level $\Gamma_1(p)$ denoted E in the introduction. It is uniquely characterized by the properties:

- $E^{p-1} = E_{p-1}$ as sections of $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}$
- The q-expansion of E at ∞, $E(q)$ (given in section 1) is congruent to 1 modulo p.

Let us fix $\mathcal{U} = \mathrm{Spf}(R) \subset \mathfrak{X}_1(Np)(w)$ an affine such that $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}$ has a generator ω. Proposition 5.2 implies that the element δ of proposition 2.6 can be chosen to be $E(\mathcal{E}/R,\omega,\psi) \in R$. Here ψ is the level $\Gamma_1(Np)$-structure of the restriction of $\mathcal{E}$ to $\mathcal{U}$. In particular $\mathcal{F}_{\mathcal{U}}$ is the free R-submodule of $\omega_{\mathcal{E}/\mathcal{U}}$ generated by the differential, which we'll call **standard differential**:

$$\omega^{\mathrm{std}} := E(\mathcal{E}/R,\psi) = E(\mathcal{E}/R,\omega,\psi)\omega, \text{ for every generator } \omega \in \omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}(\mathcal{U}).$$

As $E(\mathcal{E}/\mathfrak{X}_1(Np)(w),\psi)$ is a canonical global section of $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}$, it follows that $\mathcal{F}$ is a free $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-submodule of $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}$. Recall that $\mathcal{F}'_v$ is the inverse image under $\mathcal{F} \longrightarrow \big(\mathcal{E}[p]/C\big) \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}/p^{1-v}\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$ of the constant sheaf (of sets) $\big(\mathcal{E}[p]/C\big) - \{0\}$. Then $\mathcal{F}'$ is an S-torsor, generated by the standard differential ω^{std}. In particular, Ω^{κ}_v admits the following generator $X_{\kappa,v}$. For every $\mathcal{U} = \mathrm{Spf}(R)$ and every $u \in S(\mathcal{U}) = \mathbb{Z}_p^{\times}(1+p^{1-v}R)$, define

$$X_{\kappa,v}(u\omega^{\mathrm{std}}) := u^{-\kappa} \in R.$$

We thus deduce:

Corollary 3.6. *The $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module Ω^{κ}_v is a free $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module with basis element $X_{\kappa,v}$. These elements are compatible via the morphisms $\rho_{v,v'}$ defined in 3.4 i e., $\rho_{v,v'}(X_{\kappa,v'}) = X_{\kappa,v}$.*

3.1 The sheaves Ω^{κ} are universal modular sheaves

In this section we'd like to show how one can express the elements of $\mathcal{M}(\Gamma_1(Np),\kappa,p^w)$ defined in the previous section more concretely.

We fix p,N,K, w and v as in the previous section and we first describe our test objects: they are sequences $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$, where

- R is a p-adically complete, separated and normal $\mathcal{O}_K$-algebra, in which p is not a zero divisor.
- $\mathcal{E} \longrightarrow \mathrm{Spec}(R)$ is an elliptic curve over R such that $\omega_{\mathcal{E}/R}$ is a free R-module of rank 1.
- ψ_N is a $\Gamma_1(N)$-level structure for $\mathcal{E}/R$ and ψ_p is a $\Gamma_1(p)$-level structure of $\mathcal{E}_K/R_K$.
- Y is a global section of $\omega_{\mathcal{E}/R}^{1-p}$ such that $YE_{p-1}(\mathcal{E}/R) = p^w$.

We still have to say what α is. We assume that $(\mathcal{E}/R, \psi_N, \psi_p, Y)$ as above exist and let η be any generator of $\omega_{\mathcal{E}/\mathcal{U}}$. We denote by $T_{\mathcal{E}/R} := S_R \cdot \omega^{\mathrm{std}} \subset \omega_{\mathcal{E}/R}$, where $S_R = \mathbb{Z}_p^\times(1 + p^{1-v}R)$ and $\omega^{\mathrm{std}} := E(\mathcal{E}/R, \eta, \psi_p)\eta \in \omega_{\mathcal{E}/R}$, where $E(\mathcal{E}/R, \eta, \psi_p) = E(\mathcal{E}_K/R_K, \eta, \psi_p) \in R \subset R_K$. We have

Lemma 3.7. *$T_{\mathcal{E}/R}$ is an S_R-torsor functorial both in R and $\mathcal{E}$, i.e. commutes with base change and is preserved by isogenies between elliptic curves, whose kernels do not intersect the canonical subgroup of the domain.*

Proof. By "universality" of $\mathcal{E} \longrightarrow \mathfrak{X}_1(Np)(w)$ (i.e. lemma 3.1) it is enough to prove the lemma if $\mathrm{Spf}(R)$ is an affine open of $\mathfrak{X}_1(Np)(w)$ and $\mathcal{E}$ is the restriction of the universal elliptic curve. In that case the functoriality of $T_{\mathcal{E}/R}$ follows from lemma 2.7.

□

So finally a test object is a quintuple $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$, where $(\mathcal{E}/R, \psi_N, \psi_p, Y)$ have already been defined and $\alpha \in T_{\mathcal{E}/R}$. Then we have the following

Lemma 3.8. *An element of $\mathcal{M}(\Gamma_1(Np), \kappa, p^w)$ is a rule which assigns to every test object $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$, an element $f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y) \in R$ such that:*

- *$f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ only depends on the isomorphism class of $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$.*
- *If $\varphi : R \longrightarrow R'$ is an $\mathcal{O}_K$-algebra homomorphism and we denote by $(\mathcal{E}_\varphi/R', \alpha_\varphi, \psi_{N,\varphi}, \psi_{p,\varphi_K}, Y_\varphi)$ the base change of $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$, then we have $f(\mathcal{E}_\varphi/R', \alpha_\varphi, \psi_{N,\varphi}\psi_{p,\varphi_K}, Y_\varphi) = \varphi\big(f(\mathcal{E}/R, \alpha, \psi, Y)\big)$.*
- *$f(\mathcal{E}/R, u\alpha, \psi_N, \psi_p, Y) = u^{-\kappa} f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ for $u \in S_R$.*

Proof. Everything follows from the "universality" of the family $\mathcal{E} \longrightarrow \mathfrak{X}_1(Np)(w)$ (see lemma 3.1) and the functoriality of Ω_v^κ. □

We define in an analogue way the elements of $\mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K$. More precisely the test objects are sequences $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ as above.

Lemma 3.9. *An element of $\mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K$ can be seen as a rule which assigns to every test object $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ an element $f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y) \in R_K$ satisfying the properties*

- *$f(\mathcal{E}/R, \alpha, \psi, Y)$ only depends on the isomorphism class of $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$.*
- *If $\varphi : R \longrightarrow R'$ is an $\mathcal{O}_K$-algebra homomorphism and we denote by $(\mathcal{E}_\varphi/R', \alpha_\varphi, \psi_{N,\varphi}, \psi_{p,\varphi_K}, Y_\varphi)$ the base change of $(\mathcal{E}/R, \alpha, \psi, Y)$, then we have $f(\mathcal{E}_\varphi/R', \alpha_\varphi, \psi_{N,\varphi}, \psi_{p,\varphi_K}, Y_\varphi) = \varphi_K\big(f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)\big)$.*
- *$f(\mathcal{E}/R, u\alpha, \psi_N, \psi_p, Y) = u^{-\kappa} f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ for $u \in S_R$.*

Proof. Let $f \in \mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K = H^0\big(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa\big) \otimes_{\mathcal{O}_K} K$ and let a test object $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ be as in the statement of lemma 3.9. Then there is a unique morphism $\varphi : \mathrm{Spf}(R) \longrightarrow \mathfrak{X}_1(Np)(w)$ such that $(\mathcal{E}/R, \psi_N, \psi_p, Y)$ are inverse images of the universal ones. It follows that $\alpha = u\omega^{\mathrm{std}} =$

$u\varphi^*(\omega_{\rm univ}^{\rm std})$ as both are expressed in terms of the Eisenstein series E, where $u \in S_R$. Finally $f = aX_{\kappa,v}$, where $a \in H^0(\mathfrak{X}_1(Np)(w), \mathcal{O}_{\mathfrak{X}_1(Np)(w)}) \otimes_{\mathcal{O}_K} K$ and $X_{\kappa,v}$ was defined in corollary 3.6.

We set:

$$f(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y) := u^{-\kappa} X_{\kappa,v}(\omega_{\rm univ}^{\rm std})\varphi_K^*(a) = u^{-\kappa}\varphi_K^*(a).$$

Clearly this rule satisfies the desired properties. We leave the converse as an exercise to the reader. □

3.1.1 Hecke operators

We will define Hecke operators acting on $\mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K$ for $0 \le w \le p/(p+1)$. For this let ℓ be a prime (it may be p) and let $X(N,p,\ell)(w)$ denote the rigid analytic space over K which represents the functor sending a K-rigid analytic space S to a quintuple $(\mathcal{E}/S, \psi_S, C, H, Y)$, where $\mathcal{E} \longrightarrow S$ is an elliptic curve, ψ_S is a $\Gamma_1(Np)$-level structure, $C \subset \mathcal{E}[p]$ is a canonical subgroup and $H \subset \mathcal{E}[\ell]$ is a locally free subgroup scheme, finite of order ℓ such that $C \cap H = \{0\}$ (the last condition is automatic if $\ell \neq p$) and Y is a global section of $\omega_{\mathcal{E}/S}^{1-p}$ such that $YE_{p-1}(\mathcal{E}/S, \psi_S) = p^w$.

We have natural morphisms $p_1 : X(N,p,\ell)(w) \longrightarrow X_1(Np)(w)$ and $p_2 : X(N,p,\ell)(w) \longrightarrow X_1(Np)(w')$ where $w' = w$ if $\ell \neq p$ and $w' = pw$ if $\ell = p$ (in which case we will assume that $0 \le w \le 1/(p+1)$.) These morphisms are defined (on points) as follows: $p_1(\mathcal{E}, \psi, C, H, Y) = (\mathcal{E}, \psi, C, Y) = (\mathcal{E}, \psi, Y) \in X_1(Np)(w)$ and $p_2(\mathcal{E}, \psi, C, H, Y) = (\mathcal{E}/H, \psi', C', Y') = (\mathcal{E}/H, \psi', Y') \in X_1(Np)(w)$ where ψ' and Y' are the induced $\Gamma_1(Np)$-level structure and global section associated to $\mathcal{E}/H$. The morphism p_1 is finite and étale while p_2 is an isomorphism of K-rigid spaces.

We denote by $\mathfrak{X}(N,p,\ell)(w)$ the normalization of $\mathfrak{X}_1(Np)(w)$ in $X(N,p,\ell)(w)$, using p_1, denote by $\mathfrak{p}_1 : \mathfrak{X}(N,p,\ell)(w) \longrightarrow \mathfrak{X}_1(Np)(w)$ the natural induced morphism whose genric fiber is p_1 and by $\mathcal{E} \longrightarrow \mathfrak{X}(N,p,\ell)(w)$ the semiabelian scheme which is pull back via $\mathfrak{p}_1$ of $\mathcal{E} \longrightarrow \mathfrak{X}_1(Np)(w)$. We also consider the morphism $X(N,p,\ell)(w) \longrightarrow \mathfrak{X}_1(Np)(w')$ which is p_2 composed with the natural specialization $X_1(Np)(w) \longrightarrow \mathfrak{X}_1(Np)(w)$ which makes $\mathfrak{X}_1(Np)(w)$ a formal model of $X(N,p,\ell)(w')$.

Lemma 3.10. *Let R be a p-adically complete, separated and normal $\mathcal{O}_K$-algebra. Then there is a bijective correspondence between the set of points $\mathfrak{X}(N,p,\ell)(w)(R) - \{\text{cusps}\}$ and the set of isomorphism classes of sequences: $(\mathcal{E}/R, \psi_N, \psi_p, C, H, Y)$, where $\mathcal{E} \longrightarrow$ Spec(R) is an elliptic curve, ψ_N, C, H, Y are respectively a level $\Gamma_1(N)$-structure, the canonical subgroup, a subgroup scheme of order ℓ of $\mathcal{E}[\ell]$ such that $C \cap H = \{0\}$ and a global section of $\omega_{\mathcal{E}/R}^{1-p}$ such that $YE_{p-1}(\mathcal{E}/R) = p^w$, and ψ_p is a $\Gamma_1(p)$-level structure of $\mathcal{E}_K/R_K$.*

Proof. The proof is very similar to the proof of lemma 3.1 and is left to the reader. □

Let us now observe that we have a natural morphism $\mathfrak{p}_2 : \mathfrak{X}(N,p,\ell)(w') \longrightarrow \mathfrak{X}_1(Np)(w)$ given on points by: $\mathfrak{p}_2(\mathcal{E}, \psi_N, \psi_p, C, H, Y) := (\mathcal{E}/H, \psi_N', \psi_p', C', Y')$ where ψ_N', C', Y' are the induced objects on $\mathcal{E}/H$ and ψ_p' is the $\Gamma_1(p)$-level structure induced on $\mathcal{E}_K/H_K$ by ψ_p. Therefore we have the following natural isogeny $\pi_\ell : \mathcal{E} \longrightarrow \mathcal{E}/H$ over $\mathfrak{X}(N,p,\ell)(w)$, which by the functoriality of the sheaves Ω^κ, v, induces a natural morphism of sheaves on $\mathfrak{X}(N,p,\ell)(w)$:

$$\pi_\ell^\kappa : \mathfrak{p}_2^*(\Omega_w^\kappa) \longrightarrow \mathfrak{p}_1^*(\Omega_{w'}^\kappa).$$

For every $0 \leq w \leq p/(p+1)$ and prime integer ℓ such that $(\ell, Np) = 1$ we define the Hecke operator $T_\ell : \mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K \longrightarrow \mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K$ by $1/(\ell+1)$ multiplied the composition:

$$H^0\big(\mathfrak{X}_1(Np)(w), \Omega_w^\kappa \otimes_{\mathcal{O}_K} K\big) \longrightarrow H^0\big(\mathfrak{X}(N,p,\ell)(w), \mathfrak{p}_2^*(\Omega_w^\kappa) \otimes_{\mathcal{O}_K} K\big) \xrightarrow{\pi_\ell^\kappa}$$

$$\xrightarrow{\pi_\ell^\kappa} H^0\big(\mathfrak{X}(N,p,\ell)(w), \mathfrak{p}_1^*(\Omega_w^\kappa) \otimes_{\mathcal{O}_K} K\big) = H^0\Big(X(N,p,\ell)(w), p_1^*\big(\Omega_w^\kappa\big) \otimes_{\mathcal{O}_K} K\big)\Big) \xrightarrow{\mathrm{Tr}_{p_1}}$$

$$\xrightarrow{\mathrm{Tr}_{p_1}} H^0\big(X_1(Np)(w), \Omega_w^\kappa \otimes_{\mathcal{O}_K} K\big).$$

Moreover, for $0 \leq w \leq 1/(p+1)$ we define the operator $U_p : \mathcal{M}(\Gamma_1(Np), \kappa, p^{pw})_K \longrightarrow \mathcal{M}(\Gamma_1(Np), \kappa, p^{pw})$ to be: $1/p$ multiplied the composition:

$$H^0\big(\mathfrak{X}_1(Np)(pw), \Omega_{pw}^\kappa \otimes_{\mathcal{O}_K} K\big) \xrightarrow{\rho_{pw,w}} H^0\big(\mathfrak{X}_1(Np)(w), \Omega_w^\kappa \otimes_{\mathcal{O}_K} K\big) \longrightarrow$$

$$\longrightarrow H^0\big(\mathfrak{X}(N,p,p)(w), \mathfrak{p}_2^*(\Omega_w^\kappa) \otimes_{\mathcal{O}_K} K\big) \xrightarrow{\pi_p^\kappa} H^0\big(\mathfrak{X}(N,p,p)(pw), \mathfrak{p}_1^*(\Omega_{pw}^\kappa) \otimes_{\mathcal{O}_K} K\big) =$$

$$= H^0(\Big(X(N,p,p)(pw), p_1^*\big(\Omega_w^\kappa\big) \otimes_{\mathcal{O}_K} K\big)\Big) \xrightarrow{\mathrm{Tr}_{p_1}} H^0\big(X_1(Np)(pw), \Omega_{pw}^\kappa \otimes_{\mathcal{O}_K} K\big).$$

As the first restriction is completely continuous, it follows the U_p is completely continuous.

Finally, let us express the Hecke operators defined above in a more concrete (and hopefully more familiar) way. Let ℓ be a prime integer (may be p). Let $f \in \mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K$ and let $(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y)$ be a test object as in section §3.1. First we consider $R \longrightarrow R'$ a normal extension such that $R_K \longrightarrow R'_K$ is finite étale and such that $(\mathcal{E}_K)_{\mathbb{R}'_K}[\ell]$ contains all subgroup-schemes H_K, finite, flat of order ℓ. By theorem 10.6 of [AG] there is an admissible blow-up $\mathrm{Bl} \longrightarrow \mathrm{Spf}(R')$ such that all H_K as above extend to finite flat group-schemes H of order ℓ of $\mathcal{E}_{\mathrm{Bl}}[\ell]$ over Bl.

Then we have

a) If ℓ is a prime integer such that $(\ell, Np) = 1$:

$$(f|T_\ell)(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y) = 1/(\ell+1) \sum_{H \subset \mathcal{E}_{\mathrm{Bl}}[\ell]} f(\mathcal{E}_{\mathrm{Bl}}/\mathrm{Bl}, \alpha' \psi'_N, \psi'_p, Y'),$$

where we mean the following: we cover the scheme Bl (which is not affine) by affine opens $Spec(S)$ on which $\omega_{\mathcal{E}_S/S}$ is free and we glue along intersections the following sum:

$$(*_\ell) \quad 1/(\ell+1) \sum_{H_S \subset \mathcal{E}_S[\ell]} f(\mathcal{E}_S/S, \alpha'_S, \psi_{N,S}, \psi_{p,S}, Y_S)$$

where $\psi_{N,S}, \psi_{p,S}, Y_S$ are base-changes to $\mathcal{E}/S$ of the original objects and $\alpha'_S \in T_{(\mathcal{E}_S/H_S)/S}$ is such that $\pi_S^\kappa(\alpha'_S) = \alpha_S$ for $\pi : \mathcal{E}_S \longrightarrow \mathcal{E}_S/H_S$ the natural isogeny.

Let us remark that every term of the sum $(*_\ell)$ is an element of S_K, therefore each $f(\mathcal{E}_{\mathrm{Bl}}/\mathrm{Bl}, \alpha', \psi'_N, \psi'_p, Y') \in H^0(\mathrm{Bl}, \mathcal{O}_{\mathrm{Bl}}) \otimes_{\mathcal{O}_K} K = R'_K$ by remark 3.3. Therefore, $(f|T_\ell)(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y) \in R_K$ as it should.

b) If $\ell = p$

$$(f|U_p)(\mathcal{E}/R, \alpha, \psi_N, \psi_p, Y) = 1/p \sum_{H \subset \mathcal{E}_{\mathrm{Bl}}[p]} f(\mathcal{E}_{\mathrm{Bl}}/\mathrm{Bl}, \alpha' \psi'_N, \psi'_p, Y'),$$

where the sum runs over all subgroups H such that $H \cap C = \{0\}$, with C the canonical subgroup of $\mathcal{E}_{\rm Bl}[p]$ and the meaning of the notation is as at a) above.

3.1.2 Comparison with integral weight overconvergent modular forms after inverting p

Proposition 3.11. *For every $k \in \mathbb{Z}$ and $0 \le w \le p/(p+1), v = w/(p-1)$ we have a natural isomorphism of sheaves on $X_1(Np)(w)$ $\Omega_v^k \otimes_{\mathcal{O}_K} K \cong \omega^{\otimes k}_{\mathcal{E}_K/X_1(Np)(w)}$. It induces a K-linear isomorphism equivariant with respect to the Hecke operators $\mathcal{M}^\dagger(\Gamma_1(Np), k)_K \cong M^\dagger(\Gamma_1(Np), k)_K$.*

Proof. The isomorphism is defined by: $\Phi_k : \Omega_v^k \otimes_{\mathcal{O}_K} K \longrightarrow \omega^{\otimes k}_{\mathcal{E}_K/X_1(Np)(w)}$ given as follows. If $\mathcal{U}$ is an open of $\mathfrak{X}_1(Np)(w)$ and $f \in \Omega_v^k(\mathcal{U})$ then $\Phi_{k,\mathcal{U}}(f \otimes 1) := f(\omega^{\rm std}) \cdot (\omega^{\rm std})^{\otimes k} \otimes 1 \in \omega^{\otimes k}_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}(\mathcal{U}) \otimes_{\mathcal{O}_K} K$. As $\omega^{\rm std} \otimes 1$ is a generator of $\omega_{\mathcal{E}_K/X_1(Np)(w)}(\mathcal{U}_K)$ (but in general not of $\omega_{\mathcal{E}/\mathfrak{X}_1(Np)(w)}(\mathcal{U})$) the morphism above is an isomorphism of $\mathcal{O}_{X_1(Np)(w)}$-modules. The second statement of the proposition follows from the description of the action of Hecke oeprators on $\mathcal{M}^\dagger(\Gamma_1(Np), k)_K$ and $M^\dagger(\Gamma_1(Np), k)_K$ in the previous section. $\square$

3.1.3 Restriction to the ordinary locus

Let us suppose that $w = v = 0$ and let us denote in this section $\mathfrak{X}_1(Np)(0) =: \mathcal{Z}^{\rm ord}$ and let us remark that $\mathcal{E} \longrightarrow \mathcal{Z}^{\rm ord}$ is ordinary. In this situation $\mathcal{Z}^{\rm ord}$ is a smooth affine formal scheme defined over $\mathbb{Z}_p$, $\mathcal{Z}^{\rm ord} = {\rm Spf}(R)$. Let us recall that in this situation we have a $\mathcal{G} := \pi_1(\mathcal{Z}^{\rm ord}_{\mathbb{Q}_p}, \eta)$-equivariant exact sequence (see section 2)

$$0 \longrightarrow T^{\rm co} \longrightarrow T \longrightarrow T^{\rm et} \longrightarrow 0$$

where $T^{\rm et} \cong \mathbb{Z}_p(\phi)$, for $\phi : \mathcal{G} \longrightarrow \mathbb{Z}_p^\times$ an etale character. If $\kappa \in \mathcal{W}(K)$ the definition in [K1] of the space of p-adic modular forms of weight κ and level $\Gamma_1(Np)$ is:

$$M(R, \kappa, p^0) := \big(\widehat{\overline{R}}(\phi^\kappa)\big)^{\mathcal{G}}.$$

Proposition 3.12. *We have a natural isomorphism of R-modules*

$$\Phi^{\rm ord}_\kappa \mathcal{M}(\Gamma_1(Np), \kappa, p^0) :=: H^0(\mathcal{Z}^{\rm ord}, \Omega_0^\kappa) \cong M(\mathbb{Z}_p, \kappa, p^0).$$

Proof. We have the natural commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega^{-1}_{\mathcal{E}/R} \otimes_R \widehat{\overline{R}}(1) & \longrightarrow & T \otimes \widehat{\overline{R}} & \stackrel{\rm dlog}{\longrightarrow} & \omega_{\mathcal{E}/R} \otimes_R \widehat{\overline{R}} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \uparrow \alpha & & \\ 0 & \longrightarrow & T^{\rm co} \otimes \widehat{\overline{R}} & \longrightarrow & T \otimes \widehat{\overline{R}} & \longrightarrow & T^{\rm et} \otimes \widehat{\overline{R}} & \longrightarrow & 0 \end{array}$$

M The morphism α is an isomorphism so that $F^0 = \omega_{\mathcal{E}/R} \otimes_R \widehat{\overline{R}}$ in this case. Modulo p the morphism α induces the isomorphism $\mathcal{E}[p]^{\rm et} \otimes R \cong \omega_{\mathcal{E}/R} \otimes_R R/pR$. Let G' (resp. F') be the inverse image of $(\mathcal{E}[p]/C) - \{0\}$ under the map $\omega_{\mathcal{E}/R} \otimes \widehat{\overline{R}} \longrightarrow (\mathcal{E}[p]/C) \otimes \overline{R}/p\overline{R}$ (resp. $\omega_{\mathcal{E}/R} \longrightarrow (\mathcal{E}[p]/C) \otimes R$). Then, G' (resp. F') is a torsor under $S^{\rm ord} = \mathbb{Z}_p^\times(1 + p\widehat{\overline{R}})$ (resp. $S = \mathbb{Z}_p^\times(1 + pR)$)

and the inclusion $F' \subset G'$ induces the isomorphism $G' \cong F' \times^S S^{\rm ord}$, where the latter is the push-out torsor. Let $\widehat{\widehat{R}}^{(-\kappa)}$ (resp $R^{(-\kappa)}$) be $\widehat{\widehat{R}}$ (resp. R) with the action by $S^{\rm ord}$ (resp. S) twisted by $-\kappa$. Then, $\big(S^{\rm ord}\big)^{\mathcal{G}} = S$ and $\big(\widehat{\widehat{R}}^{(-\kappa)}\big)^{\mathcal{G}} = R^{(-\kappa)}$. Let us recall that $\Omega^\kappa(\mathcal{U}) = {\rm Hom}_S(G', R^{(-\kappa)})$ so that

$$\Omega^\kappa(\mathcal{U}) = \big({\rm Hom}_S(G', \widehat{\widehat{R}}^{(-\kappa)})\big)^{\mathcal{G}} = \big({\rm Hom}_{S^{\rm ord}}(F' \times^S S^{\rm ord}, \widehat{\widehat{R}}^{(-\kappa)})\big)^{\mathcal{G}} = {\rm Hom}_{S^{\rm ord},\mathcal{G}}\big(G', \widehat{\widehat{R}}^{(-\kappa)}\big).$$

The isomorphism $\alpha\colon T^{\rm et} \otimes \widehat{\widehat{R}} = \widehat{\widehat{R}}(\phi) \longrightarrow \omega_{\mathcal{E}/R} \otimes_R \widehat{\widehat{R}}$ gives $\alpha^{-1}(G') \cong S^{\rm ord}$, where the $\mathcal{G}$-action is defined by: if $\sigma \in \mathcal{G}$, $y \in S^{\rm ord}$ then $\sigma * y = \sigma(y)\phi(\sigma) \in S^{\rm ord}$. Therefore α induces an isomorphism

$$\Omega^\kappa(\mathcal{U}) \cong {\rm Hom}_{S^{\rm ord},\mathcal{G}}(S^{\rm ord}, \widehat{\widehat{R}}^{-\kappa}) = \big({\rm Hom}_{S^{\rm ord}}(S^{\rm ord}, \widehat{\widehat{R}}^{-\kappa}\big)^{\mathcal{G}}.$$

Let us observe that ${\rm Hom}_{S^{\rm ord}}(S^{\rm ord}, \widehat{\widehat{R}}^{(-\kappa)}) \cong \widehat{\widehat{R}}$, as $\widehat{\widehat{R}}$-modules, and the $\mathcal{G}$-action is given as follows. Let $\sigma \in \mathcal{G}$ and $g : S^{\rm ord} \longrightarrow \widehat{\widehat{R}}^{(-\kappa)}$ be an $S^{\rm ord}$-morphism. Then $(\sigma g)(u) := \sigma(g(\sigma^{-1}(u))$, in particular $(\sigma g)(1) = \sigma(g(\sigma^{-1}(1))) = \sigma(g(\phi(\sigma)^{-1} \cdot 1)) = \phi(\sigma)^\kappa g(1)$.

In other words, ${\rm Hom}_{S^{\rm ord}}(S^{\rm ord}, \widehat{\widehat{R}}^{(-\kappa)}) \cong \widehat{\widehat{R}}(\phi^\kappa)$ as $\mathcal{G}$-modules and therefore α induces an isomorphism

$$\Psi^{\rm ord}\colon \mathcal{M}(\Gamma_1(Np), \kappa, p^0) \cong M(R, \kappa, p^0).$$

□

3.1.4 q-Expansions of overconvergent modular forms

Definition 3.13. Let us fix $N, p \geq 5, w$ as in section §3.1.2 and suppose K is such that contains an element of valuation $v = w/(p-1)$ and a primitive root of unity of order Np. Let $f \in H^0(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa$. Then we define its q-expansion at the cusp ψ, denoted $f(q,\psi)$ as $f({\rm Tate}(q^{Np}), \omega_{\rm can} = dz/z, \psi) \in \mathcal{O}_K[\![q]\!]$. Let us recall that because $E({\rm Tate}(q^{Np}), \omega_{\rm can}, \psi) \in 1 + p\mathcal{O}_K[\![q]\!] \subset S_{\mathcal{O}_K[\![q]\!]}$, we have $\omega_{\rm can} \in T_{{\rm Tate}(q^{Np})/\mathcal{O}_K[\![q]\!]}$.

Example 3.14. *1) Let us recall the overconvergent modular form* $X_{\kappa,v} \in H^0(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa)$ *defined in section 3 and let us calculate its q-expansion. We have:*

$$X_{\kappa,v}(q,\psi) = X_\kappa({\rm Tate}(q^{Np}), \omega_{\rm can}, \psi) = X_\kappa({\rm Tate}(q^{Np}), E({\rm Tate}(q^{Np}), \omega_{\rm can}, \psi)^{-1}\omega_{\rm can}, \psi) =$$

$$= \big(E({\rm Tate}(q^{Np}), \omega_{\rm can}, \psi)\big)^\kappa = \big(E(q,\psi)\big)^\kappa.$$

2) Let $f \in H^0(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa)$, *let us denote by* $f^{\rm ord} = f|_{\mathcal{Z}^{\rm ord}}$ *and let* $g = \Psi^{\rm ord}(f^{\rm ord}) \in M(\mathcal{O}_K, p^0, \kappa)$. *Then the same calculation as above shows that*

$$f(q,\psi) = f^{\rm ord}({\rm Tate}(q^{Np}), \omega_{\rm can}, \psi) = g({\rm Tate}(q^{Np}), \omega_{\rm can}, \psi) = g(q,\psi).$$

Theorem 3.15. *Let* $E_\kappa \in M(\mathcal{O}_K, p^0, \kappa)$ *denote the normalized p-adic Eisenstein series of weight* κ *whose q-expansion at the cusp* ∞ *was given in section 1 and let* $Z_\kappa^{\rm ord} = (\Psi^{\rm ord})^{-1}(E_\kappa) \in \Omega^\kappa(\mathcal{Z}^{\rm ord})$. *Then there is a unique section* $Z_\kappa \in H^0(\mathfrak{X}_1(Np)(w), \Omega^\kappa(w))$ *such that* $Z_\kappa|_{\mathcal{Z}^{\rm ord}} = Z_\kappa^{\rm ord}$. *Moreover* Z_κ *generates* $H^0(\mathfrak{X}_1(Np)(w), \Omega_v^\kappa) \otimes_{\mathcal{O}_K} K$.

Proof. Let us denote by $X_\kappa^{\rm ord} := X_\kappa|_{\mathcal{Z}^{\rm ord}}$. Then by example 3.14 and [C1], $h^{\rm ord} := Z_\kappa^{\rm ord}/X_\kappa^{\rm ord} \in R = \mathcal{O}_{\mathcal{Z}^{\rm ord}}(\mathcal{Z}^{\rm ord})$ has the property that there is a unique $h \in \mathcal{O}_{\mathcal{Z}}(\mathcal{Z})$ such that $h|_{\mathcal{Z}^{\rm ord}} = h^{\rm ord}$. It is enough to denote by $Z_\kappa := hX_\kappa \in \Omega^\kappa(\mathcal{Z})$ and observe that it has the desired property. □

An immediate consequence of theorem 3.15 is the following result announced in section 1.

Corollary 3.16. *We have a natural K-linear, Hecke-equivariant isomorphism,*

$$\Phi_\kappa : \mathcal{M}^\dagger(\Gamma_1(Np), \kappa)_K \cong M^\dagger(\Gamma_1(Np), \kappa)_K.$$

Moreover, if $\kappa = k$ is an integer then Φ_k coincides with the one defined in the proposition 3.11.

Proof. It is enough to show that we have natural, compatible isomorphisms for every $0 \leq w \leq p/(p+1)$: $\mathcal{M}(\Gamma_1(Np), \kappa, p^w)_K \cong M(\Gamma_1(Np), \kappa, p^w)_K$. Indeed, given the definition of $M(\Gamma_1(Np), \kappa, p^w)_K$ in section 1 (following [C1]) the isomorphism is defined by sending Z_kappa (defined in theorem 3.15 to E_κ. Using the description of the action of the Hecke operators on $M^\dagger(\Gamma_1(Np), \kappa)_K$ in [C1], it follows easyly that Φ_κ thus defined is Hecke-equivariant. □

3.2 The case of a general κ

In general not all $\kappa \in \mathcal{W}(K)$ are associated to a pair $(\chi, s) \in \widehat{\mu}_{p-1} \times m_K$ as above. However for a general $\kappa \in \mathcal{W}(K)$ there is $s \in m_K$, $r \in \mathbb{N}$ and a character α of $\mathbb{Z}_p^\times$ of finite order such that for all $t \in \mathbb{Z}_p^\times$ with $v(\langle t\rangle - 1) > r$ we have $\kappa(t) = \alpha(t) \cdot \langle t\rangle^s$. For $r \in \mathbb{N}$ as above let $w \in \mathbb{Q}$, $0 < w < p/(p+1)$ be such that for every point of $X_1(Np)(w)(\mathbb{C}_p)$ the corresponding elliptic curve has a canonical subgroup of order $r+1$. Then proposition 2.4, proposition 2.6 and corollary 2.9 can be redone using these canonical subgroups to providea a torsor $\mathcal{F}'_{r,w}$ on $\mathfrak{X}_1(Np)(w)$ for the sheaf of groups $S_{r,w} := \mathbb{Z}_p^\times\big(1 + p^{r+1-v}\mathcal{O}_{\mathfrak{X}_1(Np)(w)}\big)$.

Moreover, if $x = a \cdot b$ is a section of $S_{r,w}$ we denote by $x^\kappa := \kappa(a) \cdot b^s$ which is again a section of $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$. We denote by $\mathcal{O}^{(\kappa)}_{\mathfrak{X}_1(Np)(w)}$ the sheaf $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$ with the action by the sheaf $S_{r,w}$ twisted by κ.

We define

$$\Omega^\kappa_w := \mathfrak{Hom}_{S_{r,w}}\big(\mathcal{F}'_{r,w}, \mathcal{O}^{(-\kappa)}_{\mathfrak{X}_1(Np)(w)}\big).$$

This is a locally free $\mathcal{O}_{\mathfrak{X}_1(Np)(w)}$-module on $\mathfrak{X}_1(Np)(w)$ of rank 1. We denote by $\mathcal{M}(\Gamma_1(Np), \kappa) := H^0(\mathfrak{X}_1(Np)(w), \Omega^\kappa_w)$ and by $\mathcal{M}(\Gamma_1(Np), \kappa)_K := H^0(X_1(Np)(w), \Omega^\kappa_{w,K})$.

Everething that was done for a weight κ associated to a pair (χ, s) as above can be redone for an arbitrary weight κ, under the restriction that the degree of overconvergence $w > 0$ depends on κ.

3.3 The sheaves $\Omega_{r,w}$

Let us fix an integer $r > 0$, let $\mathcal{W}_r \subset \mathcal{W}$ be the respective affinoid as defined in section 1 and $A_r = A(\mathcal{W}_r)$ be the affinoid algebra of $\mathcal{W}_r$. We have a continuous universal character $\psi_r : \mathbb{Z}_p^\times \longrightarrow A_r^\times$ defined by: $\psi_r(t)(\kappa) := \kappa(t)$, for $t \in \mathbb{Z}_p^\times$ and $\kappa \in \mathcal{W}_r$.

Let $0 < w < p/(p+1)$ be a rational number such that for every point of $X_1(Np)(w)$ the corresponding elliptic curve has a canonical subgroup of order $r+1$.

Now we consider the natural projection $\mathcal{W}_r \times \mathfrak{X}_1(Np)(w) \longrightarrow \mathfrak{X}_1(Np)(w)$ and the inverse images under it of the sheaf of groups $S_{r,w} := Z_p^\times(1 + p^{r+1-v}\mathcal{O}_{\mathfrak{X}_1(Np)(w)})$ and of the $S_{r,w}$-torsor $\mathcal{F}'_{r,w}$ on $\mathfrak{X}_1(Np)(w)$, which we denote by the same names.

Now we'd like to show that we have an action of $S_{r,w}$ on $\mathcal{O}_{\mathcal{W}_r} \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}$. Let $x = a \cdot b$ be a section of $S_{r,w}$ and $A \otimes B$ a section of $\mathcal{O}_{\mathcal{W}_r} \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}$.

We define $x \cdot (A \otimes B)$ as the map which sends $\kappa \in \mathcal{W}_r$ to

$$\big(x \cdot (A \otimes B)\big)(\kappa) := \kappa(a)A(\kappa) \cdot b^\kappa B,$$

where b^κ was defined in section 3.2. This is clearly analytic in κ therefore defines a section of $\mathcal{O}_{\mathcal{W}_r} \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}$.

Now we define the sheaf

$$\Omega_{r,w} := \mathfrak{Hom}_{S_{r,w}}\big(\mathcal{F}'_{r,w}, \mathcal{O}_{\mathcal{W}_r} \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)}\big).$$

As $\mathcal{F}'_{r,s}$ is an $S_{r,w}$-torsor it follows that $\Omega_{r,s}$ is a locally free (in fact free) $\mathcal{O}_{W_r} \otimes \mathcal{O}_{\mathfrak{X}_1(Np)(w)})$-module of rank 1.

4 The sheaves Ω^κ for Hilbert modular forms

Let F denote a totally real number field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$ and different ideal $\mathcal{D}_F$. Fix an integer $N \geq 4$, a prime $p \geq 3$ and a rational numer $0 \leq w \leq 1/p$ and assume that p is unramified in F and does not divide N. Define the weight space for this setting as $\mathcal{W}(K) := \mathrm{Hom}_{\mathrm{cont}}\big((\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^*, K^*\big)$.

Let $\mathfrak{M}(\mathcal{O}_K, \mu_N)$ be a toroidal compactification of the Hilbert modular scheme over $\mathcal{O}_K$ classifying abelian schemes $A \to S$ of relative dimension g over $\mathcal{O}_K$, together with an emebdding $\iota\colon \mathcal{O}_F \subset \mathrm{End}_S(A)$, a closed immersion $\Psi\colon \mu_N \otimes \mathcal{D}_F^{-1} \to A$ compatible with $\mathcal{O}_F$-actions and satisfying Rapoport condition that $\omega_{A/S}$ is a locally free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$-module of rank 1. See [AG, §3] for details. It is a smooth scheme over $\mathcal{O}_K$ and it admits a universal semiabelian scheme with real mulpiplication A. Let $\mathfrak{M}(\mathcal{O}_K, \mu_N)(w)$ be the formal scheme defining the strict neighborhood of the ordinary locus of $\mathfrak{M}(\mathcal{O}_K, \mu_N)_k$ of width p^w. Then, the canonical subgroup H_K exists in the universal abelian scheme over the rigid analytic fiber $\mathfrak{M}(\mathcal{O}_K, \mu_N)(w)_K$.

Lemma 4.1. *The group scheme H_K is isomorphic to the constant group scheme $\mathcal{O}_F/p\mathcal{O}_F$ over a finite and ètale covering $Z(w)_K \to \mathfrak{M}(\mathcal{O}_K, \mu_N)(w)_K$.*

Proof. We first consider the finite and étale cover Z' where H_K becomes constant. We have to prove that finite ètale locally over Z' such group is isomorphic fo $\mathcal{O}_F/p\mathcal{O}_F$. It suffices to prove it for every point of Z'. For points speciis is clear Let $K \subset L$ be a finite extension and let $\mathbb{F}$ be its residue field and consider an L-valued point x of Z'. If x specializes to a cusp then the canonical subgroup of the underlying abelian variety over L is $\mu_p \otimes \mathcal{O}_F$ and the claim is clear. If not, the pull-back A_x of A to the ring of integers of L is an abelian scheme. Then, $A_{x,K}$ admits a canonical subgroup $H_{x,K}$ which is a constant group. In particular, it is a $\mathcal{O}_F/p\mathcal{O}_F$-module, of dimension p^g as $\mathbb{F}_p$-vector space. We need to prove that it is a free $\mathcal{O}_F/p\mathcal{O}_F$-module. It suffices to show that given a non-trivial element e of $\mathcal{O}_F/p\mathcal{O}_F$ then e does not annihilate $H_{x,K}$. We let $H_x \subset A_x$ be the schematic closure of $H_{x,K}$ in A_x. Its special fiber $H_{x,\mathbb{F}}$ coincides with the kernel of Frobenius on $A_{x,\mathbb{F}}$. In particular, its module of invariant differentials coincide with $\omega_{A_{x,\mathbb{F}}/\mathbb{F}}$ which is a free $\mathcal{O}_F\otimes\mathbb{F}$-module so that e does not annihilate it. Then, e does not annihilate $H_{x,K}$ either. $\square$

Write $Z(w)$ for the normalization of $\mathfrak{M}(\mathcal{O}_K,\mu_N)(w)$ in $Z(w)_K$. Assuming that $\mathcal{O}_K$ contains a primitive p-th root of 1, it follows from 4.1 that also have that the cartier dual C_K^vee is a free $\mathcal{O}_F/p\mathcal{O}_F$-module of rank 1. Choose $\mathcal{O}_F/p\mathcal{O}_F$-generator e. It coincides with $A^\vee[p]/D$ over $Z(w)_K$ in the notations of 2.6. Since $F_0/p^{1-v}F_0\cong\big(A^\vee[p]/D\big)\otimes\mathcal{O}_{\mathcal{Z}(w)}/p^{1-v}\mathcal{O}_{\mathcal{Z}(w)}$ as $\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{Z}(w)}/p^{1-v}\mathcal{O}_{\mathcal{Z}(w)}$-modules, it follows that F_0 is a locally free $\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{Z}(w)}$-module of rank 1. Define $\mathcal{F}'_v$ as the inverse image of $\big(A^\vee[p]/D\backslash\{0\}$ in F_0. It then a torsor under the sheaf of groups $S_v:=\big(\mathcal{O}_F\otimes\mathbb{Z}_p\big)^*\cdot\big(1+p^{1-v}\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{Z}(w)}\big)$.

Definition 4.2. We define, for every $\kappa\in\mathcal{W}(K)$ a sheaf of $\mathcal{O}_{\mathcal{Z}(w)}$-modules on $\mathcal{Z}(w)$ by

$$\Omega_v^\kappa:=\mathfrak{H}om_S\big(\mathcal{F}'_v,\mathcal{O}_{\mathcal{Z}(w)}^{(-\kappa)}\big).$$

where $\mathcal{O}_{\mathcal{Z}(w)}^{(-\kappa)}$ is $\mathcal{O}_{\mathcal{Z}(w)}$ with action of S twisted by $-\kappa$.

Define $\Gamma\big(\mathcal{Z}(w),\Omega_v^\kappa\big)$ to be the Hilbert overconvergent modular forms of level $\Gamma_1(Np)$, weight κ and degree of overconvergence p^w.

We state several properties which are proven as in the elliptic case

(1) the sheaf Ω_v^κ is a locally free $\mathcal{O}_{\mathcal{Z}(w)}$-module of rank one;

(2) for $w'\geq w$ we have a natural isomorphism $\rho_{v,v'}\colon f_{w,w'}^*(\Omega_{v'}^\kappa)\cong\Omega_{v'}^\kappa$ of $\mathcal{O}_{\mathcal{Z}(w')}$-modules which is the identity for $w;=w$ and satisfies the usual cocycle condition for $w''\geq w'\geq w$. See 3.4.

Note that $\mathfrak{M}(\mathcal{O}_K,\mu_N)(0)$ is the ordinary locus. Let $\mathfrak{M}\big(\mathcal{O}_K,\mu_{Np^\infty}\big)$ be the formal affine scheme defined by the Igusa tower over $\mathfrak{M}(\mathcal{O}_K,\mu_N)(0)$: it classifies ordinary abelian schemes with real multplication by cO_F and a μ_{Np^∞} structure and it is Galois with group $(\mathcal{O}_F\otimes\mathbb{Z}_p)^*$ over $\mathfrak{M}(\mathcal{O}_K,\mu_N)(0)$. Let $\mathcal{G}$ be the Galois group of $\mathfrak{M}\big(\mathcal{O}_K,\mu_{Np^\infty}\big)$ over $\mathfrak{M}\big(\mathcal{O}_K,\mu_{Np}\big)$. Following [K2, §1.9], we define the p-adic modular forms à la Katz of level $\Gamma_1(Np)$ and weight κ to be the space of eigenfunctions on $\mathfrak{M}\big(\mathcal{O}_K,\mu_{Np^\infty}\big)$ on which $\mathcal{G}$ acts via the character κ. Then, as in 3.12 one proves:

(3) $\Gamma\big(\mathcal{Z}(0),\Omega_0^\kappa\big)$ is isomorphic to the space of Katz's p-adic modular forms of level $\Gamma_1(Np)$ and weight κ.

Given fractional ideal $\mathfrak{A}$ and $\mathfrak{B}$ of $\mathcal{O}_F$ and a notion of positivity on $\mathfrak{A}\mathfrak{B}$, one can construct a Tate object Tate$(\mathfrak{A},\mathfrak{B})$. It admits a canonical generator of the differentials and a canonical μ_{pN}-level structure. Then,

(4) evaluation at the Tate object Tate$(\mathfrak{A},\mathfrak{B})$ provides a q-expansion map

$$\Gamma\big(\mathcal{Z}(w),\Omega_v^\kappa\big)\longrightarrow\mathcal{O}_K\big[\big[q^\alpha|\alpha\in\mathfrak{A}\mathfrak{B}^+\cup\{0\}\big]\big].$$

5 Appendix: The map dlog

Congruence group schemes: Assume that K contains a p–th root of 1, let R be a p-adically complete and separated flat $\mathcal{O}_K$-algebra which is an integral domain and let $\lambda\in R$ such that $\lambda^{p-1}\in pR$.

Let $G_\lambda=\mathrm{Spec}(A_\lambda)$, with $A_\lambda=R[T]/\big(P_\lambda(T)\big)$, be the finite and flat group scheme over R as in [AG, Def. 5.1]. Here, $P_\lambda(T):=\frac{(1+\lambda T)^p-1}{\lambda^p}$ and the group scheme structure is given as follows. The comultpilication is $T\mapsto T\otimes 1+1\otimes T+\lambda T\otimes T$, the counit $T\mapsto 0$, the coinverse by $T\mapsto -T(1+\lambda T)^{-1}$. This makes sense since A_λ is p-adically complete and separated so that $1+\lambda T$ is a unit in A_λ.

Homomorphisms between congruence group schemes: If μ is ana element of R dividing λ the $\mathbb{F}_p$-vector space $\mathrm{Hom}_R(G_\lambda,G_\mu)$ is of dimension 1 generated by the map $\eta_{\lambda,\mu}\colon G_\lambda\to\mu_p$ sending $Z\mapsto 1+\lambda\mu^{-1}T$; see [AG, §5.3]. If $\nu\in R$ divides μ one has $\eta_{\nu,\mu}\circ\eta_{\lambda,\mu}=\eta_{\lambda,\nu}$. In particular, if μ is a unit then there is a canonical isomorphism $G_\mu\cong\mu_p$ by [AG, Ex. 5.2(b)] and we put $\eta_\lambda:=\eta_{\lambda,\mu}$. It follows that $\mathrm{Hom}_R(G_\lambda,G_\mu)=0$ if μ does not divide λ.

Relation to Oort-Tate theory: In terms of Oort–Tate theory G_λ corresponds to the group scheme $G_{(a,c)}=\mathrm{Spec}\big(R[Y]/(Y^p-aY)\big)$ where $ac=p$ and $c=c(\lambda)=\lambda^{p-1}(1-p)^{p-1}w_{p-1}^{-1}$. There is a canonical isomorphism $G_{(a,c)}\cong G_\lambda$ sending $T\mapsto\sum_{i=1}^{p-1}\lambda^{i-1}\frac{Y^i}{w_i}$ where $w_1,\ldots,w_{p-1}$ are the universal constants of Oort-Tate; see [AG, §5.4]. We remark for later purposes that $a=p\lambda^{1-p}$ up to unit so that $a=0$ modulo $p\lambda^{1-p}$. Recall also that $w_i\equiv i!$ modulo p for $i=1,\ldots,p-1$ so that $T\mapsto\sum_{i=1}^{p-1}\lambda^{i-1}\frac{Y^i}{i!}$ modulo p. In particular,

$$1+\lambda T\mapsto\sum_{i=0}^{p-1}\lambda^{i-1}\frac{Y^i}{i!}$$

and at the level of differentials we have $dT\mapsto\sum_{i=1}^{p-1}\lambda^{i-1}\frac{Y^{i-1}}{(i-1)!}dY=\beta dY$ with $\beta=\sum_{i=1}^{p-2}\lambda^{i-1}\frac{Y^i}{i!}$. In particular, $\beta=1$ modulo λY and $(1+\lambda T)^{-1}dT\mapsto\big(1-\lambda^{p-1}Y^{p-1}\big)dY$ modulo $\lambda^{p-1}\lambda Y^p=\lambda^p aY$ which is a multiple of λp and hence is 0 modulo p.

Differentials: Since $P_\lambda(T):=\frac{(1+\lambda T)^p-1}{\lambda^p}$ the derivative of $P_\lambda(T)$ is $p\lambda^{1-p}(1+\lambda T)^{p-1}$ which is a up to unit. Hence, we have $\Omega_{G_\lambda/R}\cong\lambda dT/aA_\lambda dT$ with $a=0$ modulo $p\lambda^{1-p}$. In this case, In particular, $\Omega_{G_\lambda/R}$ is free of rank 1 as $A_\lambda/(p\lambda^{1-p})$-module so that also the invariant differentials $\omega_{G_\lambda/R}$ of G_λ is a free $R/(p\lambda^{1-p})$-module of rank 1. The image of the invariant differential of μ_p via η_λ is then $\lambda(1+\lambda T)^{-1}dT$ i.e.,

$$\mathrm{dlog}\colon G_\lambda^\vee\longrightarrow\omega_{G_\lambda},\qquad \eta_\lambda\mapsto\lambda(1+\lambda T)^{-1}.$$

Let p be a prime number ≥ 3. Let $0\leq w\leq\frac{1}{p}$ be a rational number. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and containing the p-roots of unity. We fix such a root ζ_p so that over $\mathcal{O}_K$ we have a canonical homomorphism of group schemes $\mathbf{Z}/p\mathbf{Z}\to\mu_p$, sending $1\mapsto\zeta_p$, which is an isomorphism over K. Normalize the induced discrete valuation on K so that p has valuation 1. Assume that there exists an element $p^w\in\mathcal{O}_K$ of normalized valuation w. Let R be a normal and flat $\mathcal{O}_K$-algebra, which is an integral domain and it is p-adically complete and separated. Let $\pi\colon A\longrightarrow\mathcal{U}:=\mathrm{Spec}(R)$ be an abelian scheme of relative dimension $g\geq 1$. Assume that the determinant ideal of the Frobenius $\varphi\colon R^1\pi_*(A_1)\longrightarrow R^1\pi_*(A_1)$ contains p^w. Then, it follows by the main theorem [AG, Thm. 3.5] that A over $\mathcal{U}_K$ admits a canonical subgroup C. Let $D\subset A[p]_K^\vee\cong A^\vee[p]_K$ be the Cartier dual of $A[p]_K/C$ over $\mathcal{U}_K$. Then,

Proposition 5.1. *(1) The map* $\mathrm{dlog}\colon A^\vee[p]_K \longrightarrow \omega_{A/R}\otimes_R \overline{R}/p\overline{R}$ *has* D *as kernel.*

(2) The cokernel of the $\overline{R}$*-linear extension* $A^\vee[p]_K\otimes\overline{R}/p\overline{R}\longrightarrow \omega_{A/R}\otimes_R\overline{R}/p\overline{R}$ *of* dlog *is annihilated by* $p^{\frac{w}{p-1}}$*.*

Proof. To prove (1) it suffices to prove that for every S-valued point x of $A^\vee[p]$, where $S=\mathrm{Spec}(R')$ and R' is a finite normal extension of R, we have $x_K\in D(S_K)$ if and only if $x_K\in\mathrm{Ker}\big(\mathrm{dlog}\big)$. Replacing R' with the completion at its prime ideals above p, we may assume that R' is a complete dvr. Passing to a faithfully flat extension we may further replace R' with its normalization $\overline{R}'$ in an algebraic closure of the fraction field of R'. Replace S with $S:=\mathrm{Spec}(\overline{R}')$.

To prove (2) we remark that $A^\vee[p]_K\otimes\overline{R}/p\overline{R}$ and $\omega_{A/R}\otimes_R\overline{R}/p\overline{R}$ are free $\overline{R}/p\overline{R}$-modules of the same rank. Take the determinant of the $\overline{R}$-linear extension of dlog and call it $d\in\overline{R}/p\overline{R}$. Then, d annihilates the cokernel. We may assume that $d\in R'/pR'$ for some $R\subset R'$ finite and normal and $R'\subset\overline{R}$. Since R' is normal, to prove that $d=\alpha p^{\frac{gw}{p-1}}$ for some $\alpha\in\mathbb{R}'$ it suffices to show that this holds after localizing at the prime ideals of R' over p. As before, passing to a faithfully flat extension we may further replace R' with its normalization $\overline{R}'$ in an algebraic closure of the fraction field of R'.

As proven in [AG, prop. 13.5] the map dlog modulo p can also be defined in terms of torsors and corresponds to the following map $rmdLog\colon A^\vee(S)=\mathrm{H}^1_{\mathrm{fppf}}(A_S,\mu_p)\to\omega_{A_S/S}/p\omega_{A_S/S}$. A μ_p-torsor over A_{S_K} extends to a μ_p-torsor $Y\to A_S$ is defined by giving a Zariski affine cover $\mathcal{U}_i$ of A_S and units u_i in $\Gamma(U_i,\mathcal{O}_{U_i})$ so that the $Y|_{U_i}$ is defined by the equation $Z_i^p-u_i$. Then, $rmdLog(Y)$ is defined by $u_i^{-1}du_i\in\Gamma(U_i,\Omega^i_{U_i/S})/(p)$ which one verifies defines a global section of $\omega_{A_S/S}/p\omega_{\mathcal{E}_S/S}$. Let $\lambda\in\overline{R}'$ be an element of valuation $\frac{1-w}{p-1}$. It follows from [AG, Prop. 13.4] and [AG, Prop. 12.1] that the kernel of $rmdLog$ has dimension g and is isomorphic to $\mathrm{H}^1_{\mathrm{fppf}}(A_S,G_\lambda)$. Note that $\mathrm{H}^1_{\mathrm{fppf}}(A_S,G_\lambda)\cong\mathrm{Hom}_S\big(G_\lambda^\vee,A_S^\vee\big)$ by Cartier duality [AG, §5.12] so that we get a map $\Psi\colon G_\lambda^{\vee,g}\longrightarrow A_S^\vee$ which is a closed immersion after inverting p. Let $D\subset A_S^\vee$ be the schematic closure of Ψ_K. Then, by [AG, Def. 12.4] it is the Cartier dual of A_S/C. This concludes the proof of (1).

(2) Consider on $E=A_S^\vee/D$ an increasing filtration by g finite and flat subgroup schemes Fil^iE such that $E_i=\mathrm{Fil}^{i+1}E/\mathrm{Fil}^iE$ is of order p. Such filtration exists over K with $E_{i,K}\cong\mathbb{Z}/p\mathbb{Z}$ since $E_K\cong(\mathbb{Z}/p\mathbb{Z})^g$. One then defines $\mathrm{Fil}^1E=E_1$ to be the schematic closure of $E_{1,K}$ in E and this is a finite and flat subgroup scheme of E sing R' is a dvr. One lets E_{i+1} be the schematic closure of $E_{i+1,K}$ in E/E_i and one puts Fil^iE to be the inverse image of E_{i+1} via the quotient map $E\to E/E_i$. In particular, $E_i\cong G_{\lambda_i}^\vee$ for some $\lambda_i\in R$. The invariant differentials $\omega_{G_{\lambda_i}}$ of G_{λ_i} define a free rank 1 module over $\overline{R}'/\big(p\lambda_i^{1-p}\big)$.

It follows from [Fa] that D is the canonical subgroup of $A^\vee$ and that $\prod_{i=1}^g\lambda_i^{1-p}$ divides p^w. Hence, each $\omega_{G_{\lambda_i}}/(p^{1-w})$ is a free $\overline{R}'/p^{1-w}\overline{R}'$-module. The image of the map $\mathrm{dlog}\colon G_{\lambda_i}^\vee\otimes\overline{R}'\longrightarrow\omega_{G_{\lambda_i}}$ is generated by λ_i so that its cokernel is annihilated by λ_i. Since $E^\vee\subset A_S$ is a closed immersion, the invariant differentials $\omega_{E^\vee}$ of $E^\vee$ modulo p^{1-w} are a quotient of the invariant differentials of A_S which is a free $\overline{R}'$-module of rank g. Note that $\omega_{E^\vee}/(p^{1-w})\cong\omega_{A_S}/(p^{1-w})$ admits a filtration, with graded pieces $\omega_{E_i^\vee}/(p^{1-w})$, compatible with the given filtration on E. Due to the functoriality of dlog the map $\mathrm{dlog}\colon E\otimes\overline{R}'\longrightarrow\omega_{E^\vee}/(p^{1-w})$ preserves the filtrations and we conclude that $\prod_{i=1}^g\lambda_i$ annihilates its image. In particular, $p^{\frac{w}{p-1}}$ annihilates the cokernel as claimed. □

We now pass to the case of elliptic curves. Let N be an integer prime to p. Let $\pi\colon \mathcal{E} \to Y_1(N,p)$ be the universal relative elliptic curve where $Y_1(N,p)$ is the modular curve associated to the congruence subgroup $\Gamma_1(N)\cap\Gamma_0(p)$. Write $\mathcal{E}_1$ for the mod p reduction of $\mathcal{E}$. Let $F\colon \mathcal{E}_1 \to \mathcal{E}_1^{(p)}$ be the Frobenius isogeny and let

$$\varphi\colon \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1} \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1}$$

be the induced σ-linear morphism on cohomology. This defines a modular form H of weight $p-1$ on the mod p-reduction of $Y_1(N,p)$ called the Hasse invariant. It coincides with the modular form E_{p-1} modulo p for $p \geq 5$. Let $U(w)$ be the formal subscheme of $Y_1(N,p)$ defined by $|H| \leq w$. This makes sense: locally one lifts H so that $|H| \leq w$ makes sense in unequal characteristic and then one shows that the formals scheme does not depend on the choice of the lift. Let $Z(w)$ be the normalization of the inverse image of $U(w)$ in the p-adic formal scheme associated to $Y_1(Np)$. Its rigid analytic geometric fibre is finite and étale of degree $p-1$ over $U(w)_{\mathbf{Q}_p}$. Recall that we have a map

$$d\log\colon \mathcal{E}[p] \longrightarrow \omega_{\mathcal{E}_1}$$

defined as follows. Consider a formal scheme S over $U(w)$ and an S-valued point x of $\mathcal{E}[p]$. Via the canonical isomorphism $\mathcal{E}[p] \cong \mathcal{E}[p]^\vee$ it defines an S-valued point of $\mathcal{E}[p]^\vee$ i. e., a group scheme homomorphism $f_x\colon \mathcal{E}[p]_S \to \mathbf{G}_{m,S}$. Then, the $d\log(x)$ is the invariant differential on $\mathcal{E}_1$ given by the inverse image via f_x of the standard invariant differential $Z^{-1}dZ$ on $\mathbf{G}_{m,S}$. In general, a similar construction provides for every finite and locally free group scheme G over a base S a map $d\log\colon G^\vee \to \omega_{G/S}$ where $\omega_{G/S}$ is the sheaf of invariant differentials of G.

Proposition 5.2. *Over $Z(w)$ we have an exact sequence*

$$0 \longrightarrow C \longrightarrow \mathcal{E}[p] \xrightarrow{d\log} \omega_{\mathcal{E}_1},$$

where C is the canonical subgroup of $\mathcal{E}[p]^\vee$. Moreover,

1.) *H admits a unique $p-1$-root $H^{\frac{1}{p-1}}$ on $Z(w)_1$ which extends at the cusps and has q-expansion 1. If $p \geq 5$ then $H^{\frac{1}{p-1}}$ lifts uniquely to a $p-1$-th root $E_{p-1}^{\frac{1}{p-1}}$ of E_{p-1} on $Z(w)$ i. e., a weight 1 modular form whose $p-1$-th power is E_{p-1};*

2.) *$\mathcal{E}[p]/C$ admits a canonical $Z(w)$-section, which we denote by γ and which is a generator over $Z(w)_{\mathbb{Q}_p}$. Modulo p^{1-w} the image of γ via $d\log$ is the weight 1 modular form $H^{\frac{1}{p-1}}$. There is $\alpha \in \mathcal{O}_{Z(w)_1}$ such that $\alpha \cdot H^{\frac{1}{p-1}} = p^{\frac{w}{p-1}}$.*

Proof. The first statement follwos from 5.1. We simply write Z for $Z(w)$.

(1) Choose a basis e_R of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}}$ on an open formal subscheme $\mathcal{U} = \mathrm{Spf}(R) \subset Z$ and denote by $A \in R$ an element such that $F(e_R) = Ae_R$ modulo p. For $p \geq 5$ we take A so that $E_{p-1}(\mathcal{E}, e_R^\vee) = Ae_R^{\vee,p-1}$. Here we identify the element $\Omega_R := e_R^\vee$ with a generator of the invariant differentials $\omega_{\mathcal{U}}$ via the isomorphism $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}}^\vee \cong \omega_{\mathcal{U}}$ given by Serre's duality. We may then write $\varphi - p^w\colon \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_{1-w}} \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1}$ on $\mathcal{U}$ as the map $S_1 \to S_1$ sending $X \to HX^p - p^wX$. For every ring extension $R \subset R'$ we let $\mathbf{Z}_{R'}(A)$ be the set of solutions of the equation $X \to AX^p - p^wX$ in

R'. We let $\mathbf{Z}_{R',1}(A)$ and $\mathbf{Z}_{R',1-w}(A)$ be the solutions in R'_1 and R'_{1-w}. Write $\mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big)$ for the image of $\mathbf{Z}_{R',1}(A)$ in $\mathbf{Z}_{R',1-w}(A)$. It coincides with the set of solutions of $R'_{1-w} \ni X \mapsto AX^p - p^w X \in R'_1$. Then, [AG, Lemma 9.5] asserts that if R' is normal, noetherian, p-torsion free and p-adically complete and separated the natural map

$$\mathbf{Z}_{R'}(A) \longrightarrow \mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big)$$

is a bijection. By [AG, Thm. 8.1 & Def. 12.4] we have an isomorphism

$$\mathcal{E}[p]/C(R'_K) \cong C^\vee(R'_K) \cong \mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big).$$

Thus, set $\mathbf{Z}_{R'}(A)$ is an $\mathbf{F}_p$-vectors space of dimension ≤ 1 and it is of dimension 1 if and only if $C^\vee(R'_K)$ becomes constant over R'_K.

Since by construction the canonical subgroup exists and has a generator c over R_K and since by assumption $\mu_{p,K} \cong \mathbb{Z}/p\mathbb{Z}$, then also $C^\vee(R'_K)$ admits a canonical generator $c^\vee$. Then, $\mathbf{Z}_R(A)$ has dimension 1 as $\mathbf{F}_p$-vector space and the image of $c^\vee$ defines a basis element. Such image is of the form $p^{\frac{w}{p-1}}\delta^{-1}$ where δ is a given $p-1$–root of A in R. This already implies that H admits a $p-1$-root on R_1 defined by δ and it also implies the last claim in (2).

Assume that $\mathcal{U}$ is contained in the ordinary locus of $Z(w)$. Then, the canonical subgroup is canonically isomorphic to μ_p and we can take the invariant differential of μ_p as a generator of $\omega_c U$ modulo p and, hence, of $\mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}_1}$. With respect to this basis H is 1 and, hence, $\delta = 1$; see [AG, Prop. 3.4]. Such construction, applied to the Tate curve, implies the claim on the q-expansion.

Assume that $p \geq 5$. Two different local trivializations of $\omega_{\mathcal{E}}$ on $\mathcal{U}$ differ by a unit u. Thus, we get two different functions A and B such that $B = u^{p-1}A$. In particular, multiplication by u defines a bijection from $\mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(A)\big)$ to $\mathrm{red}_{1,1-w}\big(\mathbf{Z}_{R',1}(B)\big)$ and the root $p^{\frac{w}{p-1}}\delta_A^{-1}$ is sent to $p^{\frac{w}{p-1}}\delta_B^{-1} \cdot u$. This implies that over $Z(w)$ the modular form $p^w E_{p-1}^{-1}$, and hence the modular form E_{p-1}, admits a globally defined $p-1$-root as wanted.

(2) Choose $\mathcal{U}$, e_R and δ as in the proof of (1). Let G_δ be the group scheme introduced at the beginninig of this section. The canonical subgroup of $\mathcal{E}[p]$ is isomorphic to the subgroup scheme $G_{(-p\delta^{1-p}, -\delta^{p-1})}$ by [C, Thm. 2.1]. It is denoted by B_{-u} with $u = p\delta^{1-p}$ in loc. cit. using the relation between Coleman's approach and Oort-Tate description given the proof of [C, Prop. 1.1]. Such group scheme is isomorphic to $G_{(a,c)}$ modulo p with a and c as at the beginning of the section. We remark that in this case $a = p^{1-w}(p^w\delta^(1-p)$ up to unit so that $a \equiv 0$ modulo p^{1-w}. By loc. cit. the immersion $h\colon G_{(a,c)} \subset \mathcal{E}$ has the property that $h^*(\Omega_R) = (1 - \delta^{p-1}Y^{p-1})dY$ modulo p. The latter is equivalent to $(1 - \delta^{p-1}Y^{p-1})dY$ and hence to $(1 + \delta T)^{-1}dT$ modulo p.

By the first claim of the proposition and identifying $\mathcal{E}[p]$ with $\mathcal{E}[p]^\vee$ via the principal polarization on $\mathcal{E}$, the map $d\log$ factors via the map $\mathcal{E}[p]^\vee \to G_\delta^\vee$ and by functoriality of $d\log$ it is compatible with the map $d\log$ for G_δ. Moreover, the morphism $\eta_\delta\colon G_\delta \to \mu_p$ introduced at the beginning of the section defines a canonical section of $\mathcal{E}[p]^\vee/C \cong G_\delta^\vee$ which is a generator over $\mathbf{Q}_p$. This defines the section γ of $\mathcal{E}[p]/C$ claimed in (2). Denote by $\mathcal{U}_{1-w}$ and $\mathcal{E}_{1-w}$ the reduction of $\mathcal{U}$ and $\mathcal{E}$ modulo $p^{1-w}\mathcal{O}_K$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}[p]^\vee & \xrightarrow{d\log} & \omega_{\mathcal{E}_{1-w}/\mathcal{U}_{1-w}} \\ \downarrow & & \downarrow \\ G_\delta^\vee & \xrightarrow{d\log} & \omega_{G_\delta/\mathcal{U}_{1-w}}. \end{array}$$

Since $G_\delta \subset \mathcal{E}[p]$ is a closed immersion, the natural map $\omega_{\mathcal{E}_{1-w}/\mathcal{U}_{1-w}} \longrightarrow \omega_{G_\delta/\mathcal{U}_{1-w}}$ is surjective. Since $\omega_{G_\delta/\mathcal{U}_{1-w}}$ is free as $\mathcal{O}_{\mathcal{U}_{1-w}}$-module, such map is an isomorphism. The map $d\log\colon G_\delta^\vee \longrightarrow \omega_{G_\delta/\mathcal{U}_{1-w}}$ sends η_δ to the pull–back of the invariant differential of μ_p in $\omega_{G_\delta/\mathcal{U}_1}$ which is $\delta(1+\delta T)^{-1}dT$ i. e., $\delta\Omega_R$ via the isomorphism $\omega_{\mathcal{E}_{1-w}/\mathcal{U}_{1-w}} \cong \omega_{G_\delta/\mathcal{U}_{1-w}}$. Hence, $d\log(\eta_\delta)$ is equal to $H^{\frac{1}{p-1}}$ modulo p^{1-w} concluding the proof of (2). $\square$

In [C1, Lemma 9.2] Coleman introduces a weight 1 overconvergent modular form D_p of level $\Gamma_1(Np)$ whose $p-1$-power is E_{p-1} and, from the proof of the Lemma, it has q-expansion 1 modulo p. These two properties characterize such modular form. Our modular form $E_{p-1}^{\frac{1}{p-1}}$ is a $p-1$-root of E_{p-1} and it has q-expansion 1 modulo p by 5.2. We deduce:

Corollary 5.3. *The overconvergent modular form defined by $E_{p-1}^{\frac{1}{p-1}}$ over $Z(w)_{\mathbb{Q}_p}$ is the weight* 1 *overconvergent modular form D_p of level $\Gamma_1(Np)$ introduced by Coleman.*

In particular, our approach can be seen as a refinement of [C1, Lemma 9.2] providing a formal model for D_p.

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