SINGULAR SRB MEASURES FOR A NON 1–1 MAP OF THE UNIT SQUARE

PAWEŁ GÓRA, ABRAHAM BOYARSKY, AND ZHENYANG LI

Abstract. We consider a map of the unit square which is not 1–1, such as the memory map studied in [8] Memory maps are defined as follows: $x_{n+1} = M_\alpha(x_{n-1}, x_n) = \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1})$, where τ is a one-dimensional map on $I = [0, 1]$ and $0 < \alpha < 1$ determines how much memory is being used. In this paper we let τ to be the symmetric tent map. To study the dynamics of M_α, we consider the two-dimensional map

$$G_\alpha : [x_{n-1}, x_n] \mapsto [x_n, \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1})] .$$

The map G_α for $\alpha \in (0, 3/4]$ was studied in [8]. In this paper we prove that for $\alpha \in (3/4, 1)$ the map G_α admits a singular Sinai-Ruelle-Bowen measure. We do this by applying Rychlik's results for the Lozi map. However, unlike the Lozi map, the maps G_α are not invertible which creates complications that we are able to overcome.

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1. Introduction

Let τ be a piecewise, expanding map on $I = [0, 1]$. We consider a process

$$x_{n+1} = M_\alpha(x_n) \equiv \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1}) \ , \ 0 < \alpha < 1,$$

which we call a map with memory since the next state x_{n+1} depends not only on current state x_n but also on the past x_{n-1}. Note that M_α is a map from $[0, 1]^2$ to $[0, 1]$ and hence is not a dynamical system.

A natural method to study the long term behaviour of the process M_α, is to study the invariant measures of the two dimensional transformation

$$G_\alpha : [x_{n-1}, x_n] \mapsto [x_n, M_\alpha(x_n)] = [x_n, \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1})] .$$

In [8] we studied G_α with the tent map $\tau(x) = 1 - 2|x - 1/2|$, $x \in I$, for $\alpha \in (0, 3/4]$. For $0 < \alpha \leq 0.46$, we proved that G_α admits an absolutely continuous invariant measure (acim). We conjecture that acim exists also for $\alpha \in [0.46, 1/2)$. As α approaches $1/2$ from below, that is, as we approach a balance between the

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memory state and the present state, the support of the acims become thinner until at $\alpha = 1/2$, all points have period 3 or eventually possess period 3. We proved that for $\alpha = 1/2$ all points (except two fixed points) are eventually periodic with period 3. For $\alpha = 3/4$ we proved that all points of the line $x + y = 4/3$ (except the fixed point) are of period 2 and all other points (except $(0,0)$) are attracted to this line. For $1/2 < \alpha < 3/4$, we prove the existence of a global attractor: for all starting points in the square $[0,1]^2$ except $(0,0)$, the orbits are attracted to the fixed point $(2/3, 2/3)$.

In this paper, we continue the study of transformation G_α for $\alpha \in (3/4, 1)$ and prove the existence of a singular Sinai-Ruelle-Bowen measure μ_α. The invariant measure is singular with respect to Lebesgue measure since for $\alpha \in (3/4, 1)$ the determinants of the derivative matrices of G_α are less than one, hence the support of the invariant measure is of Lebesgue measure 0. The invariant measure has two main properties: for Lebesgue almost every point $x \in [0,1]^2$ and any continuous function $g : [0,1]^2 \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(G_\alpha^k x) = \mu_\alpha(g),$$

and the conditional measures induced by μ_α on segments with expanding directions are one-dimensional absolutely continuous measures.

Our method follows closely the techniques in Rychlik [10] for the Lozi map. The most important difference between the Lozi map and the G_α's is the fact that our maps are not invertible. For maps that are invertible or locally invertible, there are results known [1, 2, 3, 6, 7, 11, 13, 14, 15, 16]. However, to the best of our knowledge the existence of a singular SRB measure has, until now, not been proven for any non-invertible map.

2. Abstract Reduction Theorem

Similarly as Rychlik in [10], we will start with abstract considerations. Sections 2 and 3 are taken from [10] almost without any changes. We present them here for completeness, to introduce the notation and the results we need in the following sections.

Let (X, Σ, m) be a Lebesgue measure with a σ-algebra Σ and a probability measure m. Let $T : X \to X$ be a measurable, nonsingular mapping, i.e., $T_* m \ll m$. We define the Frobenius-Perron operator induced by T as

$$P_T f = \frac{d(T_*(fm))}{dm} \quad (\text{Radon} - \text{Nikodym derivative}),$$

for $f \in L^1(X, \Sigma, m)$ and we have $P_T f \in L^1(X, \Sigma, m)$. Equivalently, we can define $P_T f$ as the unique element of $L^1(X, \Sigma, m)$ satisfying

$$\int_X (h \circ T) \cdot f dm = \int_X h \cdot P_T f dm,$$

for all $h \in L^\infty(X, \Sigma, m)$. This means that operator P_T is the conjugate of the Koopman operator $K_T h = h \circ T$ acting on $L^\infty(X, \Sigma, m)$.

A measurable, countable partition β of X is called regular iff for every $A \in \beta$, $T(A)$ is Σ-measurable and $T_{|A}$ maps $(A, \Sigma_{|A})$ onto $(T(A), \Sigma_{|T(A)})$ isomorphically.

For any regular partition β we define $g_T : X \to \mathbb{R}^+$ as follows:

$$g_T(x) = \frac{d(T_*(\chi_A m))}{dm}(Tx) \ , \ \text{for } x \in A \in \beta. \tag{2.1}$$

We can write $g_T = \sum_{A\in\beta} K_T(P_T\chi_A)\cdot\chi_A$. The function g_T is determined up to a set of measure 0 and does not depend on the choice of partition β. For piecewise differentiable map T the function g_T is the reciprocal of the Jacobian. Using g_T we can express P_T as follows

$$P_T f(x) = \sum_{y\in T^{-1}(x)} g_T(y)\cdot f(y) \ , \ x \in X. \tag{2.2}$$

Equality (2.2) holds m almost everywhere.

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{T} & X \end{array}$$

Now, we consider the case when T is a factor of another mapping $S : Y \to Y$, where (Y, Σ_Y, ν) is a Lebesgue space. We assume that S is nonsingular. By ξ we denote the measurable partition of Y which is S-invariant, i.e., $S^{-1}\xi \le \xi$. Let $X = Y/\xi$ and let $T = S_\xi$ be the factor map. We assume that $m = \pi_*(\nu)$ or $m = \nu\circ\pi$. We denote the natural projection by $\pi : Y \to X$. Let $C(x)$ denote the element $\pi^{-1}(x) \in \xi$. We have $S(C(x)) \subset C(Tx)$. We will find the relation between P_T and P_S.

$$\begin{array}{ccc} L^1(Y,\Sigma_Y,\nu) & \xrightarrow{P_S} & L^1(Y,\Sigma_Y,\nu) \\ \Big\downarrow{\scriptstyle E_\nu(\cdot|\xi)} & & \Big\downarrow{\scriptstyle E_\nu(\cdot|\xi)} \\ L^1(X,\Sigma,m) & \xrightarrow{P_T} & L^1(X,\Sigma,m) \end{array}$$

Proposition 1. *(Rychlik [10], Proposition 1) Let $E_\nu(\cdot|\xi) : L^1(Y,\Sigma_Y,\nu) \to L^1(X,\Sigma,m)$ be the operator of conditional expectation with respect to the σ-algebra generated by the partition ξ, see [4]. For any $f \in L^1(Y,\Sigma_Y,\nu)$ we have*

$$P_T(E_\nu(f|\xi)) = E_\nu(P_S f|\xi).$$

Proof. Let $h \in L^\infty(X,\Sigma,m)$. Then:

$$\begin{aligned} &\int_X h\cdot P_T(E_\nu(f|\xi))dm = \int_X (h\circ T)E_\nu(f|\xi)dm \\ &= \int_Y (h\circ T\circ\pi)(E_\nu(f|\xi)\circ\pi)d\nu = \int_Y (h\circ\pi\circ S)fd\nu \\ &= \int_Y (h\circ\pi)\cdot P_S f d\nu = \int_Y (h\circ\pi)(E_\nu(P_S f|\xi)\circ\pi)d\nu \\ &= \int_X h\cdot E_\nu(P_S f|\xi)dm. \end{aligned} \tag{2.3}$$

We used two properties of the conditional expectation:

(a) If $g \in L^\infty(Y, \Sigma_Y, \nu)$ is ξ-measurable, then $E_\nu(gf|\xi) = gE_\nu(f|\xi)$.
(b) $\int_Y E_\nu(f|\xi)d\nu = \int_Y f d\nu$. □

We assume that S has a regular partition $\mathcal{P}$ with the property

$$S^{-1}\xi \vee \mathcal{P} = \xi. \tag{2.4}$$

We will consider $\xi = \mathcal{P}^- = \bigvee_{k=0}^\infty S^{-k}\mathcal{P}$ so this property will holds automatically.

Lemma 1. *(Rychlik* [10]*, Lemma 1) The family $\beta = \{\pi(A)\}_{A\in\mathcal{P}}$ is a T-regular partition of X.*

Proof. By assumption (2.4) $\mathcal{P} \le \xi$ and thus $\pi^{-1}(\pi A) = A$ for every $A \in \mathcal{P}$. Hence, β is a partition. Also, $\pi^{-1}(T(\pi A)) = S(A)$. Then, $T(\pi A)$ is measurable, by the definition of the factor space and regularity of $\mathcal{P}$. Then, $T_{|\pi(A)} : \pi(A) \to T(\pi(A))$ is the factor $S_{|A} : A \to S(A)$. Moreover, by (2.4) $T_{|\pi(A)}$ is 1-1 (since $S_{|A}$ is almost everywhere 1-1) and $(T_{|\pi(A)})^{-1}$ is the factor of $(S_{|A})^{-1}$. So, $T_{|\pi(A)}$ is an isomorphism of $(\pi(A), \Sigma_{|\pi(A)})$ and $(T(\pi(A)), \Sigma_{|T(\pi(A))})$. □

Let $\{\nu_C\}_{C\in\xi}$ be the family of conditional measures of ν with respect to ξ. In the following proposition we relate g_S, g_T and $\{\nu_C\}_{C\in\xi}$.

Proposition 2. *(Rychlik* [10]*, Proposition 2) For almost every $x \in X$ and $\nu_{C(x)}$-almost every $y \in C(x)$, we have*

$$g_T(x) = g_S(y)\frac{d((S_{|A})_*^{-1}\nu_{C(Tx)})}{d\nu_{C(x)}}(y), \tag{2.5}$$

where A is the element of $\mathcal{P}$ which contains $C(x)$. Note, that $C(x) = S^{-1}(C(Tx)) \cap A$. In particular, $(S_{|A})_*^{-1}\nu_{C(Tx)}$ is equivalent to $\nu_{C(x)}$ for almost every $x \in X$.

Proof. Let $h \in L^\infty(Y, \Sigma_Y, \nu)$. Then,

$$E_\nu(P_S h|\xi)(x) = \int_{C(x)} \sum_{A\in\mathcal{P}} (h \cdot g_S) \circ (S_{|A})^{-1} d\nu_{C(x)} = \int h d\sigma_{1,x}, \tag{2.6}$$

where

$$\sigma_{1,x} = \sum_{z\in T^{-1}(x)} g_S \cdot ((S_{|C(z)})_*^{-1}\nu_{C(x)}).$$

The first inequality in (2.6) follows by the definition of $P_S h$ and the fact that $E_\nu(P_S h|\xi)$ is almost surely constant on elements of ξ. The second, by the definition of g_S. Also, we have

$$P_T E_\nu(h|\xi)(x) = \sum_{z\in T^{-1}(x)} \int_{C(z)} h d\nu_{C(z)} g_T(z) = \int h d\sigma_{2,x}, \tag{2.7}$$

where

$$\sigma_{2,x} = \sum_{z\in T^{-1}(x)} g_T(z)\, \nu_{C(z)}.$$

In view of Proposition 1 since h is arbitrary the measures $\sigma_{1,x}$ and $\sigma_{2,x}$ are equal for almost every x. Since the measures $\nu_{C(x)}$ have disjoint supports and since functions g_T and g_S are positive almost everywhere the equality (2.5) is proved. □

We will consider situation when the elements of ξ are endowed with some natural measure. Let $\{\ell_C\}_{C\in\xi}$ be a family of such measures such that for any $C\in\xi$ the measure ℓ_C is equivalent to ν_C and the Radon-Nikodym derivative

$$\rho = \frac{d\nu_C}{d\ell_C},$$

defined on Y is Σ_Y-measurable. Then, for almost every $x\in X$, $(S_{|C(x)})_*^{-1}\ell_{C(Tx)}$ is also equivalent to $\ell_{C(x)}$. Also, the function

$$\lambda(y) = \frac{d((S_{|C(x)})_*^{-1}\ell_{C(Tx)})}{d\ell_{C(x)}}(y)\ ,\ y\in C(x),$$

is Σ_Y-measurable.

Let us now consider the situation when $S:[0,1]^2\to[0,1]^2$ preserves two families of cones, the cone of stable directions and the cone of unstable directions. Let $\xi=\mathcal{P}^-$ be the S-invariant partition which consists of pieces of lines with stable directions. On each element $C\in\xi$ we have Lebesgue measure ℓ_C. If ν is the Lebesgue measure on $[0,1]^2$, then we proved in Proposition 7 that for almost all C, the conditional measure ν_C is equivalent to ℓ_C.

By Proposition 2 we have

$$g_T\circ\pi = g_S\frac{\rho\circ S}{\rho}\lambda.$$

For y,y' belonging to the same C it gives

$$\frac{\rho(y)}{\rho(y')} = \frac{g_S(y)}{g_S(y')}\frac{\rho(Sy)}{\rho(Sy')}\frac{\lambda(y)}{\lambda(y')},$$

and by induction

$$\frac{\rho(y)}{\rho(y')} = \left(\prod_{k=0}^{n-1}\frac{g_S(S^ky)}{g_S(S^ky')}\frac{\lambda(S^ky)}{\lambda(S^ky')}\right)\frac{\rho(S^ny)}{\rho(S^ny')}. \tag{2.8}$$

Formula (2.8) proves the following:

Proposition 3. *(Rychlik's Proposition 3) For almost every $x\in X$ and for $\nu_{C(x)}$-almost every $y,y'\in C(x)$, the following conditions are equivalent:*

(a)

$$\lim_{n\to\infty}\frac{\rho(S^ny)}{\rho(S^ny')} = 1;$$

(b)

$$\frac{\rho(y)}{\rho(y')} = \prod_{k=0}^{\infty}\frac{g_S(S^ky)}{g_S(S^ky')}\frac{\lambda(S^ky)}{\lambda(S^ky')}.$$

Remark 1. *If (a) holds almost everywhere, then (b) and the condition $\int\rho d\ell_C=1$ determine ρ completely.*

3. Existence of Absolutely Continuous Invariant Measures.

As before, let T be a nonsingular map of a Lebesgue space (X,Σ,m). Let β be a regular partition of X such that $\beta^- = \bigvee_{k=0}^{\infty}T^{-k}(\beta)$ is a partition into points, i.e., β is a generator for T. We will give conditions which prove that T admits an invariant measure absolutely continuous with respect to m. We introduce the following notations: $g=g_T$, $g_n=g_{T^n}$, $P=P_T$, for any $A\in\Sigma$, $\beta(A)=\{B\in\beta:$

$m(B\cap A)>0\}$. By supremum and infimum we understand the essential supremum or minimum.

(I) Distortion condition:

$$\exists_{\ d>0}\ \forall_{\ n\geq 1}\ \forall_{\ B\in\beta^n}\ \sup_B g_n \leq d\cdot \inf_B g_n;$$

(II) Localization condition:

$$\exists_{\varepsilon>0}\ \exists_{\ 0<r<1}\ \forall_{\ n\geq 1}\ \forall_{\ B\in\beta^n}\ m(T^nB)<\varepsilon \Longrightarrow \sum_{B'\in\beta(T^nB)} \sup_{B'} g \leq r;$$

(III) Bounded variation condition:

$$\sum_{B\in\beta} \sup_B g < +\infty.$$

Remark 2. *If conditions (I) and (III) hold for T, β and g, then they also hold for T^N, β^N and g_N. Condition (I) holds with the same value of d. Moreover,*

$$\sum_{B\in\beta^N} \sup_B g_N \leq \left(\sum_{B\in\beta} \sup_B g\right)^N .$$

Theorem 1. *(Rychlik [10], Theorem 1) Let (I)–(III) be satisfied. Then, the sequence $(P^n\mathbf{1})_{n\geq 1}$ is bounded in $L^\infty(X,\Sigma,m)$ and the averages $\frac{1}{n}\sum_{k=0}^{n-1}P^k\mathbf{1}$ converge in $L^1(X,\Sigma,m)$ to some $\phi\in L^\infty(X,\Sigma,m)$ such that $P\phi=\phi$.*

Proof. For the proof we refer to [10] or to [5]. □

Theorem 1 gives the existence of an absolutely continuous invariant measure. To improve on this result we introduce the following condition:

(IV) Expanding condition:

$$\exists_{\ r\in(0,1)}\ \sup_X g \leq r.$$

We can assume that r is chosen to satisfy both (II) and (IV).

Theorem 2. *(Rychlik [10], Theorem 2) Let (I)–(IV) be satisfied. Then, there exists a bounded, finite dimensional projection $Q: L^1(X,\Sigma,m)\to L^\infty(X,\Sigma,m)$ such that*

(1) $Q(L^1(X,\Sigma,m))\subset L^\infty(X,\Sigma,m)$ and Q is bounded as an operator from $L^1(X,\Sigma,m)$ to $L^\infty(X,\Sigma,m)$;

(2) for every $f\in L^1(X,\Sigma,m)$ the averages $\frac{1}{n}\sum_{k=0}^{n-1}P^kf$ converge in $L^1(X,\Sigma,m)$ to Qf.

(3) The range $\mathcal{R}(Q)$ of Q consists of all eigenvectors of P corresponding to the eigenvalue 1 and of 0 function.

(4) There exist non-negative functions $\phi_1,\phi_2,\dots,\phi_s\in\mathcal{R}(Q)$ which span $\mathcal{R}(Q)$ and $\phi_i\wedge\phi_j=0$ as $i\neq j$. Moreover, $\int_X\phi_i dm=1$, $i=1,\dots,s$ and if

$$C_i=\bigcup_{n=0}^{\infty}T^{-n}\{x:\phi_i(x)>0\}$$

is the basin of attraction of the measure $\phi_i m$, then Q can be represented as

$$Qf=\sum_{i=1}^{s}\int_{C_i} f\,dm\cdot\phi_i.$$

Moreover, $\bigcup_{i=1}^{s} C_i = X$ *up to a set of measure 0 and* $\{\phi_i\}_{i=1}^{s}$ *are the only functions* $\phi \in L^1(X, \Sigma, m)$ *such that* $\phi \cdot m$ *is a* T*-invariant, ergodic, probabilistic measure.*

Proof. For the proof we refer to [10]. □

4. Preliminary Results for Maps G_α when $\alpha \in (\frac{3}{4}, 1)$.

Recall the map $G = G_\alpha$:

$$G(x, y) = [y, \tau(\alpha y + (1-\alpha)x)] \ , \ (x, y) \in [0, 1]^2,$$

where τ is the tent map $x \mapsto 1 - 2|x - 1/2|$. L denotes the line $\alpha y + (1-\alpha)x = 1/2$ which divides $[0, 1]^2$ into two domains on which G is 1–1. A more explicit formula for G is

$$G(x, y) = \begin{cases} (y, 2(\alpha y + (1-\alpha)x) & , \text{ if } y \text{ is below } L; \\ (y, 2 - 2(\alpha y + (1-\alpha)x) & , \text{ if } y \text{ is above } L. \end{cases}$$

We have two possibilities for the Jacobian matrices:

$$A_\pm = DG = \begin{bmatrix} 0 & 1 \\ (1-\alpha)\,\tau'\,(\alpha y + (1-\alpha)x) & \alpha\tau'\,(\alpha y + (1-\alpha x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \pm 2\,(1-\alpha) & \pm 2\alpha \end{bmatrix},$$

with $+$ sign for $(x, y) \in A_1$, the region below line L and $-$ sign for $(x, y) \in A_2$, the region above line L. Similarly, when we consider the inverse branches G_1^{-1} and G_2^{-1} , we have two Jacobian matrices:

$$B_\pm = DG^{-1} = \begin{bmatrix} \frac{-\alpha}{1-\alpha} & \pm\frac{1}{2(1-\alpha)} \\ 1 & 0 \end{bmatrix}.$$

We now construct invariant cones of directions in the tangent spaces as in [10].

For $A_\pm$, we consider the direction vector in the form $(u, 1)$. Then,

$$\begin{bmatrix} 0 & 1 \\ \pm 2\,(1-\alpha) & \pm 2\alpha \end{bmatrix} \begin{bmatrix} u \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \pm 2u\,(1-\alpha) \pm 2\alpha \end{bmatrix}$$
$$= (\pm 2u\,(1-\alpha) \pm 2\alpha) \begin{bmatrix} \frac{1}{\pm 2u(1-\alpha) \pm 2\alpha} \\ 1 \end{bmatrix}.$$

Let

$$S_\pm(u) = \frac{1}{\pm 2u\,(1-\alpha) \pm 2\alpha}, \tag{4.1}$$

be the corresponding transformation on directions.

For $B_\pm$, we consider the direction vector in the form $(1, v)$. Then,

$$\begin{bmatrix} \frac{-\alpha}{1-\alpha} & \pm\frac{1}{2(1-\alpha)} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \end{bmatrix} = \begin{bmatrix} \frac{-\alpha}{1-\alpha} \pm \frac{v}{2(1-\alpha)} \\ 1 \end{bmatrix}$$
$$= (\frac{-\alpha}{1-\alpha} \pm \frac{v}{2(1-\alpha)}) \begin{bmatrix} 1 \\ \frac{1}{\frac{-\alpha}{1-\alpha} \pm \frac{v}{2(1-\alpha)}} \end{bmatrix}.$$

Let

$$T_\pm(v) = \frac{1}{\frac{-\alpha}{1-\alpha} \pm \frac{v}{2(1-\alpha)}} = \frac{2(1-\alpha)}{-2\alpha \pm v}, \tag{4.2}$$

be the corresponding transformation on directions.

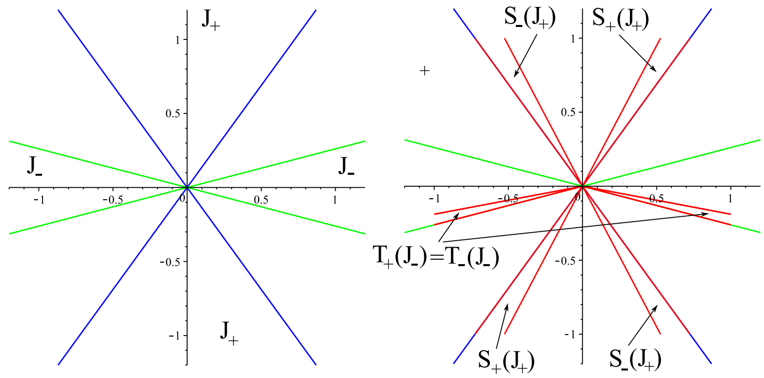

FIGURE 1. Invariant cones J_+ and J_- and their images for $\alpha = 0.82$.

Lemma 2. *(Rychlik* [10]*, Lemma 3) Let* $\theta_0 = \alpha - \sqrt{\alpha^2 + 2\alpha - 2}$, $J_+ = \left\{u \in \mathbb{R} \middle| |2u\,(1-\alpha)\,| \le \theta_0\right\}$, $J_- = \left\{v \in \mathbb{R} \middle| |v| \le \theta_0\right\}$. *Then,* J_+ *and* J_- *are* $S_\pm-$ *and* $T_\pm-$*invariant, respectively.*

Proof. First, note that $\theta_0 < \alpha$. We will prove the case of S_+. The case for S_- is similar. It follows from $|2u\,(1-\alpha)\,| \le \theta_0$ that

$$2\alpha - \theta_0 \le 2\alpha + 2u\,(1-\alpha) \le 2\alpha + \theta_0.$$

Thus,

$$|2(1-\alpha)S_+(u)| \le \frac{2(1-\alpha)}{2\alpha - \theta_0} \le \theta_0, \tag{4.3}$$

where the last inequality follows from the definition of θ_0.

Now we prove the case of T_+. The case of T_- is similar. It follows from $|v| \le \theta_0$ that

$$-3\alpha + \sqrt{\alpha^2 + 2\alpha - 2} \le -2\alpha + v \le -\alpha - \sqrt{\alpha^2 + 2\alpha - 2}.$$

Thus,

$$\begin{aligned} |T_+(v)| &= \left|\frac{2(1-\alpha)}{-2\alpha + v}\right| = \frac{2(1-\alpha)}{|-2\alpha + v|} \\ &\le \frac{2(1-\alpha)}{\alpha + \sqrt{\alpha^2 + 2\alpha - 2}} = \theta_0. \end{aligned} \tag{4.4}$$

□

Remark 3. *We also have* $S_\pm(J_+) \subseteq \left\{u \in \mathbb{R} \middle| |2u\,(1-\alpha)\,| \ge \frac{2(1-\alpha)}{2\alpha+\theta_0}\right\} = \left\{u \in \mathbb{R} \middle| |u| \ge \theta_1\right\}$, $T_\pm(J_-) \subseteq \left\{v \in \mathbb{R} \middle| |v| \ge 2(1-\alpha)\theta_1\right\}$, *where* $\theta_1 = \frac{1}{2\alpha+\theta_0}$.

Lemma 3. *(Rychlik* [10]*, Lemma 4) Let* $\kappa = \frac{\alpha - \sqrt{\alpha^2+2\alpha-2}}{\alpha+\sqrt{\alpha^2+2\alpha-2}}$, *which is less than 1 (actually it is less than 0.5 and decreasing with respect to* α*). Then,* $\sup_{J_+} |S'_\pm(u)| = \sup_{J_-} |T'_\pm(v)| = \kappa$.

Proof. It follows from (4.3) that

$$|S'_{\pm}(u)| = 2(1-\alpha)S^2_{\pm}(u) \leq 2(1-\alpha)\left(\frac{\theta_0}{2(1-\alpha)}\right)^2 = \kappa.$$

And, it follows from (4.4) that

$$|T'_{\pm}(v)| = \frac{1}{2(1-\alpha)}T^2_{\pm}(v) \leq \frac{1}{2(1-\alpha)}\theta^2 = \kappa.$$

□

Using Lemma 2 and Lemma 3, we see that for any sequence $(\varepsilon_0, \varepsilon_1, \dots) \in \{+,-\}^\infty$, we have $|(T_{\varepsilon_{n-1}} \circ T_{\varepsilon_{n-2}} \circ \dots \circ T_{\varepsilon_1} \circ T_{\varepsilon_0})(J_-)| \leq \kappa^n |J_-|$ so the set

$$\bigcap_{n=1}^{\infty} (T_{\varepsilon_{n-1}} \circ T_{\varepsilon_{n-2}} \circ \dots \circ T_{\varepsilon_1} \circ T_{\varepsilon_0})(J_-),$$

consists of exactly one point which can be expressed as a continued fraction:

$$\frac{2(1-\alpha)}{-2\alpha + \frac{\epsilon_0 2(1-\alpha)}{-2\alpha + \frac{\epsilon_1 2(1-\alpha)}{-2\alpha + \frac{\epsilon_2 2(1-\alpha)}{-2\alpha + \cdots}}}}.$$

Similarly, for any sequence $(\eta_0, \eta_1, \dots) \in \{+,-\}^\infty$, we have $|(S_{\eta_{n-1}} \circ S_{\eta_{n-2}} \circ \dots \circ S_{\eta_1} \circ S_{\eta_0})(J_-)| \leq \kappa^n |J_-|$ so the set

$$\bigcap_{n=1}^{\infty} (S_{\eta_{n-1}} \circ S_{\eta_{n-2}} \circ \dots \circ S_{\eta_1} \circ S_{\eta_0})(J_-),$$

consists of exactly one point which can be expressed as a continued fraction:

$$\frac{\eta_0}{2\alpha + 2(1-\alpha)\frac{\eta_1}{2\alpha + 2(1-\alpha)\frac{\eta_2}{2\alpha + \cdots}}}.$$

They are both convergent since $\kappa < 1$.

Now, using the above construction we define invariant directions for G. For points $p \in U^s = [0,1]^2 \setminus \bigcup_{n=0}^{\infty} G^{-n}(L)$, setting $\varepsilon_i = +$ or $\varepsilon_i = -$, depending on whether $G^i(p)$ is below or above the line L, we obtain the invariant stable direction $v(p) \in J_-$,

$$v(p) = \frac{2(1-\alpha)}{-2\alpha + \frac{\epsilon_0 2(1-\alpha)}{-2\alpha + \frac{\epsilon_1 2(1-\alpha)}{-2\alpha + \frac{\epsilon_2 2(1-\alpha)}{-2\alpha + \cdots}}}}.$$

To construct an invariant unstable direction for a point p we have to use G-preimages of p. Since G is not invertible the "invariant" direction will depend on the chosen admissible past of the point p. Some points have only one admissible past, for example, for the fixed point $(2/3, 2/3)$ the only admissible past is $(\dots, 2, 2, \dots, 2, 2)$ and it has the unique well defined unstable direction. Other points have finite number or infinitely many admissible pasts. The richest case happens when the directions in the set of "invariant" directions form a Cantor set, namely the attractor of the Iterated Function System $\{S_+, S_-\}$. For a point $p \in U^u = [0,1]^2 \setminus \bigcup_{n=0}^{\infty} G^n(L)$ with specified past $(\dots, k_{n-1}, \dots, k_1, k_0) \in \{1,2\}^\infty$ we

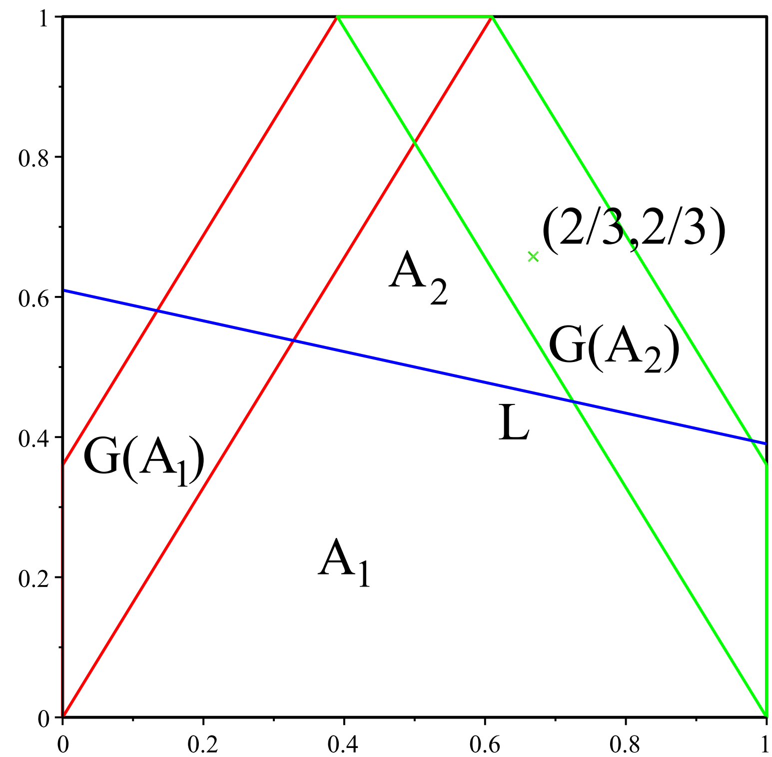

FIGURE 2. Partition line L, regions A_1, A_2 and their images, fixed point $(2/3, 2/3)$ for $\alpha = 0.82$.

choose $\eta_i = +$ when $k_i = 1$ or $\eta_i = -$ when $k_i = 2$, and obtain the invariant stable direction $u(p) \in J_+$,

$$u(p) := \frac{\eta_0}{2\alpha + 2(1-\alpha)\frac{\eta_1}{2\alpha+2(1-\alpha)\frac{\eta_2}{2\alpha+\cdots}}}.$$

We now compute $\lambda^s(p)$ and $\lambda^u(p)$, which represent the rates of change on the length along directions of E^s and E^u, respectively. For directions in E^u the rate is independent of the chosen past of the point.

Lemma 4. *(Rychlik* [10], *Lemma 5)*

$$\lambda^s(p) = |v(p)|\frac{h_1(p)}{h_1(G(p))}, \quad \lambda^u(p) = |u(p)|\frac{h_2(p)}{h_2(G^{-1}(p))},$$

where,

$$h_1(p) = \frac{1}{\sqrt{(v(p))^2+1}}, \quad h_2(p) = \frac{1}{\sqrt{(u(p))^2+1}}.$$

Proof.

$$\begin{aligned}
DG(p)\begin{bmatrix} 1 \\ v(p) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \pm 2\,(1-\alpha) & \pm 2\alpha \end{bmatrix}\begin{bmatrix} 1 \\ v(p) \end{bmatrix} \\
&= \begin{bmatrix} v(p) \\ \pm 2\,(1-\alpha) \pm 2\alpha v(p) \end{bmatrix} = v(p)\begin{bmatrix} 1 \\ \pm\frac{2(1-\alpha)}{v(p)} \pm 2\alpha \end{bmatrix} \\
&= v(p)\begin{bmatrix} 1 \\ T_{\pm}^{-1}(v(p)) \end{bmatrix} = v(p)\begin{bmatrix} 1 \\ v(G(p)) \end{bmatrix}.
\end{aligned}$$

Thus,

$$\lambda^s(p) = \frac{\|v(p)(1, v(G(p)))\|}{\|(1, v(p))\|} = |v(p)|\frac{h_1(p)}{h_1(G(p))}.$$

Similarly,

$$\begin{aligned}
DG^{-1}(p)\begin{bmatrix} u(p) \\ 1 \end{bmatrix} &= \begin{bmatrix} \frac{-\alpha}{1-\alpha} & \pm\frac{1}{2(1-\alpha)} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u(p) \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\alpha}{1-\alpha}u(p) \pm \frac{1}{2(1-\alpha)} \\ u(p) \end{bmatrix} = u(p)\begin{bmatrix} \frac{-\alpha}{1-\alpha} \pm \frac{1}{2(1-\alpha)u(p)} \\ 1 \end{bmatrix} \\
&= u(p)\begin{bmatrix} S_{\pm}^{-1}(u(p)) \\ 1 \end{bmatrix} = u(p)\begin{bmatrix} u(G^{-1}(p)) \\ 1 \end{bmatrix},
\end{aligned}$$

and thus,

$$\lambda^u(p) = \frac{\|u(p)(u(G^{-1}(p)), 1)\|}{\|(u(p), 1)\|} = |u(p)|\frac{h_2(p)}{h_2(G^{-1}(p))}.$$

□

We need the conditions that both θ_0 and $\frac{\theta_0}{2(1-\alpha)}$ are less than 1, which hold since $\alpha \in (\frac{3}{4}, 1)$.

Now we present a proposition analogous to Proposition 5 in Rychlik [10].

Proposition 4. *Let $\lambda_+ = \frac{\theta_0}{2(1-\alpha)}$, $\lambda_- = \theta_0$. Then both $\lambda_+, \lambda_- \in (0, 1)$. And there exists a constant $C > 0$ such that $|\lambda_n^s(p)| \leq C\lambda_-^n$ if $p \in U^s$, $|\lambda_n^u(p)| \leq C\lambda_+^n$ if $p \in U^u$.*

Proof. Using Lemma 2 and the invariant sets J_+ and J_-, it follows that h_1 and h_2 are bounded, i.e. there exists numbers c_1 and c_2 such that $0 < c_1 \leq h_i \leq c_2$, $i = 1, 2$. Thus, by Lemma 4

$$\begin{aligned}
\lambda_n^s(p) &= \lambda^s(G^{n-1}(p)) \cdot \lambda^s(G^{n-2}(p)) \cdots \lambda^s(G(p))\lambda^s(p) \\
&= |v(G^{n-1}(p))|\frac{h_1(G^{n-1}(p))}{h_1(G^n(p))} \cdot |v(G^{n-2}(p))|\frac{h_1(G^{n-2}(p))}{h_1(G^{n-1}(p))} \cdots \\
&\quad \cdot |v(G(p))|\frac{h_1(G(p))}{h_1(G^2(p))} \cdot |v(p)|\frac{h_1(p)}{h_1(G(p))} \\
&= |v(G^{n-1}(p))| \cdot |v(G^{n-2}(p))| \cdots |v(p)|\frac{h_1(p)}{h_1(G^n(p))} \\
&\leq \frac{c_2}{c_1}\theta_0^n.
\end{aligned}$$

Similarly, we have

$$\lambda_n^u(p) \leq \frac{c_2}{c_1}\left(\frac{\theta_0}{2(1-\alpha)}\right)^n .$$

□

Let $\mathcal{P} = \mathcal{P}^{(1)}$ be the partition of the square $[0,1]^2$ into the regions of definition of the map G, i.e., $A_1 = \{(x,y) : \alpha y + (1-\alpha)x \leq 1/2\}$ and $A_2 = \{(x,y) : \alpha y + (1-\alpha)x \geq 1/2\}$. These regions intersect, but the intersection is a negligible set both in a measure-theoretic and topological sense. We define $\mathcal{P}^{(n)} = \mathcal{P} \bigvee G^{-1}(\mathcal{P}) \bigvee G^{-2}(\mathcal{P}) \bigvee \cdots \bigvee G^{n-1}(\mathcal{P})$. $\mathcal{P}^{(n)}$ is the defining partition for the map G^n.

Let L denote the partition line

$$L = \{p = (x,y) : \alpha y + (1-\alpha)x = 1/2\}.$$

Lemma 5. *(Rychlik* [10]*, Lemma 8) For every $N \geq 1$ there is an open cover $\mathcal{U}_N$ of the unit square such that every element of $\mathcal{U}_N$ intersects no more than $2N$ elements of $\mathcal{P}^{(N)}$.*

The proof is exactly the same as in [10].

Proposition 5. *(Rychlik* [10]*, Proposition 7) There exist constants $F > 0$ and $0 < r < 1$ such that for any segment I with the direction from the unstable cone J_+ we have*

$$\Gamma_n(I) = \sum_{J \in \mathcal{P}^{(n)}|I} \frac{|J|}{|G^n(J)|} \leq F(r^n + |I|), \tag{4.5}$$

where $|\cdot|$ denotes the length of the segment.

Proof. The proof follows closely the proof from [10]. We choose N in such a way that

$$r_0 = 2NC(\lambda_+)^N < 1,$$

where C and λ_+ are from Proposition 4. Let ε_0 be the Lebesgue constant of the cover $\mathcal{U}_N$ from Lemma 5. Let us define

$$\gamma_n = \sum_{J \in \mathcal{P}^{(nN)}|I} \frac{|J|}{|G^{nN}(J)|}, \quad n = 1, 2, \ldots \tag{4.6}$$

We will show that

$$\gamma_{n+1} \leq r_0 \gamma_n + \frac{1}{\varepsilon_0} R_0 |I|, \tag{4.7}$$

where

$$R_0 = \sup_I \gamma_1 = \sup_I \sum_{J \in \mathcal{P}^{(N)}|I} \frac{|J|}{|G^N(J)|},$$

and I is any segment with the direction from the unstable cone J_+.

Let $J \in \mathcal{P}^{(nN)}|I$. Either $|J| < \varepsilon_0$ or $|J| \geq \varepsilon_0$. In the first case $\mathcal{P}^{((n+1)N)}|J$ consists of not more that $2N$ elements (Lemma5) as the partition $\mathcal{P}^{((n+1)N)}$ is obtained from $\mathcal{P}^{(nN)}$ in N steps. Thus,

$$\begin{aligned}\sum_{J'\in\mathcal{P}^{((n+1)N)}|J} \frac{|J'|}{|G^{(n+1)N}(J')|} &\le 2N \max_{J'} \frac{|J'|}{|G^{(n+1)N}(J')|}\\ &= 2N \ \max_{J'}\left(\frac{|J'|}{|G^{nN}(J')|}\ \frac{|G^{nN}(J')|}{|G^{(n+1)N}(J')|}\right)\\ &= 2N\ \frac{|J|}{|G^{nN}(J)|}\ \max_{J'}\frac{|G^{nN}(J')|}{|G^{(n+1)N}(J')|}\\ &\le 2N\frac{|J|}{|G^{nN}(J)|}\ C(\lambda_+)^N = r_0\frac{|J|}{|G^{nN}(J)|}.\end{aligned} \tag{4.8}$$

We have used the fact that $J' \subset J$ and G^{nN} is a linear transformation on J, so the expansion rate is uniform on J.

In the second case we have

$$\begin{aligned}\sum_{J'\in\mathcal{P}^{((n+1)N)}|J} \frac{|J'|}{|G^{(n+1)N}(J')|} &= \sum_{J'\in\mathcal{P}^{((n+1)N)}|J}\left(\frac{|J'|}{|G^{nN}(J')|}\ \frac{|G^{nN}(J')|}{|G^{(n+1)N}(J')|}\right)\\ &= \frac{|J|}{|G^{nN}(J)|}\ \sum_{J'\in\mathcal{P}^{((n+1)N)}|J}\frac{|G^{nN}(J')|}{|G^{(n+1)N}(J')|}\\ &\le \frac{|J|}{|G^{nN}(J)|}R_0 \le R_0\frac{1}{\varepsilon_0}|J|.\end{aligned} \tag{4.9}$$

We used again the linearity of G^{nN} on J. Moreover

$$\sum_{J'\in\mathcal{P}^{((n+1)N)}|J}\frac{|G^{nN}(J')|}{|G^{(n+1)N}(J')|} \le \sum_{K\in\mathcal{P}^{(N)}|G^{nN}(J)}\frac{|K|}{|G^{N}(K)|} \le R_0,$$

since intervals $G^{nN}(J')$ are elements of $\mathcal{P}^{(N)}|G^{nN}(J)$ and $G^{nN}(J)$ has the direction from J_+. Also $|G^{nN}(J)| > |J| > \varepsilon_0$.

Summing up (4.8) and (4.9) over all $J \in \mathcal{P}^{(nN)}|I$, we obtain

$$\begin{aligned}\sum_{J'\in\mathcal{P}^{((n+1)N)}|I} \frac{|J'|}{|G^{(n+1)N}(J')|} &\le \sum_{J\in\mathcal{P}^{(nN)}|I}\left(r_0\frac{|J|}{|G^{nN}(J)|} + R_0\frac{1}{\varepsilon_0}|J|\right)\\ &= r_0\gamma_n + R_0\frac{1}{\varepsilon_0}|I|,\end{aligned} \tag{4.10}$$

and (4.7) is proved. To obtain inequality (4.5) from (4.7) we proceed as follows. Since $\gamma_1 \le R_0$ by definition, the inequality (4.7) implies

$$\gamma_n \le r_0^n R_0 + \frac{R_0}{\varepsilon_0(1-r_0)}|I|, \quad n = 1, 2, \ldots, \tag{4.11}$$

or, using capital gamma notation

$$\Gamma_{nN}(I) \le r_0^n R_0 + \frac{R_0}{\varepsilon_0(1-r_0)}|I|, \quad n = 1, 2, \ldots.$$

Let us define

$$\bar{R}_i = \sup_I \Gamma_i(I) = \sup_I \sum_{J\in\mathcal{P}^{(i)}|I}\frac{|J|}{|G^i(J)|}, \quad i = 1, 2, \ldots, N,$$

where sup is taken over all segments with the direction in the expanding cone J^+. Of course $\bar{R}_N = R_0$. Let $R = \max\{\bar{R}_1, \bar{R}_2, \dots, \bar{R}_N\}$. Let us consider arbitrary $n \geq 1$ and represent it as $n = k \cdot N + \ell$, $0 < \ell \leq N$. Similarly as above,using in all considerations as the initial partition $\mathcal{P}^{(\ell)}|I$ instead of $\mathcal{P}^{(N)}|I$, we can prove that

$$\Gamma_n(I) \leq r_0^k \bar{R}_\ell + \frac{\bar{R}_\ell}{\varepsilon_0(1-r_0)}|I|.$$

To make these estimates independent of ℓ we can write

$$\Gamma_n(I) \leq r_0^k R + \frac{R}{\varepsilon_0(1-r_0)}|I|.$$

Now, let $r = (r_0)^{1/N}$ and $F = \max\{\frac{R}{r^{N-1}}, \frac{R}{\varepsilon_0(1-r_0)}\}$. We obtain inequality (4.5). $\square$

We define $\mathcal{P}^- = \bigvee_{n=0}^{\infty} G^{-n}(\mathcal{P})$. Elements of $\mathcal{P}^-$ are either segments with direction from the stable cone or points. Let $\xi(p) \in \mathcal{P}^-$ denote an element of $\mathcal{P}^-$ containing point p.

Lemma 6. *(corresponds to Lemma 9 of* [10]*) Let*

$$D^s(\delta) = \{p \in [0,1]^2 : \mathrm{dist}(G^n p, L) \geq \delta \lambda_n^s(p), \text{for } n = 0, 1, 2, \dots\}. \tag{4.12}$$

For every $p \in D^s(\delta)$ the distance from p to the endpoints of $\xi(p)$ is not smaller than δ. In particular, $|\xi(p)| \geq 2\delta$.

Proof. Assume that the distance from p to one of the endpoints of $\xi(p)$ called q is $\mathrm{dist}(p,q) < \delta$. Since endpoints of elements ξ belong to preimages $G^{-n}(L)$, there is an integer $k \geq 0$ such that $q \in G^{-k}(L)$. Then,

$$\mathrm{dist}(G^k p, L) \leq \mathrm{dist}(G^k p, G^k q) \leq \lambda_k^s(p)\mathrm{dist}(p,q) < \delta \lambda_k^s(p),$$

which contradicts $p \in D^s(\delta)$. $\square$

Lemma 7. *(corresponds to Lemma 10 of* [10]*) Let $(\lambda_n) = (\lambda_n)_{n=0}^{\infty}$ be a sequence of positive numbers such that $Z = \sum_{n=0}^{\infty} \lambda_n < +\infty$. Let*

$$D^s(\delta, (\lambda_n)) = \{p \in [0,1]^2 : \mathrm{dist}(G^n p, L) \geq \delta \lambda_n, \text{for } n = 0, 1, 2, \dots\}. \tag{4.13}$$

Let I be a segment with direction from unstable cone. Then, there is a constant A_1 such that $|I \setminus D^s(\delta, (\lambda_n))| \leq A_1 \cdot Z \cdot \delta$.

Proof. We follow closely Rychlik [10]. Let

$$C(t) = \{q : \mathrm{dist}(q, L) \leq t\},$$

where $t \geq 0$. Let $p \in I \setminus D^s(\delta, (\lambda_n))$. There exists $n \geq 0$ such that $\mathrm{dist}(G^n p, L) < \delta \lambda_n$. Let $J \in \mathcal{P}^{(n)}|I$ be the subinterval containing point p. Then, $G^n p$ belongs to the interval $G^n J$ such that

$$|G^n J \cap C(\delta \lambda_n)\}| \leq A_0 \cdot \delta \lambda_n,$$

for some constant A_0 independent of δ and n, as $G^n J$ has a direction from the expanding cone and thus, the angle between $G^n J$ and line L is bounded away from 0. Thus, $p \in J \cap G^{-n}(C(\delta \lambda_n))$ and

$$|J \cap G^{-n}(C(\delta \lambda_n))| \leq \frac{A_0 \cdot \delta \lambda_n}{|G^n J|} \cdot |J|.$$

By Proposition 5, this gives

$$|I \cap G^{-n}(C(\delta \lambda_n))| \leq A_0 \cdot \delta \lambda_n \cdot F(r^n + |I|) \leq A_0 F (1 + \mathrm{diam}([0,1]^2)) \delta \lambda_n.$$

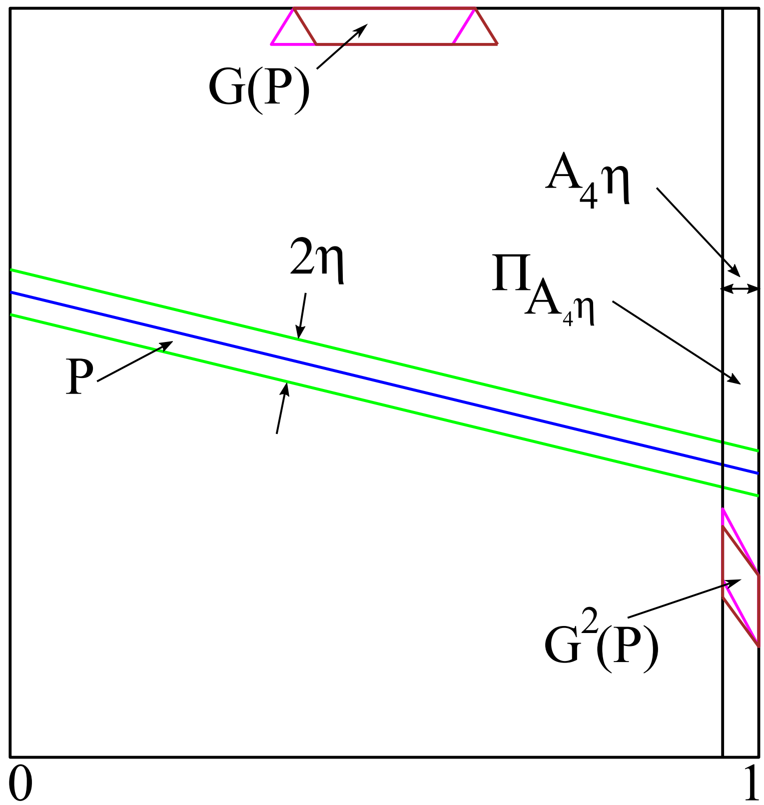

FIGURE 3. Two images of a neighbourhood of the partition line L. Both $G(P)$ and $G^2(P)$ are unions of two parallelograms.

Summing up over all n, we obtain

$$|I \setminus D^s(\delta, (\lambda_n))| \leq A_0 F(1+\sqrt{2})\delta Z.$$

The Lemma is proved with $A_1 = A_0 F(1+\sqrt{2})$. □

Corollary 1. *For any interval I with the direction from the expanding cone we have*

$$|I \setminus D^s(\delta)| \leq A_2 \cdot \delta,$$

where $A_2 = A_1 \sum_{n=0}^{\infty} C\lambda_- = A_1 C/(1-\lambda_-)$.

Proof. Let $\lambda_n = C\lambda_-^n$, $n = 0, 1, 2, \ldots$ Since $\lambda_n^s \leq C\lambda_-^n$ we have $D^s(\delta) \supset D^s(\delta, (\lambda_n))$. This proves the claim. □

Let ν denote the normalized Lebesgue measure on $[0,1]^2$.

Corollary 2. *The set $\tilde{D}^s = \bigcup_{\delta>0} D^s(\delta)$ is of full ν-measure in $[0,1]^2$. Moreover*

$$\nu([0,1]^2 \setminus D^s(\delta)) \leq A_2 \cdot \delta.$$

Proof. Follows by Corollary 1 and Fubini's Theorem. □

Let us consider the function $1/D(p)$ where $D(p) = |\xi(p)|$. We will prove that it is integrable.

Proposition 6. *(corresponds to Proposition 8 of [10]) There is a constant $A_3 > 0$ such that for an arbitrary $\delta > 0$,*

$$\nu(\{p \in [0,1]^2 : D(p) < \delta\}) \le A_3 \delta^2.$$

Proof. If $p \in \{p \in [0,1]^2 : D(p) < \delta\}$, then $\mathrm{dist}(G^n p, L) < \delta \lambda_n^s(p)$ at least for two $n_1 < n_2$ since both ends of $\xi(p)$ have to be trimmed (Lemma 6). This means that p is less that δ close to preimage $G^{-n_1}(L)$ and $\mathrm{dist}(G^{n_1}p, L) < \delta\lambda_{n_1}^s(p) = \eta$. Then,

$$G^{n_1+2}p \in \Pi_{A_4\eta} = \{(x,y) : 1 - A_4\eta \le x \le 1, 0 \le y \le 1\},$$

for some constant $A_4 > 0$ independent of δ and n_1. See Figure 3. Also, $G^{n_1+2}p \in [0,1]^2 \setminus D^s(\eta)$ since $\mathrm{dist}(G^{n_2}p, L) < \delta\lambda_{n_2}^s(p) < \eta$, and $G^{n_1+1}p$ is far from the line L. Let $n = n_1 + 2$. We have $G^n p \in \Pi_{A_4\eta} \setminus D^s(\eta)$. Since the vertical direction is in the expanding cone, by Corollary 1 and Fubini's Theorem, we have

$$\nu(\Pi_{A_4\eta} \setminus D^s(\eta)) \le A_2 \cdot A_4 \cdot \eta^2.$$

Thus,

$$\nu(\{p \in [0,1]^2 : D(p) < \delta\}) \le \sum_{n=0}^{\infty} \nu(G^{-n}(\Pi_{A_4\eta} \setminus D^s(\eta)) \le \sum_{n=0}^{\infty} A_2 \cdot A_4 \cdot \eta^2 (2 \cdot \mathrm{Jac}^{-1}(\alpha))^n,$$

where $\mathrm{Jac}(\alpha) = 2(1-\alpha)$ is the Jacobian of both G_1 and G_2. We need the multiplier 2 because G is a 2 to 1 map.(This is different from the Lozi map studied in [10].) By Lemma 4 and Proposition 4 we have

$$\lambda_n^s(p) \le C\lambda_-^n,$$

where

$$\lambda_- = \alpha - \sqrt{\alpha^2 + 2\alpha - 2}.$$

We have

$$\nu(\{p \in [0,1]^2 : D(p) < \delta\}) \le A_2 \cdot A_4 \cdot C^2 \cdot \delta^2 \cdot \sum_{n=0}^{\infty} \left(\frac{2(\alpha - \sqrt{\alpha^2 + 2\alpha - 2})^2}{2(1-\alpha)} \right)^n.$$

It can be easily proved that for $3/4 < \alpha < 1$ we have $\frac{(\alpha - \sqrt{\alpha^2+2\alpha-2})^2}{(1-\alpha)} < 1$. Thus, the series converges to some constant $A(\alpha)$, and setting $A_3 = A_2 \cdot A_4 \cdot C^2 \cdot A(\alpha)$ completes the proof of the proposition. $\square$

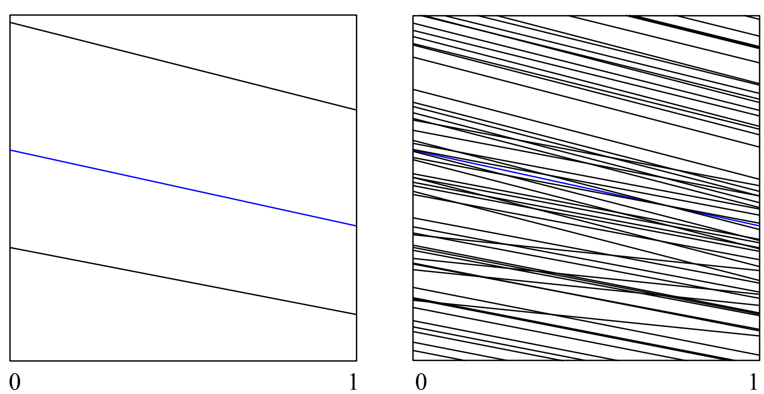

FIGURE 4. Partitions $\mathcal{P}^{(2)}$ and $\mathcal{P}^{(6)}$ for $\alpha = 0.82$.

Corollary 3. *(corresponds to Corollary 3 of* [10]*) The function $p \mapsto 1/D^\beta(p)$ is integrable on $[0,1]^2$ for any $\beta \in [1,2)$.*

Proof. We will use the following identity for positive random variables

$$E(X) = \int_0^\infty P(X > t)\, dt \tag{4.14}$$

which can be found, e.g., in [4], page 275. We have

$$\begin{aligned} &\int (1/D)^\beta d\nu = \int_0^\infty \nu(\{D^{-\beta} > \gamma\})d\gamma \le 1 + \int_1^\infty \nu(\{D^{-\beta} > \gamma\})d\gamma \\ &\le 1 + \int_1^\infty \nu(\{D < \gamma^{-1/\beta}\})d\gamma \le 1 + \int_1^\infty A_3 \gamma^{-2/\beta} d\gamma < +\infty \end{aligned} \tag{4.15}$$

□

In the following proposition we will discuss the family of conditional measures $\{\nu_C\}_{C \in \mathcal{P}^-}$ of measure ν on elements of the partition $\mathcal{P}^-$. The theory of conditional measures can be reviewed by referring to [12] or [9]. Let $\{\ell_C\}_{C \in \mathcal{P}^-}$ be the family of one-dimensional Lebesgue measures on the elements of $\mathcal{P}^-$.

Proposition 7. *(corresponds to Proposition 9 of* [10]*) For almost every $C \in \mathcal{P}^-$, measure ν_C is absolutely continuous with respect to ℓ_C and the Radon-Nikodym derivative $\frac{d\nu_C}{d\ell_C}$ is constant on C, equal to $1/|C|$.*

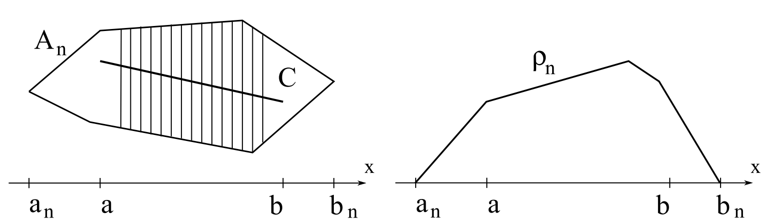

FIGURE 5. A polygon A_n of the partition $\mathcal{P}^n$ and the density ρ_n.

Proof. We follow closely [10]. Let A_n be the polygon of the partition $\mathcal{P}^n$, $n \ge 1$ containing $C \in \mathcal{P}^-$. See Figure 5. Since A_n is convex, the projection of the measure $\frac{1}{\nu(A_n)} v_{|A_n}$ onto the x-axis is a measure absolutely continuous with respect to Lebesgue measure with density ρ_n which is positive on some interval (a_n, b_n) and zero outside of this interval. Since $\rho_n(t)$ is proportional to the length of the intersection of the vertical line $x = t$ with the polygon A_n the density ρ_n is concave on (a_n, b_n). We have $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$ and $a_n \to a$, $b_n \to b$ where a and b are the end points of the projection of C onto the x-axis. Since $\rho_n(a_n) = \rho_n(b_n) = 0$, $\int_{a_n}^{b_n} \rho_n = 1$, and ρ_n are concave the family $\{\rho_n\}_{n \ge 1}$ is uniformly bounded by $2/(b-a)$. Since they are concave their variations are also uniformly bounded by $4/(b-a)$. By Helly's theorem ([4]), there exists a subsequence ρ_{n_k} convergent to some density ρ almost everywhere. ρ is concave as a limit of concave functions. Projecting ρ onto C we obtain $\frac{d\nu_C}{d\ell_C}$ which is also concave. We will denote it again by ρ.

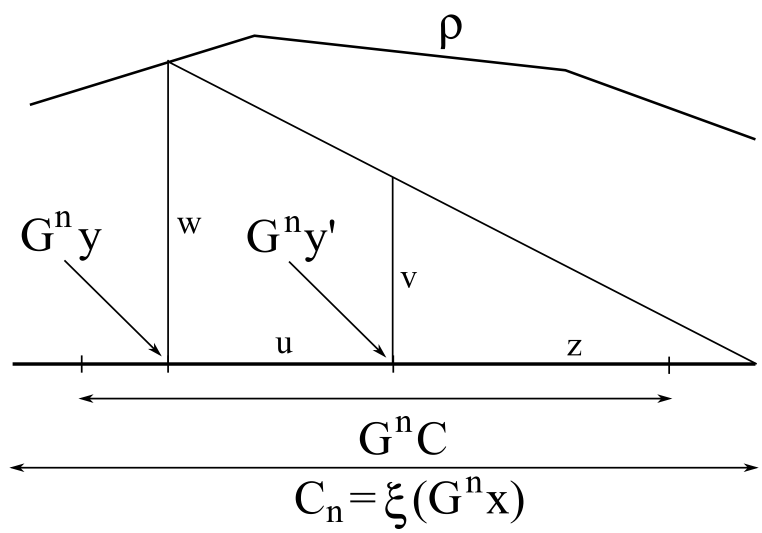

FIGURE 6. Using the concavity of $\rho_{|\xi(G^n x)}$.

To use Proposition 3 we will prove that for almost every $x \in [0,1]^2$ and almost every $y, y' \in C(x) = C$, we have

$$\lim_{n\to\infty} \frac{\rho(G^n y)}{\rho(G^n y')} = 1. \tag{4.16}$$

By Lemma 7 we assume that $\operatorname{dist}(G^n C, L) > \delta\lambda^n$, where $\delta > 0$ and $\lambda \in (\lambda_-, 1)$. Now, we use the concavity of $\rho_{|\xi(G^n x)}$. See Figure 6. From the triangles in Figure 6 we have $\frac{w}{v} = \frac{u+z}{z} = 1 + \frac{u}{z}$. Thus,

$$\frac{\rho(G^n y)}{\rho(G^n y')} \le \frac{w}{v} = 1 + \frac{u}{z} \le 1 + \frac{\operatorname{dist}(G^n y, G^n y')}{\operatorname{dist}(G^n C, \partial(\xi(G^n x)))} \le 1 + \frac{C\lambda_-^n}{\delta\lambda^n},$$

which goes to 1 as $n \to \infty$. Thus, $\limsup_{n\to\infty} \frac{\rho(G^n y)}{\rho(G^n y')} \le 1$. By symmetry, we obtain (4.16). By Proposition 3 we have

$$\frac{\rho(y)}{\rho(y')} = \prod_{k=0}^{\infty} \frac{g_G(G^k y)}{g_G(G^k y')} \frac{\lambda(G^k y)}{\lambda(G^k y')}.$$

Since the Jacobian of G is constant and G is piecewise linear and in particular linear on every $C \in \xi$, the right hand side of the above formula is constant. □

5. Applying the Abstract Theorems to Transformations G_α.

We will use the notation introduced in Section 2. Let $Y = [0,1]^2$, $S = G$ and ν be Lebesgue measure on $[0,1]^2$. The map T is the factor map induced by G on the space $X = [0,1]^2/\mathcal{P}^-$.

By formula (2.5) we have

$$g_T(x) = g_G(y) \frac{d((G_{|A})_*^{-1} \nu_{C(Tx)})}{d\nu_{C(x)}}(y),$$

for almost every x and almost every y, where A is an element of the partition $\mathcal{P}$.

Lemma 8. *We can rewrite g_T as follows:*

$$g_T = \frac{1}{\mathrm{Jac}_G}\lambda^s \frac{D}{D\circ T}, \tag{5.1}$$

where λ^s is defined in Lemma 4 and $D(x) = |C(x)|$.

Proof. We can write

$$\frac{d((G_{|A})_*^{-1}\nu_{C(Tx)})}{d\nu_{C(x)}} = \frac{d((G_{|A})_*^{-1}\nu_{C(Tx)})}{d((G_{|A})_*^{-1}\ell_{C(Tx)})}\frac{d((G_{|A})_*^{-1}\ell_{C(Tx)})}{d\ell_{C(x)}}\frac{d\ell_{C(x)}}{d\nu_{C(x)}}.$$

In view of Proposition 7 this gives the required formula for g_T. □

Since g_T given by formula (5.1) is very discontinuous we will replace it by considering instead of Lebesgue measure on $[0,1]^2$ an equivalent measure $\nu = \frac{1}{D}\bar\nu$, where $\bar\nu$ is the Lebesgue measure. Then, we define $m = \frac{1}{d}\bar m$, where $\bar m$ is the factor of the Lebesgue measure on $X = [0,1]^2/\mathcal{P}^-$.

Proposition 8. *(Rychlik* [10]*, Proposition 10) If we apply the results of Section 2 to the measure $\nu = \frac{1}{D}\bar\nu$, then*

$$g_T = \frac{1}{\mathrm{Jac}_G}\lambda^s. \tag{5.2}$$

Proof. Let $A \in \mathcal{P}^-$. By the definition of g_T (2.1), we have

$$\begin{aligned} g_T &= \frac{d(T_*(\chi_A\cdot m))}{dm}\circ T = \frac{d(T_*(\chi_A\cdot \frac{1}{D}\cdot \bar m))}{d(\frac{1}{D}\cdot\bar m)}\circ T \\ &= \frac{\frac{1}{D}\circ (T_{|A})^{-1}\cdot dT_*(\chi_A\bar m)}{\frac{1}{D}d\bar m}\circ T \\ &= \frac{\frac{1}{D}}{\frac{1}{D}\circ T}\cdot\frac{1}{\mathrm{Jac}_G}\lambda^s\frac{D}{D\circ T} = \frac{1}{\mathrm{Jac}_G}\lambda^s. \end{aligned} \tag{5.3}$$

□

We will now verify assumptions (I)–(IV).

Lemma 9. *(Rychlik* [10]*, Lemma 12) Condition (I) is satisfied.*

Proof. Let $n \geq 1$ and $x_1, x_2 \in B \in \beta^{(n)}$. We can treat x_1, x_2 as elements of the Lebesgue space X and also as points in $[0,1]^2$ or elements of $\mathcal{P}^-$. The points $G^k x_1$ and $G^k x_2$ are on the same side of the partition line L for $k = 1, 2, \ldots, n-1$. Since Jac_G is constant, we need only to find a universal constant d such that

$$\frac{1}{d} \leq \frac{\lambda_n^s(x_1)}{\lambda_n^s(x_2)} \leq d\,.$$

By Lemma 4 we have $\lambda^s(p) = |v(p)|\frac{h_1(p)}{h_1(G(p))}$, so

$$\lambda_n^s(p) = |v(p)|\cdot|v(Gp)|\cdot\dots\cdot|v(G^{n-1}p)|\frac{h_1(p)}{h_1(G^n(p))}.$$

By Lemma 3 we have $|v(Gp) - v(Gp')| \leq \kappa|v(p) - v(p')|$, where $0 < \kappa < 1$. Thus, $|v(G^k x_1) - v(G^k x_2)| \leq \kappa^{n-k}|J_-|$, for $k = 1, 2, \ldots, n-1$. Thus, there exists a constant d_0 such that

$$\exp\left(-d_0\kappa^{n-k}\right) \leq \frac{\lambda^s(G^k x_1)}{\lambda^s(G^k x_2)} \leq \exp\left(d_0\kappa^{n-k}\right).$$

Then, for some constant d_1 (we have to include the fractions $h_1(x_1)/h_1(G^n(x_1))$ and $h_1(x_2)/h_1(G^n(x_2)))$, we obtain

$$\exp\left(-d_1\sum_{k=0}^{n-1}\kappa^{n-k}\right)\le\frac{\lambda_n^s(x_1)}{\lambda_n^s(x_2)}\le\exp\left(d_1\sum_{k=0}^{n-1}\kappa^{n-k}\right).$$

Letting $d=\exp(d_1/(1-\kappa))$, completes the proof. □

Lemma 10. *(Rychlik* [10]*, Lemma 13) Conditions (II) and (IV) are satisfied for some iteration T^N, $N\ge 1$.*

Proof. Condition (IV) is satisfied because $g_n=(\mathrm{Jac}_G)^{-n}\cdot\lambda_n^s\le(\mathrm{Jac}_G)^{-n}\cdot(C\theta_0^n)=C(\theta_0/\mathrm{Jac}_G)^n$. For large n, $g_n<1$, since $\theta_0/\mathrm{Jac}_G<1$. We used Proposition 4 and $\theta_0=\alpha-\sqrt{\alpha^2+2\alpha-2}$, $\mathrm{Jac}_{G_\alpha}=2(1-\alpha)$. For $3/4<\alpha<1$, we have $\theta_0/\mathrm{Jac}_G<1$.

Now, we will prove that condition (II) is satisfied for some iterate of T. The proof is similar to that of Proposition 5. Let N be such that $r_0=2NC(\lambda_+)^N<1$ and let ε_0 be the Lebesgue constant of the cover $\mathcal{U}_N$ of Lemma 5. By Proposition 4, $\lambda_+=\theta_0/\mathrm{Jac}_G$ so $g_n\le C\cdot\lambda_+$ for $n\ge 1$.

Let $B\in\beta^{(N)}$ and $A=\pi^{-1}(B)$. Then, $T^N(B)=\pi(G^NA)$. G^NA is a convex polygon such that

$$\partial(G^N(A))\subset\cup_{k=0}^N G^k(L).$$

Except for L itself and the first image $G(L)\subset\{y=1\}$, all subsequent images $G^k(L)$, $k\ge 2$, consist of segments with directions from the unstable cone J_+. Also, all images of the sides of $[0,1]^2$ have this property. We can assume ε_0 is much smaller than the distance between L and $G(L)$. There are two possibilities:

(1) $\mathrm{diam}(G^NA)\ge\varepsilon_0$. Then, G^NA contains a segment I with the unstable direction (from J_+) of length $A_5\cdot\varepsilon_0$, for some $A_5\le 1$. If all sides of G^NA belong to $\cup_{k=2}^N G^k(L)$, i.e., they have directions from J_+, then obviously G^NA contains a segment I with the direction from J_+ of length ε_0. If one of the sides belongs to L and another to $G(L)$, then G^NA also contains such segment since ε_0 is small. If only one side of G^NA belongs to L or $G(L)$, then the worst case scenario is a triangle with two remaining sides with directions from J_+. Since the angle between directions from J_+ and L or $G(L)$ is separated from zero, G^NA contains a segment I with the direction from J_+ of length $A_5\cdot\varepsilon_0$, for some $A_5\le 1$.

By Corollary 1, for arbitrary $\delta>0$, $|I\backslash D^s(\delta)|\le A_2\delta$. The set $\bar{A}=\pi^{-1}\pi(G^NA)=\pi^{-1}(T^NB)$ has measure larger than $A_4^{-1}(1-A_2\delta)\cdot A_5\cdot\varepsilon_0$, where A_4 will be found in the following Lemma 11. So, we put $\delta=\frac{1}{2}A_2^{-1}$ and $\varepsilon=A_4^{-1}(1-A_2\delta)\cdot A_5\cdot\varepsilon_0=\frac{1}{2}A_4^{-1}\cdot A_5\cdot\varepsilon_0$. Then, $\nu(\bar{A})>\varepsilon$ and $m(T^NB)=\nu(\bar{A})>\varepsilon$, since $m=\pi_*\nu$.

(2) $\mathrm{diam}(G^NA)<\varepsilon_0$. Then, T^NB is contained in no more than $2N$ elements of β^N and

$$\sum_{B'\in\beta^N(T^NB)}\sup_{B'}g_N\le(2N)(C\lambda_+^N)=r_0.$$

Thus, condition (II) is satisfied for T^N with $r=r_0$. □

Lemma 11. *(Rychlik* [10]*, Lemma 14) Let I be a segment with the direction from J_+ and let ℓ_I be the Lebesgue measure on I. Then, the measure $\pi_*(\ell_I)$ is absolutely continuous with respect to m and*

$$\frac{d\pi_*(\ell_I)}{dm}(x)=\frac{1}{\sin\angle(I,x)},\tag{5.4}$$

for $x \in X$, where $\angle(I, x)$ is the angle between segment I and segment $C(x) \in \mathcal{P}^-$. In particular, for some $A_4 > 0$ we have

$$\frac{1}{A_4} \leq \frac{d\pi_*(\ell_I)}{dm} \leq A_4. \tag{5.5}$$

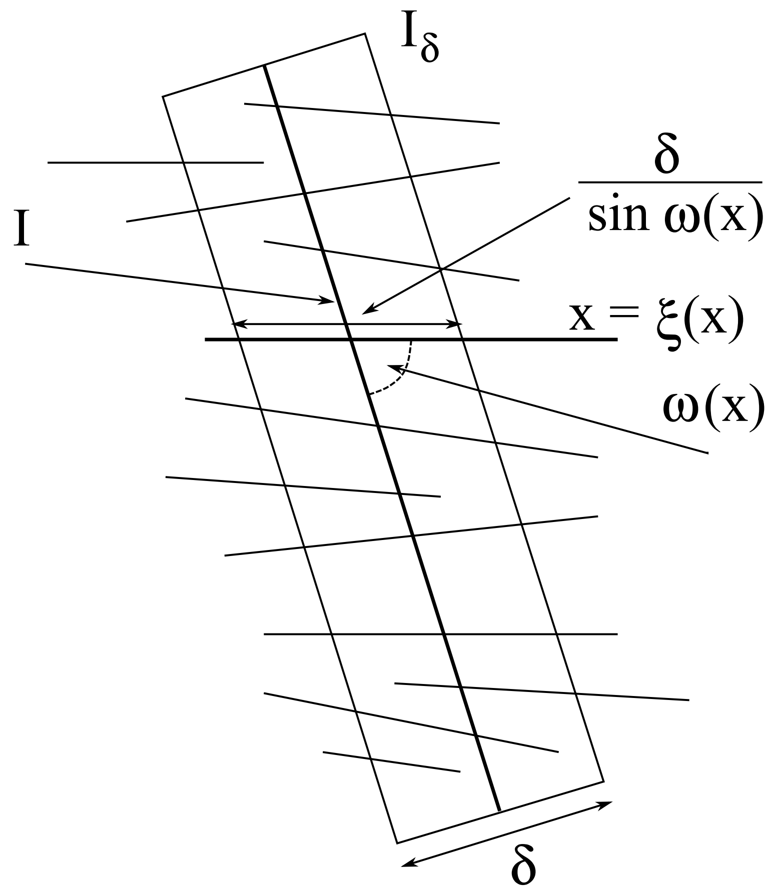

FIGURE 7. Strip I_δ for the proof of Lemma 11.

Proof. Fix some small $\delta > 0$. Let I_δ be a strip of width δ ($\delta/2$ on each side of I). We note that if $x \in D^s(\delta) \cap I$ and $\mathrm{dist}(x, \partial I) > \delta$, then $\xi(x) \cap I_\delta$ is an interval of length $\delta/\sin\omega(x)$, where $\omega(x) = \angle(I, x)$. See Figure 7. So

$$\nu_x(I_\delta) = \frac{\delta}{\sin\omega(x) \cdot D(x)}.$$

Let E be a subinterval of I. If $\bar{\nu}$ is the Lebesgue measure on $[0,1]^2$, we have

$$\bar{\nu}(\pi^{-1}(\pi E) \cap I_\delta) = \int_{\pi(E)} \nu_x(x \cap I_\delta) d\bar{m} = \delta \int_{\pi(E)} \frac{1}{\sin\omega(x)} dm.$$

On the other hand, by Corollary 1,

$$\bar{\nu}(\pi^{-1}(\pi E) \cap I_\delta) = \ell_I(E) \cdot \delta + o(\delta).$$

This proves (5.4). Since the angles between directions from J_- and J_+ are separated from zero the inequality (5.5) is also proved. $\square$

Remark 4. *Condition (III) holds since β is finite.*

Thus, we checked the assumptions of Theorems 1 and 2. Hence, we have

Theorem 3. *The results of Theorems 1 and 2 apply to G_α maps for $3/4 < \alpha < 1$.*

6. Invariant measures for maps G.

We proved the existence of the invariant measures of the form $\phi \cdot m$ for the factor map T. Now, we will construct a G-invariant measure μ such that the projection $\pi_*(\mu)$ onto X coincides with $\phi \cdot m$.

Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function. We will define $\mu(f)$. Let

$$f^<(p) = \inf_{\xi(p)} f \ , \ p \in [0,1]^2,$$

and

$$f^>(p) = \sup_{\xi(p)} f \ , \ p \in [0,1]^2.$$

Both, $f^<$ and $f^>$ are Ξ-measurable (Ξ is the σ-algebra generated by the partition $\xi = \mathcal{P}^-$). We define

$$\mu(f) = \lim_{n\to\infty} \tilde{\mu}((f \circ G^n)^<),$$

where $\tilde{\mu} = \phi \cdot m$.

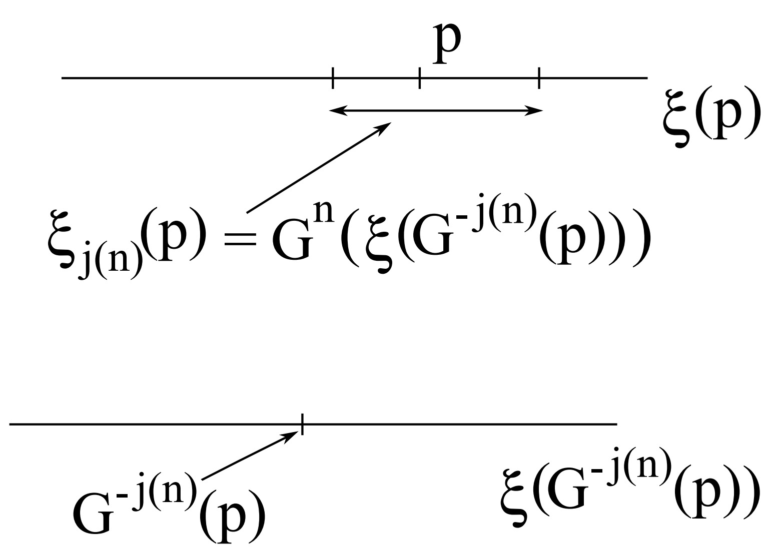

Figure 8. Definition of the function $f^<_{n|j(n)}$.

Lemma 12. *The limits $\lim_{n\to\infty} \tilde{\mu}((f \circ G^n)^<)$ and $\lim_{n\to\infty} \tilde{\mu}((f \circ G^n)^>)$ exist and are equal.*

Proof. This proof follows the proof of Lemma 15 in Rychlik [10], but we have to deal with the fact that G is not invertible. This causes a need for more complicated notation. The map G has two invertible "branches" $G_1 = G_{|A_1}$ and $G_2 = G_{|A_2}$. Corresponding inverses are G_1^{-1} and G_2^{-1}. Let $j(n) = (i_1, i_2, \ldots, i_n) \in \{1,2\}^n$. Then, $G^{j(n)} = G_{i_n} \circ \cdots \circ G_{i_2} \circ G_{i_1}$ and $G^{-j(n)} = G_{i_1}^{-1} \circ G_{i_2}^{-1} \circ \cdots \circ G_{i_n}^{-1}$. Let us also introduce the notation $(j(n), i_{n+1}) = (i_1, i_2, \ldots, i_n, i_{n+1})$, for $i_{n+1} \in \{1,2\}$.

We define $f^<_{n|j(n)}(p) = (f \circ G^n)^< \circ G^{-j(n)}(p)$, where $j(n)$ is such that $p \in G^{j(n)}([0,1]^2)$. Note that

$$f^<_{n|j(n)}(p) = \inf_{q \in \xi(G^{-j(n)}(p))} f(G^n(q)) = \inf_{s \in G^n(\xi(G^{-j(n)}(p)))} f(s)$$

is constant on $\xi(G^{-j(n)}(p))$, see Figure 8. Thus, $f^<_{n|j(n)}$ is Ξ-measurable. Also, $f^<_{n|j(n)} \le f^<_{n+1|(j(n),i_{n+1})}$ since G contracts segments $\xi \in \mathcal{P}^-$.

We define

$$f^<_n = \min_{j(n)} f^<_{n|j(n)}.$$

Now, $f^<_n$ is Ξ measurable and $f^<_n \le f^<_{n+1}$.

Similarly, we define $f^>_{n|j(n)} = (f \circ G^n)^> \circ G^{-j(n)}$ and

$$f^>_n = \max_{j(n)} f^>_{n|j(n)}.$$

The functions $f^>_n$ are Ξ measurable and $f^>_n \ge f^>_{n+1}$.

We have $f \ge f^<_n$ for all $n \ge 1$ and $f^<_1 \le f^<_2 \le \cdots \le f^<_n \le \ldots$. Similarly, $f \le f^>_n$ for all $n \ge 1$ and $f^>_1 \ge f^>_2 \ge \cdots \le f^>_n \ge \ldots$. Also, if $\xi_n = G^n \xi$ is a partition of $G^n([0,1]^2)$, then

$$f^>_n(p) - f^<_n(p) = \sup_{\xi_n(p)} f - \inf_{\xi_n(p)} f \le \omega_{\delta_n}(f),$$

where $\delta_n = \sup_p \operatorname{diam}(\xi_n(p))$ and

$$\omega_\delta(f) = \sup_{\operatorname{dist}(x,y) < \delta} |f(x) - f(y)|,$$

is the modulus of continuity of f. Thus, $f^>_n - f^<_n \to 0$ as $n \to \infty$ and, consequently, $f^>_n \searrow f$ and $f^<_n \nearrow f$ uniformly as $n \to \infty$.

We have

$$\begin{aligned} (f \circ G^n)^<(p) &= \inf_{q \in \xi(p)} f(G^n(q)) = \inf_{G^n(\xi(p))} f \\ &\ge \min_{j(n)} \inf_{G^{j(n)}\left(\xi\left(G^{-j(n)}(G^n(p))\right)\right)} = (f^<_n \circ G^n)(p), \end{aligned} \tag{6.1}$$

so $(f \circ G^n)^< \ge f^<_n \circ G^n$. Similarly, $(f \circ G^n)^> \le f^>_n \circ G^n$. We have

$$\begin{aligned} &|\tilde{\mu}((f \circ G^n)^>) - \tilde{\mu}((f \circ G^n)^<)| \le \sup |(f \circ G^n)^> - (f \circ G^n)^<| \\ &\le \sup |f^>_n \circ G^n - f^<_n \circ G^n| \le \sup |f^>_n - f^<_n| \le \omega_{\delta_n}(f), \end{aligned} \tag{6.2}$$

which goes to 0 as $n \to \infty$. Thus, both limits are the same if they exist. To show existence we write

$$f^<_n \circ G^n \le (f \circ G^n)^< \le (f \circ G^n)^> \le f^>_n \circ G^n,$$

which implies

$$\tilde{\mu}\left(f^<_n \circ G^n\right) \le \tilde{\mu}\left((f \circ G^n)^<\right) \le \tilde{\mu}\left((f \circ G^n)^>\right) \le \tilde{\mu}\left(f^>_n \circ G^n\right).$$

By the T-invariance of $\tilde{\mu}$ we have

$$\tilde{\mu}\left(f_n^{<} \circ G^n\right) = \tilde{\mu}\left(f_n^{<} \circ T^n\right) = \tilde{\mu}\left(f_n^{<}\right),$$

and similarly $\tilde{\mu}\left(f_n^{>} \circ G^n\right) = \tilde{\mu}\left(f_n^{>}\right)$. Since both sequences $\{f_n^{<}\}$ and $\{f_n^{>}\}$ converge uniformly to the same limit we have $\lim_{n\to\infty} \tilde{\mu}(f_n^{<}) = \lim_{n\to\infty} \tilde{\mu}(f_n^{>})$, which completes the proof. □

Proposition 9. *(Rychlik [10], Proposition 11) Let $\tilde{\mu}$ be an arbitrary measure on X which is T-invariant and such that the sets of Σ are $\tilde{\mu}$ measurable. Then, there exists a unique measure on Y such that μ is S-invariant and $\pi_*(\mu) = \tilde{\mu}$.*

Proof. Let μ be constructed as in Lemma 12 and let η be some other S-invariant measure such that $\pi_*(\eta) = \tilde{\mu}$. For every continuous function f on Y we have $\eta(f^{<}) \le \eta(f) \le \eta(f^{>})$. Since $\eta(f^{<}) = (\pi_*\eta)(f^{<}) = \tilde{\mu}(f^{<})$ (and similarly for $f^{>}$) for any function f and in particular for $f \circ S^n$, we get

$$\tilde{\mu}((f \circ S^n)^{<}) \le \eta(f) \le \tilde{\mu}((f \circ S^n)^{>}),$$

as $\eta(f \circ S^n) = \eta(f)$. Going to the limit completes the proof. □

Corollary 4. *In view of Theorem 2, we can construct G-invariant measures $\mu_1, \mu_2, \ldots, \mu_s$ such that $\pi_*(\mu_i) = \phi_i \cdot m$, $i = 1, 2, \ldots, s$.*

Theorem 4. *(Rychlik [10], Theorem 4) Let μ be a Borel, regular measure on $[0,1]^2$ such that $\pi_*\mu$ is absolutely continuous with respect to m. Then,*

$$\frac{1}{n}\sum_{k=0}^{n-1} G_*^k \mu \xrightarrow{n\to\infty} \sum_{i=1}^{s} \mu(\bar{C}_i) \cdot \mu_i \ ,$$

where $\bar{C}_i = \pi^{-1}(C_i)$, $C_1, C_2, \ldots, C_s$ as in Theorem 2, $\mu_1, \mu_2, \ldots, \mu_s$ are as above, and the convergence is in $*$-weak topology of measures.

Proof. We refer to [10]. □

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