

# EXISTENCE OF ACIM FOR PIECEWISE EXPANDING $C^{1+\varepsilon}$ MAPS

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**ABSTRACT.** In this paper, we establish Lasota-Yorke inequality for the Frobenius-Perron Operator of a piecewise expanding  $C^{1+\varepsilon}$  map of an interval. By adapting this inequality to satisfy the assumptions of the Ionescu-Tulcea and Marinescu ergodic theorem [3], we demonstrate the existence of an absolutely continuous invariant measure (ACIM) for the map. Furthermore, we prove the quasi-compactness of the Frobenius-Perron operator induced by the map. Additionally, we explore significant properties of the system, including weak mixing and exponential decay of correlations.

## 1. INTRODUCTION

In this paper, we study a piecewise expanding map with the derivative satisfying Hölder condition. We explore the Lasota-Yorke inequality for functions with generalized bounded  $p$ -variation, which we denote as  $BV_{p,1/p}$ . This builds on the earlier work of Keller [2, 4], who focused on a specific case,  $BV_{1,1/p}$ . By taking a simpler approach, we show that our method leads to better results for a wider range of  $p$  values. This not only broadens the usefulness of the Lasota-Yorke inequality but also deepens our understanding of the dynamics of piecewise expanding maps under consideration.

We work within the space  $L_m^1$  of all integrable functions defined on the unit interval  $I$  with respect to the Lebesgue measure  $m$ . Let  $\tau : I \rightarrow I$  be defined on  $I = [0, 1]$ . We called it piecewise  $C^{1+\varepsilon}$  expanding transformation if there exists a partition  $\mathcal{P} = \{I_i = (a_{i-1}, a_i), i = 1, 2, 3 \dots N\}$  of  $I = [0, 1]$  and an  $0 < \varepsilon \leq 1$  such that the transformation  $\tau : I \rightarrow I$  satisfies the following conditions:

- (1)  $\tau|_{I_i}$  is monotonic,  $C^1$  function, which can be extended to a  $C^1$  function on  $\bar{I}_i, i = 1, 2, \dots, N$ .
- (2) For  $0 < \varepsilon \leq 1$ ,  $\tau'_i(x)$  is Hölder continuous for each  $i = 1, 2, \dots, N$ , i.e. there exist constants  $M_i$  such that  $|\tau'_i(x) - \tau'_i(y)| \leq M_i|x - y|^\varepsilon$ , for all  $x, y \in I_i$ . When  $\varepsilon = 1$ ,  $\tau'_i(x)$  is Lipschitz function.
- (3)  $|\tau'_i(x)| \geq s_i > 1$  for all  $x \in I_i$ .

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Let  $M = \max_{I \leq i \leq N} M_i$  and  $s = \min_{I \leq i \leq N} s_i$ . We say  $\tau \in \mathcal{T}([0, 1])$ , the class of piecewise expanding  $C^{(1+\varepsilon)}$  maps on  $I$ .

A measurable transformation  $\tau : I \rightarrow I$  is called nonsingular if, for any measurable set  $A \subset I$ , the condition  $m(A) = 0$  implies that  $m(\tau^{-1}(A)) = 0$  as well. The piecewise  $C^{1+\varepsilon}$  expanding maps are nonsingular.

For  $\tau$  piecewise monotonic, the Frobenius-Perron Operator  $P_\tau : L_m^1 \rightarrow L_m^1$  is defined by,

$$P_\tau f(x) = \sum_{y \in \tau^{-1}(x)} f(y) \cdot g(y) = \sum_{i=1}^N \frac{f(\tau^{-1}(x))}{|\tau'(\tau^{-1}(x))|} \cdot \chi_{\tau(I_i)}(x),$$

where  $g(y) = \frac{1}{|\tau'(y)|}$  and  $\sup_{I_i} |g| < \frac{1}{s_i} < 1$ . We know that  $P_\tau$  is a linear and continuous operator and has the following properties:

- (1)  $P_\tau$  is positive, It means  $f \geq 0 \implies P_\tau f \geq 0$ ;
- (2)  $P_\tau$  preserves integrals, i.e.  $\int_I P_\tau f dm = \int_I f dm$  for  $f \in L_1$ ;
- (3)  $P_\tau$  satisfies the composition property,  $P_\tau^n = P_\tau^n$  where  $n$  is the  $n$ th iterate of  $\tau$ ;
- (4)  $P_\tau f = f$  if and only if the measure  $d\mu = f dm$  is  $\tau$  invariant, i.e. for each measurable set  $A$ ,  $\mu(\tau^{-1}(A)) = \mu(A)$ . The Perron-Frobenius operator  $P_\tau$  is a powerful tool for capturing the ergodic properties of the dynamical system  $(\tau, \mu)$ , particularly in terms of:

- Existence of an absolutely continuous invariant measure (ACIM),
- Weak mixing and decay of correlations,
- Quasi-compactness.

In [2, 4] Keller introduced the concept of generalized bounded variation in a broader context, which allows for the study of quasi-compact topological spaces. In this work, we focus on its application within the interval  $[0, 1]$ . This simplifies our setting significantly, as we now work with the standard Borel  $\sigma$ -algebra on  $[0, 1]$  and the normalized Lebesgue measure  $m$ .

**Definition 1.1** (Oscillation of a function, [4]). For an arbitrary function  $f : I \rightarrow \mathbb{C}$  and  $\varepsilon > 0$ , define the oscillation  $\text{Osc}(f, r, x) : I \rightarrow [0, \infty]$  by

$$\text{Osc}(f, r, x) = \begin{cases} \text{ess sup}_{y_1, y_2 \in S_r(x)} \{|f(y_1) - f(y_2)| \mid y_1, y_2 \in S_r(x)\}, & \text{if } m(S_r(x)) > 0, \\ 0, & \text{if } m(S_r(x)) = 0, \end{cases}$$

where,  $S_r(x) = \{y \in I \mid d(x, y) < r\}$ . It is easy to see that  $\text{Osc}(f, r, x)$  is lower semi-continuous and hence measurable. We define  $\text{Osc}_p(f, r) = \|\text{Osc}(f, r, \cdot)\|_p$ , for  $1 \leq p \leq \infty$ .

The function  $\text{Osc}_p(f, r)$  is an isotonic function in the  $r$ -variable from  $(0, A]$  to  $[0, \infty]$ , where  $A$  is any positive constant. Following Definition (1.9) in [4], for  $p \geq 1$  we define:

(1) For  $f : I \rightarrow \mathbb{C}$ , set

$$\text{Var}_{p,\phi}(f) = \sup_{0 < r \leq A} \frac{\text{Osc}_p(f, r)}{\phi(r)},$$

where  $\phi(r) = r^{1/p}$  and  $A$  is any positive constant.

(2) For  $f \in BV_{p,\phi}$ , define the norm

$$\|f\|_{p,\phi} = \text{Var}_{p,\phi}(f) + \|f\|_p.$$

## 2. MAIN ESTIMATES

In this section, we present the main results of the paper.

**Lemma 2.1.** *Let  $\tau : [0, 1] \rightarrow [0, 1]$  be piecewise expanding  $C^{1+\varepsilon}$  map, where  $\varepsilon = 1/p$ , for some fixed  $p \geq 1$ . Then, for every  $f \in BV_{p,1/p}$ ,*

$$(1) \quad V_{p,1/p}(P_\tau f) \leq \alpha \|f\|_{p,1/p} + \beta \|f\|_p,$$

for  $\alpha = \left( \frac{2^{1/p} D(1+D)}{s} + \frac{(1+D)}{s} + \frac{(1+D)}{s} \right)$  and  $\beta > 0$ , where  $D = \frac{M(r/s)^\varepsilon}{s}$ .

*Proof.* We have,

$$\begin{aligned} (2) \quad \text{Osc}_p(P_\tau f, r) &= \left( \int_I |\text{Osc}(P_\tau f, r, x)|^p dm(x) \right)^{\frac{1}{p}} \\ &= \left( \int_I \left| \sup_{y_1, y_2 \in S(x,r)} |P_\tau f(y_1) - P_\tau f(y_2)| \right|^p dm(x) \right)^{\frac{1}{p}} \\ &= \left( \int_I \left| \sup_{y_1, y_2 \in S(x,r)} \left| \sum_{i=1}^N ((f \cdot g)(\tau_i^{-1}(y_1)) \cdot \chi_{\tau(I_i)}(y_1) - (f \cdot g)(\tau_i^{-1}(y_2)) \cdot \chi_{\tau(I_i)}(y_2)) \right| \right|^p dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Using Minkowski inequality,

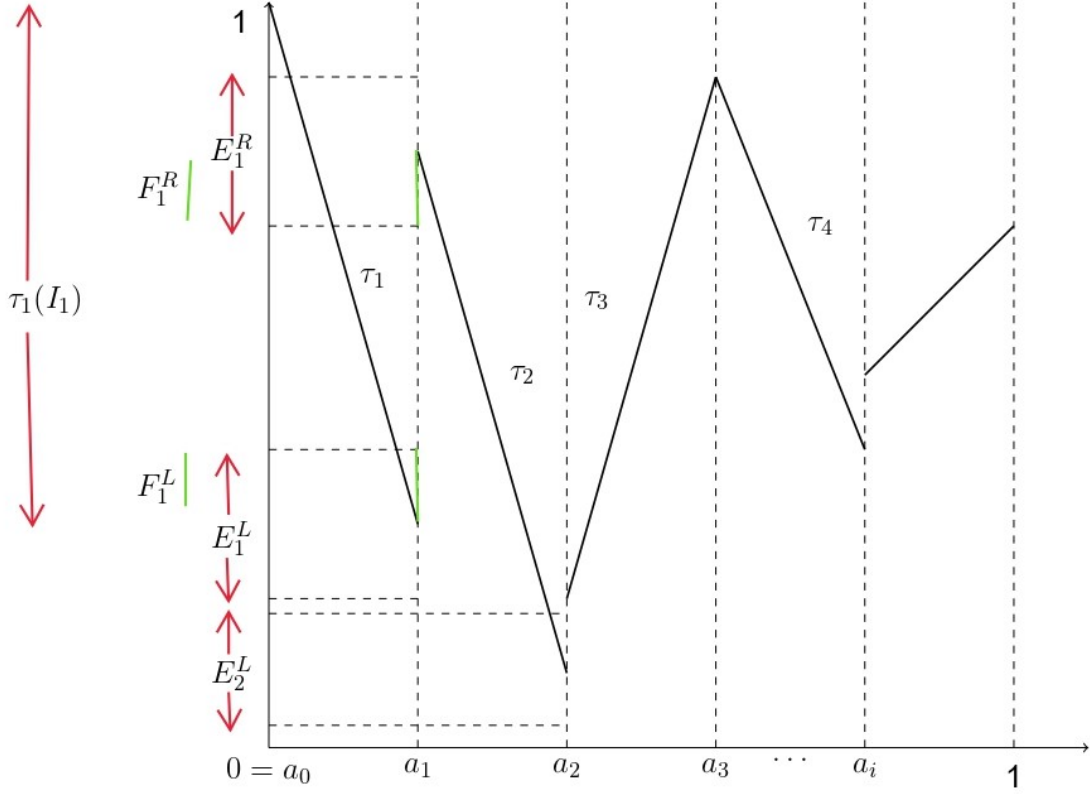


FIGURE 1. Piecewise expanding map on unit interval.

$$\begin{aligned}
&\leq \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( (f \cdot g)(\tau_i^{-1}(y_1)) - (f \cdot g)(\tau_i^{-1}(y_2)) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}} \\
&+ \left( \int_I \left| \chi_{E_0^R}(x) \sup_{y \in F_0^R} |f(\tau_1^{-1}(y)) \cdot g(\tau_1^{-1}(y))| + \chi_{E_1^L}(x) \sup_{y \in F_1^L} |f(\tau_1^{-1}(y)) \cdot g(\tau_1^{-1}(y))| \right. \right. \\
&+ \chi_{E_1^R}(x) \sup_{y \in F_1^R} |f(\tau_2^{-1}(y)) \cdot g(\tau_2^{-1}(y))| + \dots + \chi_{E_N^L}(x) \sup_{y \in F_N^L} |f(\tau_N^{-1}(y)) \cdot g(\tau_N^{-1}(y))| \left. \right|^p dm(x) \Big)^{\frac{1}{p}} \\
&= \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( (f \cdot g)(\tau_i^{-1}(y_1)) - (f \cdot g)(\tau_i^{-1}(y_2)) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}} \\
&+ \left( \int_I \left| \sup_{y \in F_{i-1}^R} \left( \sum_{i=1}^N \chi_{E_{i-1}^R}(x) |f(\tau_i^{-1}(y)) \cdot g(\tau_i^{-1}(y))| \right) \right|^p dm(x) \right)^{\frac{1}{p}} \\
&+ \left( \int_I \left| \sup_{y \in F_i^L} \left( \sum_{i=1}^N \chi_{E_i^L}(x) |f(\tau_i^{-1}(y)) \cdot g(\tau_i^{-1}(y))| \right) \right|^p dm(x) \right)^{\frac{1}{p}} .
\end{aligned}$$

We split this into two parts:

(3)

$$\Delta_1 = \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( (f \cdot g)(\tau_i^{-1}(y_1)) - (f \cdot g)(\tau_i^{-1}(y_2)) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}},$$

and

$$(4) \quad \begin{aligned} \Delta_2 &= \left( \int_I \left| \sup_{y \in F_{i-1}^R} \left( \sum_{i=1}^N \chi_{E_{i-1}^R}(x) |f(\tau_i^{-1}(y)) \cdot g(\tau_i^{-1}(y))| \right) \right|^p dm(x) \right)^{\frac{1}{p}} \\ &+ \left( \int_I \left| \sup_{y \in F_i^L} \left( \sum_{i=1}^N \chi_{E_i^L}(x) |f(\tau_i^{-1}(y)) \cdot g(\tau_i^{-1}(y))| \right) \right|^p dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

In  $\Delta_2$ , index  $i$  runs through  $i$ 's such that the image  $\tau(I_i)$  is not touching the endpoints of  $[0, 1]$ .

First, we estimate  $\Delta_1$ . Adding and subtracting  $f(\tau_i^{-1}(y_1)) \cdot g(\tau_i^{-1}(y_2))$  we get,

$$\begin{aligned} \Delta_1 &= \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( f(\tau_i^{-1}(y_1))(g(\tau_i^{-1}(y_1)) - g(\tau_i^{-1}(y_2))) \right. \right. \right. \\ &\quad \left. \left. \left. + g(\tau_i^{-1}(y_2))(f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(y_2))) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Now, using Minkowski inequality again we obtain,

(5)

$$\begin{aligned} \Delta_1 &\leq \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( f(\tau_i^{-1}(y_1))(g(\tau_i^{-1}(y_1)) - g(\tau_i^{-1}(y_2))) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}} \\ &+ \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( g(\tau_i^{-1}(y_2))(f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(y_2))) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

For the simplicity let us break (5) into two parts,

(6)

$$\Delta_{1_1} = \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( f(\tau_i^{-1}(y_1))(g(\tau_i^{-1}(y_1)) - g(\tau_i^{-1}(y_2))) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}}$$

and

(7)

$$\Delta_{1_2} = \left( \int_I \left| \sup_{y_1, y_2 \in S(x, r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( g(\tau_i^{-1}(y_2))(f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(y_2))) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}}.$$

First, we estimate  $\Delta_{1_1}$ . For any  $i = 1, 2, \dots, N$ , the derivative  $\tau'$  is  $\varepsilon$ -Hölder continuous on  $I_i$ , with constant  $M_i$  and  $|\tau'_i(x)| \geq s_i > 1$ . Let  $M = \max_{1 \leq i \leq N} M_i$  and  $s = \min_{1 \leq i \leq N} s_i$ . Also,

$\max\{|y_1 - y_2| \mid y_1, y_2 \in S_r(x)\} = 2r$ , and  $|\tau_i^{-1}(y_1) - \tau_i^{-1}(y_2)| \leq \frac{1}{s} \cdot |y_1 - y_2|$ . Then,

$$|g(\tau_i^{-1}(y_1)) - g(\tau_i^{-1}(y_2))| = \left| \frac{\tau'(\tau_i^{-1}(y_1)) - \tau'(\tau_i^{-1}(y_2))}{\tau'(\tau_i^{-1}(y_1)) \cdot \tau'(\tau_i^{-1}(y_2))} \right| \leq \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s} \right) \cdot \frac{1}{|\tau'(\tau_i^{-1}(y_1))|}$$

Hence,

$$\begin{aligned} \Delta_{1_1} &\leq \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s} \right) \left( \int_I \left| \sup_{y_1 \in S(x,r)} \left( \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) |f(\tau_i^{-1}(y_1))| \cdot \frac{1}{|\tau'(\tau_i^{-1}(y_1))|} \right) \right|^p dm(x) \right)^{\frac{1}{p}} \\ &\leq \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s} \right) \left( \int_I \left| \sup_{y_1 \in S(x,r)} \left( \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) |f(\tau_i^{-1}(y_1)) + f(\tau_i^{-1}(x)) - f(\tau_i^{-1}(x))| \right. \right. \right. \\ &\quad \left. \left. \left. \cdot |g(\tau_i^{-1}(y_1))| \right) \right|^p dm(x) \right)^{\frac{1}{p}} \\ &\leq \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s^{(2p-1)/p}} \right) \left( \int_I \left| \sup_{y_1 \in S(x,r)} \left( \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right) \right|^p \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| \right. \\ &\quad \left. \cdot |g(\tau_i^{-1}(y_1))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &+ \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s^{(2p-1)/p}} \right) \left( \int_I \left| \sum_{i=1}^N \sup_{y_1 \in S(x,r)} |f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(x))| \right|^p \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| \right. \\ &\quad \left. \cdot |g(\tau_i^{-1}(y_1))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &\leq \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s^{(2p-1)/p}} \right) \left( \int_I \left| \sup_{y_1 \in S(x,r)} \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| \right. \\ &\quad \left. \cdot |g(\tau_i^{-1}(y_1))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &+ \left( \frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s^{(2p-1)/p}} \right) \left( \int_I \left| \sum_{i=1}^N \sup_{y_1 \in S(x,r)} \text{Osc}(f(\tau_i^{-1}(x)), r, x) \right|^p \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| \right. \\ &\quad \left. \cdot |g(\tau_i^{-1}(y_1))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \end{aligned}$$

By Keller [4], Lemma (2.3) we have  $Osc(f(\tau_i^{-1}(x)), r, x) \leq Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right)$  which gives,

$$\begin{aligned} &\leq \left(\frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s^{(2p-1)/p}}\right) \left(\int_I \left| \sup_{y_1 \in S(x,r)} \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| \right. \\ &\qquad \qquad \qquad \left. \cdot |g(\tau_i^{-1}(y_1))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &+ \left(\frac{2^\varepsilon M \cdot (r/s)^\varepsilon}{s^{(2p-1)/p}}\right) \left(\int_I \left| \sum_{i=1}^N \sup_{y_1 \in S(x,r)} Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right) \right|^p \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| \right. \\ &\qquad \qquad \qquad \left. \cdot |g(\tau_i^{-1}(y_1))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \end{aligned}$$

Now, for any  $w \in S(x, r)$  let us estimate  $\left| \frac{g(\tau_i^{-1}(w))}{g(\tau_i^{-1}(x))} \right|$ ,

$$\left| \frac{g(\tau_i^{-1}(w))}{g(\tau_i^{-1}(x))} - 1 \right| = \left| \frac{\frac{1}{\tau_i'(\tau_i^{-1}(w))} - \frac{1}{\tau_i'(\tau_i^{-1}(x))}}{\tau_i'(\tau_i^{-1}(w))} \right| = \left| \frac{\tau_i'(\tau_i^{-1}(x)) - \tau_i'(\tau_i^{-1}(w))}{\tau_i'(\tau_i^{-1}(w))} \right| \leq \frac{M \cdot (r/s)^\varepsilon}{s}.$$

So,

$$(8) \quad \left| \frac{g(\tau_i^{-1}(w))}{g(\tau_i^{-1}(x))} \right| \leq \left(1 + \frac{M \cdot (r/s)^\varepsilon}{s}\right).$$

Let  $D = \left(\frac{M \cdot (r/s)^\varepsilon}{s}\right)$ . Using (8) we get,

$$(9) \quad \begin{aligned} \Delta_{11} &\leq \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left(\int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &+ \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left(\int_I \left| \sum_{i=1}^N Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right) \right|^p \cdot |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Now, we estimate  $\Delta_{12}$ ,

$$\begin{aligned} \Delta_{12} &= \left(\int_I \left| \sup_{y_1, y_2 \in S(x,r)} \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left( g(\tau_i^{-1}(y_2)) (f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(y_2))) \right) \right| \right|^p dm(x) \right)^{\frac{1}{p}} \\ &\leq \left(\int_I \left| \sup_{y_1, y_2 \in S(x,r)} \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right| |g(\tau_i^{-1}(y_2))|^p |f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(y_2))| \right|^p dm(x) \right)^{\frac{1}{p}} \\ &\leq \frac{(1+D)}{s^{(p-1)/p}} \left(\int_I \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \sup_{y_1, y_2 \in S(x,r)} |f(\tau_i^{-1}(y_1)) - f(\tau_i^{-1}(y_2))| \right|^p |g(\tau_i^{-1}(x))| dm(x) \right)^{\frac{1}{p}} \end{aligned}$$

$$\leq \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) \text{Osc} \left( f, \frac{r}{s}, \tau_i^{-1}(x) \right) \right|^p |g(\tau_i^{-1}(x))| dm(x) \right)^{\frac{1}{p}}$$

Combining  $\Delta_{1_1}$  and  $\Delta_{1_2}$  we get,

$$\begin{aligned} \Delta_1 &\leq \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &\quad + \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N \text{Osc} \left( f, \frac{r}{s}, \tau_i^{-1}(x) \right) \right|^p \cdot |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}} \\ &\quad + \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N \text{Osc} \left( f, \frac{r}{s}, \tau_i^{-1}(x) \right) \right|^p |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Now, let us estimate  $\Delta_2$ . Again, for simplicity, we divide it into two parts,

$$\begin{aligned} (10) \quad \Delta_{2_1} &= \left( \int_I \left| \sup_{y \in F_{i-1}^R} \left( \sum_{i=1}^N \chi_{E_{i-1}^R}(x) |f(\tau_i^{-1}(y)) \cdot g(\tau_i^{-1}(y))| \right) \right|^p dm(x) \right)^{\frac{1}{p}} \\ \Delta_{2_2} &= \left( \int_I \left| \sup_{y \in F_i^L} \left( \sum_{i=1}^N \chi_{E_i^L}(x) |f(\tau_i^{-1}(y)) \cdot g(\tau_i^{-1}(y))| \right) \right|^p dm(x) \right)^{\frac{1}{p}}. \end{aligned}$$

We have

$$\begin{aligned} \Delta_{2_1} &= \left( \int_I \left| \sup_{y \in F_{i-1}^R} \left( \sum_{i=1}^N |f(\tau_i^{-1}(y)) + f(\tau_i^{-1}(x)) - f(\tau_i^{-1}(x))| \cdot |g(\tau_i^{-1}(y))| \chi_{E_{i-1}^R}(x) \right) \right|^p dm(x) \right)^{\frac{1}{p}} \\ &\leq \left( \int_I \left| \sup_{y \in F_{i-1}^R} \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(y))|^p \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}} \\ &\quad + \left( \int_I \left| \sum_{i=1}^N \sup_{y \in F_{i-1}^R} |f(\tau_i^{-1}(y)) - f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(y))|^p \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}} \end{aligned}$$



$$\leq \frac{1}{s^{(p-1)/p}} \left( \int_I \left| \sup_{y \in F_{i-1}^{R_i}} \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(y))| \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}}$$

$$+ \frac{1}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N \sup_{y \in F_{i-1}^{R_i}} |f(\tau_i^{-1}(y)) - f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(y))| \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}}.$$

Multiplying by  $\left| \frac{g(\tau_i^{-1}(x))}{g(\tau_i^{-1}(x))} \right|$  in both terms and following the same technique as we used in  $\Delta_{1_1}$  we get,

$$(11) \quad \Delta_{2_1} \leq \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p |g(\tau_i^{-1}(x))| \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}}$$

$$+ \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right) \right|^p |g(\tau_i^{-1}(x))| \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}}.$$

We can estimate  $\Delta_{2_2}$  in a similar way,

$$(12) \quad \Delta_{2_2} \leq \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p |g(\tau_i^{-1}(x))| \chi_{E_i^L}(x) dm(x) \right)^{\frac{1}{p}}$$

$$+ \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right) \right|^p |g(\tau_i^{-1}(x))| \chi_{E_i^L}(x) dm(x) \right)^{\frac{1}{p}}.$$

Now, combining all the estimates we obtain,

$$(13) \quad Osc_p(P_\tau f, r) \leq \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p \cdot |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}}$$

$$+ \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right) \right|^p \cdot |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}}$$

$$+ \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N Osc\left(f, \frac{r}{s}, \tau_i^{-1}(x)\right) \right|^p |g(\tau_i^{-1}(x))| \chi_{\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)}(x) dm(x) \right)^{\frac{1}{p}}$$

$$+ \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p |g(\tau_i^{-1}(x))| \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}}$$

(14)

$$\begin{aligned}
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N \text{Osc} \left( f, \frac{r}{s}, \tau_i^{-1}(x) \right) \right|^p |g(\tau_i^{-1}(x))| \chi_{E_{i-1}^R}(x) dm(x) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N |f(\tau_i^{-1}(x))| \right|^p |g(\tau_i^{-1}(x))| \chi_{E_i^L}(x) dm(x) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \int_I \left| \sum_{i=1}^N \text{Osc} \left( f, \frac{r}{s}, \tau_i^{-1}(x) \right) \right|^p |g(\tau_i^{-1}(x))| \chi_{E_i^L}(x) dm(x) \right)^{\frac{1}{p}}.
\end{aligned}$$

Using change of variables, let  $z = \tau_i^{-1}(x)$  with  $dm(z) = \frac{1}{|\tau'(\tau_i^{-1}(x))|} dm(x) = |g(\tau_i^{-1}(x))| dm(x)$ , using Minkowski's inequality we obtain

(15)

$$\begin{aligned}
\text{Osc}_p(P_\tau f, r) & \leq \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_i^{-1}(\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L))} |f(z)|^p dm(z) \right)^{\frac{1}{p}} \\
& + \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_i^{-1}(\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L))} \left| \text{Osc} \left( f, \frac{r}{s}, z \right) \right|^p dm(z) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_i^{-1}(\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L))} \left| \text{Osc} \left( f, \frac{r}{s}, z \right) \right|^p dm(z) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_{i-1}^{-1}(E_{i-1}^R)} |f(z)|^p dm(z) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_{i-1}^{-1}(E_{i-1}^R)} \left| \text{Osc} \left( f, \frac{r}{s}, z \right) \right|^p dm(z) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_i^{-1}(E_i^L)} |f(z)|^p dm(z) \right)^{\frac{1}{p}} \\
& + \frac{(1+D)}{s^{(p-1)/p}} \left( \sum_{i=1}^N \int_{\tau_i^{-1}(E_i^L)} \left| \text{Osc} \left( f, \frac{r}{s}, z \right) \right|^p dm(z) \right)^{\frac{1}{p}}.
\end{aligned}$$

We have  $\tau_{i-1}^{-1}(E_{i-1}^R) = \tau_{i-1}^{-1}(F_{i-1}^R)$  and  $\tau_i^{-1}(E_i^L) = \tau_i^{-1}(F_i^L)$  and sets  $\tau_{i-1}^{-1}(E_{i-1}^R), \tau_i^{-1}(E_i^L)$  are all disjoint for  $i = 1, 2, \dots, N$ . Lebesgue measure of each of them is less than  $r/s$ . By Keller[4], Lemma (2.2)

$$\sup |f(z)| \leq \frac{1}{A} \int_Y \text{Osc}(f|_Y, A, z) dm(z) + \frac{1}{m(Y)} \int_Y |f| dm.$$

We choose disjoint sets  $Y_{i-1}^R, Y_i^L, i = 1, 2, \dots, N$  such that  $Y_{i-1}^R \supset \tau_{i-1}^{-1}(F_{i-1}^R), Y_i^L \supset \tau_i^{-1}(F_i^L)$  for all  $r \leq A$ . We choose  $A$  such that these sets are disjoint. We can choose them of equal length

say  $B < \frac{\gamma}{2}$ . We make  $A$  sufficiently small.

We estimate:

$$\begin{aligned} \int_{\tau_i^{-1}(E_i^L)} |f(z)|^p dm(z) &\leq \int_{\tau_i^{-1}(E_i^L)} \left( \frac{1}{A} \int_{Y_i^L} |Osc(f, A, z)|^p dm(z) + \frac{1}{B} \int_{Y_i^L} |f|^p dm. \right) dm(z) \\ &\leq \frac{r}{s} \cdot \frac{1}{A} \int_{Y_i^L} |Osc(f, A, z)|^p dm(z) + \frac{r}{s} \cdot \frac{1}{B} \int_{Y_i^L} |f|^p dm \\ &\leq \frac{r}{s} \int_{Y_i^L} \frac{|Osc(f, A, z)|^p}{A} dm(z) + \frac{r}{s} \cdot \frac{1}{B} \int_{Y_i^L} |f|^p dm, \end{aligned}$$

and similarly the integral over  $\tau_{i-1}^{-1}(E_{i-1}^R), i = 1, 2, \dots, N$ . After summing up over  $i$  and  $R, L$

$$\begin{aligned} (16) \quad &\left( \sum_{i=1}^N \int_{\tau_{i-1}^{-1}(E_{i-1}^R)} |f(z)|^p dm(z) \right)^{\frac{1}{p}} + \left( \sum_{i=1}^N \int_{\tau_{i-1}^{-1}(E_{i-1}^L)} |Osc\left(f, \frac{r}{s}, z\right)|^p dm(z) \right)^{\frac{1}{p}} \\ &+ \left( \sum_{i=1}^N \int_{\tau_i^{-1}(E_i^L)} |f(z)|^p dm(z) \right)^{\frac{1}{p}} + \left( \sum_{i=1}^N \int_{\tau_i^{-1}(E_i^R)} |Osc\left(f, \frac{r}{s}, z\right)|^p dm(z) \right)^{\frac{1}{p}} \\ &\leq \left(\frac{r}{s}\right)^{1/p} \cdot \frac{Osc_p(f, A)}{A^{1/p}} + \left(\frac{r}{s}\right)^{1/p} \cdot \frac{1}{B^{1/p}} \|f\|_p. \end{aligned}$$

Using this and since measure of  $\tau_i(I_i) \setminus (E_{i-1}^R \cup E_i^L)$  is less than 1, we get,

$$\begin{aligned} Osc_1(P_\tau f, r, ) &\leq \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} \|f\|_p + \frac{2^\varepsilon D(1+D)}{s^{(p-1)/p}} Osc_p\left(f, \frac{r}{s}\right) + \frac{(1+D)}{s^{(p-1)/p}} Osc_p\left(f, \frac{r}{s}\right) \\ &+ \frac{(1+D)}{s^{(p-1)/p}} \left( \left(\frac{r}{s}\right)^{1/p} \cdot \frac{Osc_p(f, A)}{A^{1/p}} + \left(\frac{r}{s}\right)^{1/p} \cdot \frac{1}{B^{1/p}} \|f\|_p \right). \end{aligned}$$

Note that  $\frac{D}{r^{1/p}} = \frac{M}{s^{1+1/p}}$  is independent of  $r$  and  $D \leq \frac{M \cdot A^{1/p}}{s^{1+1/p}}$ . Since,  $\varepsilon = 1/p$ , on dividing by  $r^{1/p}$  and taking supremum over  $0 < r \leq A$ , we get

$$\begin{aligned} V_{p,1/p}(P_\tau f) &\leq \frac{2^{1/p} M(1+D)}{s^{1+1/p} \cdot s^{(p-1)/p}} \|f\|_p + \frac{2^{1/p} D(1+D)}{s^{1/p} \cdot s^{(p-1)/p}} V_{p,1/p} f + \frac{(1+D)}{s^{1/p} \cdot s^{(p-1)/p}} V_{p,1/p} \\ &+ \frac{(1+D)}{s^{(p-1)/p}} \left(\frac{1}{s}\right)^{1/p} V_{p,1/p} f + \frac{(1+D)}{s^{(p-1)/p}} \left(\frac{1}{s \cdot B}\right)^{1/p} \|f\|_p, \end{aligned}$$

where now,  $D = \frac{M}{s^{1+1/p}}$ . After simplification,

$$\begin{aligned} V_{p,1/p}(P_\tau f) &\leq \left( \frac{2^{1/p} D(1+D)}{s} + \frac{(1+D)}{s} + \frac{(1+D)}{s} \right) V_{p,1/p} f \\ &+ \left( \frac{2^{1/p} M(1+D)}{s^2} + \frac{(1+D)}{s \cdot B^{1/p}} \right) \|f\|_p. \end{aligned}$$

Setting

$$\alpha = \left( \frac{2^{1/p}D(1+D)}{s} + \frac{(1+D)}{s} + \frac{(1+D)}{s} \right) \text{ and } \beta = \left( \frac{2^{1/p}M(1+D)}{s^2} + \frac{(1+D)}{s \cdot B^{1/p}} \right),$$

we obtain the inequality,

$$(17) \quad V_{p,1/p}(P_\tau f) \leq \alpha V_{p,1/p}f + \beta \|f\|_p.$$

□

**Theorem 2.2.** *Lasota Yorke Inequality.* For  $s = \min_{1 \leq i \leq N} s_i > 2$  and  $f \in BV_{p,1/p}$ , we obtain the Lasota Yorke inequality.

*Proof.* The Lasota Yorke Inequality holds for a sufficiently small  $A$ . If we choose  $A$  sufficiently small, we can make the first part of  $\alpha$  so small that  $M$  does not play any role. For  $s > 2$ , we will obtain Lasota Yorke inequality. Applying Lemma 2.1 to  $P_\tau$  gives,

$$\begin{aligned} \|P_\tau f\|_{p,1/p} &= V_{p,1/p}(P_\tau f) + \|P_\tau f\|_p \\ &\leq \alpha V_{p,1/p}f + \beta \|f\|_p + \|f\|_p \leq \alpha \|f\|_{p,1/p} + (\beta + 1) \|f\|_p \end{aligned}$$

with  $\alpha < 1$  and  $\beta > 0$ .

□

**Theorem 2.3.** *Let  $f \in BV_{p,1/p}$ . There exist a constant  $K$  for every  $n \in \mathbb{N}$  such that for any  $1 \leq p \leq \infty$ ,*

$$\|P_\tau^n f\|_{p, \frac{1}{p}} \leq K \|f\|_p,$$

where  $K = (1 + \frac{\beta}{1-\alpha})$ ,  $\alpha < 1$ .

*Proof.* From Lemma 2.1 we have,

$$(18) \quad Var_{p, \frac{1}{p}}(P_\tau f) \leq \alpha V_{p, \frac{1}{p}}f + \beta \|f\|_p$$

Since,

$$\|f\|_{p, \frac{1}{p}} = V_{p, \frac{1}{p}}f + \|f\|_p$$

Using this result for  $P_\tau$  in equation (18) we get,

$$\begin{aligned} (19) \quad \|P_\tau f\|_{p, \frac{1}{p}} - \|P_\tau f\|_p &\leq \alpha \left( \|f\|_{p, \frac{1}{p}} - \|f\|_p \right) + \beta \|f\|_p, \\ \|P_\tau f\|_{p, \frac{1}{p}} &\leq \alpha \left( \|f\|_{p, \frac{1}{p}} - \|f\|_p \right) + \beta \|f\|_p + \|P_\tau f\|_p, \\ \|P_\tau f\|_{p, \frac{1}{p}} &\leq \alpha \|f\|_{p, \frac{1}{p}} + (1 - \alpha) \|f\|_p + \beta \|f\|_p. \end{aligned}$$

Now, applying  $P_\tau f$  in above equation, we obtain

$$\begin{aligned}
(20) \quad \|P_\tau^2 f\|_{p, \frac{1}{p}} &\leq \alpha \|P_\tau f\|_{p, \frac{1}{p}} + (1 - \alpha) \|P_\tau f\|_p + \beta \|P_\tau f\|_p \\
&\leq \alpha \left( \alpha \|f\|_{p, \frac{1}{p}} + (1 - \alpha) \|f\|_p + \beta \|f\|_p \right) + (1 - \alpha) \|f\|_p + \beta \|f\|_p \\
&= \alpha^2 \|f\|_{p, \frac{1}{p}} + (1 - \alpha^2) \|f\|_p + (\beta + \alpha\beta) \|f\|_p.
\end{aligned}$$

By induction, for  $n \in \mathbb{N}$

$$\begin{aligned}
\|P_\tau^n f\|_{p, \frac{1}{p}} &\leq \alpha^n \|f\|_{p, \frac{1}{p}} + (1 - \alpha^n) \|f\|_p + (\beta + \beta\alpha + \beta\alpha^2 + \cdots + \beta\alpha^{n-1}) \|f\|_p \\
&= \alpha^n \|f\|_{p, \frac{1}{p}} + (1 - \alpha^n) \|f\|_p + \frac{\beta(1 - \alpha^n)}{1 - \alpha} \|f\|_p
\end{aligned}$$

Since  $\alpha < 1$ , as  $n \rightarrow \infty$ ,  $\alpha^n \rightarrow 0$ , we obtain

$$(21) \quad \|P_\tau^n f\|_{p, \frac{1}{p}} \leq \left( 1 + \frac{\beta}{1 - \alpha} \right) \|f\|_p.$$

□

The set  $\{f \in BV_{p, \frac{1}{p}} \mid \|f\|_{p, \frac{1}{p}} \leq c\}$  is a compact subset of  $L_m^p$  for each  $c > 0$  and  $P_\tau$  is contraction on  $L_m^p$  as shown in Theorem 1.13 of [4]. This allows us to apply Ionescu-Tulcea and Marinescu ergodic theorem [3] for this pair of spaces and obtain the following results:

**Theorem 2.4.** *Under the assumptions of Theorem 2.2 the following results hold:*

- (1)  $P_\tau : L_m^p \rightarrow L_m^p$  has a finite number of eigenvalues  $c_1, c_2, \dots, c_r$  of modulus 1.
- (2) Set  $E_i = \{f \in L_m^p \mid P_\tau f = c_i f\} \subseteq BV_{p, 1/p}$  and  $E_i$  is finite dimensional for  $i = 1, 2, \dots, r$ .
- (3)  $P_\tau = \sum_{i=1}^r c_i \Psi_i + Q$ , where  $\Psi_i$  represents the projection on eigen-spaces denoted by  $E_i$ ,  $\|\Psi_i\|_p \leq 1$  and  $Q$  is a linear operator on  $L_m^p$  with  $Q(BV_{p, 1/p}) \subseteq BV_{p, 1/p}$ ,  $\sup_{n \in \mathbb{N}} \|Q^n\|_p < \infty$  and  $\|Q^n\|_{p, 1/p} = O(q^n)$  for some  $0 < q < 1$ . Furthermore  $\Psi_i \Psi_j = 0 (i \neq j)$  and  $\Psi_i Q = Q \Psi_j = 0$  for all  $i$ .

*Proof.* The result (1), (2) and (3) are direct consequence of Ionescu-Tulcea and Marinescu ergodic Theorem [3]. This shows that  $P_\tau$  is quasicompact operator on  $(BV_{p, 1/p}, \|\cdot\|_{p, 1/p})$ . □

### 3. ROTA'S THEOREM

As  $P_\tau$  is a positive operator from  $L_m^p$  to  $L_m^p$ , we can use Rota's theorem [7] for positive operator on  $L_m^p$  which says that the set,

$$\left\{ \frac{\lambda}{|\lambda|} \mid \lambda \text{ is an eigenvalue of } P, |\lambda| = \|P\|_p \right\},$$

forms a multiplicative subgroup of unit circle.

**Proposition 3.1.** *Let  $\tau \in \mathcal{T}(I)$ . Then  $\tau$  has a finite number of ergodic components. On each component  $\tau$  has an absolutely continuous invariant measure. On each component some iterate of  $\tau$ , say  $\tau^n$  has a finite number of disjoint invariant domains and on each of them  $\tau^n$  is exact.*

*Proof.* By Ionescu-Tulcea and Marinescu theorem  $P_\tau$  has a finite number of eigenvalues of modulus 1. By Rota's theorem they form a finite number of groups of roots of unity. Each such group corresponds to an ergodic component of  $\tau$ . One of eigenvalues in the group is 1 which shows the existence of an acim. If the order of roots in the group is  $n$ , then  $\tau^n$  has all eigenvalues equal to 1 and supports of the eigenfunctions form the disjoint  $\tau^n$ -invariant domains. On each domain  $\tau^n$  is exact. This follows by Theorem 3.2.  $\square$

**Theorem 3.2.** *Let  $\tau$  be a piecewise expanding function on  $C^{1+\varepsilon}$ , and let  $\mu$  be the unique absolutely continuous  $\tau$ -invariant measure. For  $f \in L^p$  and  $g \in L^\infty$ , the correlation  $C_\mu(f, g, N)$  decays exponentially with respect to the number of iterations  $N$ .*

*Proof.* There is only one eigenvalue of modulus 1, which is 1. For  $f \in L^p$  and  $g \in L^\infty$ , the correlation after  $N$  iterations is given by

$$C_\mu(f, g, N) = \left| \int P_\tau^N f \cdot g d\mu - \int f d\mu \int g d\mu \right|.$$

By the Ionescu-Tulcea and Marinescu theorem, we know that

$$P_\tau(f) = \Psi_1(f) + Q(f),$$

where  $\Psi_1(f)$  is the projection of  $f$  on the eigenfunction (say eigenfunction is the constant function 1 so the projection  $\Psi_1(f) = \int f \cdot 1 d\mu = \mu(f) \cdot 1$ ) and  $\|Q\| \leq q < 1$ . Then, for any  $N$ ,

$$P_\tau^N(f) = \Psi_1(f) + Q^N(f).$$

Hence, the correlation becomes

$$\begin{aligned} C_\mu(f, g, N) &= \left| \int \Psi_1(f) \cdot g d\mu + \int Q^N(f) \cdot g d\mu - \mu(f) \cdot \mu(g) \right| \\ &= \left| \int \mu(f) \cdot 1 \cdot g d\mu + \int Q^N(f) \cdot g d\mu - \mu(f) \cdot \mu(g) \right| \\ &= \left| \mu(f)\mu(g) + \int Q^N(f) \cdot g d\mu - \mu(f)\mu(g) \right| \\ &= \left| \int Q^N(f) \cdot g d\mu \right|. \end{aligned}$$

Since  $\|Q^N\| \leq q^N$ , we have

$$C_\mu(f, g, N) \leq q^N \cdot \|f\|_p \|g\|_\infty.$$

As  $q < 1$ ,  $C_\mu(f, g, N)$  decays exponentially as  $N \rightarrow \infty$ . This implies that the dynamical system  $(\tau, \mu)$  is exact.  $\square$

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