Rare events and Applications (Winter 2019): Assignment 1

Due date: **January 31**

The questions below are **modifications** *of questions from the book "Large Deviations" by Frank den Hollander.*

1. Let $P(X = a) = 1$, calculate $I(z)$ for $\forall z \in \mathbf{R}$.

2. Calculate for $X \sim$ Poisson(λ), $X \sim$ Exponential(α), $X \sim \mathcal{N}(\mu, \sigma^2)$, $X \sim$ Bernoulli(p):

   (a) $\varphi(t) = E(e^t X), t \in \mathbf{R}$

   (b) distribution of the Cramer's transform $\hat{X}$ using $d\hat{F}(x) = (\varphi(\tau))^{-1} \int^x e^{\tau y} dF(y)$

   (c) $I(z)$, $z \in \mathbf{R}$

3. Prove that: (a) $I \geq 0$ with $I(z) = 0$ iff $z = \mu$; (b) $I'(\mu) = 0, I''(\mu) = \frac{1}{\sigma^2}$

4. The function "$\log \varphi$ is steep if $D_\varphi = \{t \in \mathbf{R} : \varphi(t) < \infty\}$ and

$$\lim_{t \to \partial D_\varphi} \left| (\log \varphi)'(t) \right| = \infty$$

   Give an example for which $\log \varphi$ is (i) steep; (ii) not steep.
   Determine the domain D_φ for your examples, and determine $E(X)$ and $V(X)$.

5. Prove that for any $a \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{S_n}{n} \in A\right) = -\inf_{z \in A} I(z), \quad \text{for } A = [a, \infty)$$