Rare events and Applications (Winter 2019): Assignment 2

Due date: **February 21**

Some *of the questions below are from the book "Large Deviations" by Frank den Hollander.*

1. Let
$$I_\rho(\nu) = \int_S d\nu(x) \log(\frac{d\nu}{d\rho})(x)$$
where $\nu \ll \rho \in \mathcal{M}_1(S)$. Show that $I_\rho(\nu) \geq 0$ with $I_\rho(\nu) = 0$ iff $\nu = \rho$.

2. (a) Give an example of ν, ρ such that $\nu \ll \rho$ and $\rho \ll \nu$ but $H(\nu|\rho) \neq H(\rho|\nu)$.

   (b) Show that the function $d_H = \frac{1}{2}(H(\nu|\rho)+H(\rho|\nu))$ is a metric: $d(x,y) = d(y,x), d(x,y) \geq 0, d(x,x) = 0$, $d(x,z) \leq d(x,y) + d(y,z)$ on the subset of measures in $\mathcal{M}_1(S)$ that are absolutely continuous with respect to each other.

3. Let $Y_1, Y_2, \ldots Y_n \sim \rho$ be i.i.d. random variables on $\Gamma = \{1,2\}$ with $\rho(1) = 1 - \rho(2) = 1 - p$. Let $\nu \in \mathcal{M}_1(\Gamma \times \Gamma)$ with $\bar{\nu}(j) = \sum_{i=1}^r \nu(i,j) = \sum_{i=1}^r \nu(j,i)$. Calculate the function $I_{\rho,2}(\nu)$ needed for the large deviations of the empirical measure on pairs $(Y_1, Y_2), (Y_2, Y_3), \ldots$
$$I_{\rho,2}(\nu) = \sum_{(i,j)=(1,1)}^{(r,r)} \nu(i,j) \log\left(\frac{\nu(i,j)}{\bar{\nu}(i)\rho(j)}\right)$$
(**Hint:** express ν in terms of two parameters $a, b \in [0,1]$)

4. Consider $\Gamma = \{1, 2, \ldots, r\}$

   (a) Show that converegence in $d_{TV}(\nu_n, \nu) \to 0$ is equivalent to pointwise convergence $\nu_n(i) \to \nu(i) \, \forall i \in \Gamma$, and that it is also equivalent to weak convergence: $E_{\nu_n}[f] \to E_\nu[f] \;\; \forall f$ bounded on Γ.

   (b) Consider the set $A_k = \{\nu \in \mathcal{M}_1(\Gamma) : \text{support}(\nu) \subset \{1, \ldots k\}\}$. Show that A_k^c is an open set.

   (c) Let $\rho_k \sim \text{Unif}(\{1, \ldots k\})$. Find ν^* which attains $\inf\{H(\nu|\rho) : \nu \in A_k\}$.

   (d) Consider $X_1, X_2, \ldots$ be i.i.d. $\sim \rho$ with $L_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ the empirical distribution of n-sample. Calculate
$$\lim_{n\to\infty} \frac{1}{n} \log P(L_n \in A_k)$$

5. Let $\rho \sim \mathcal{N}(0, \sigma^2)$. Show that
$$\inf\left\{H(\nu|\rho) : \nu \in \mathcal{M}_1(\mathbb{R}), \int x d\nu(x) = \mu\right\} = \frac{\mu^2}{2}$$
Use the Contraction principle to re-verify Cramer's theorem for $X_1, \ldots, X_n \sim \rho$ i.i.d.