Iterated Function Systems and Fractals

Paweł Góra

Department of Mathematics and Statistics
Concordia University, Montreal

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Dynamical Systems

Let $f : X \to X$ be a function. It can be considered as a **movement**:

$$x \to f(x) \to f(f(x)) \to f(f(f(x))) \to f(f(f(f(x)))) \to \dots$$

Contractions

$f : X \to X$ is a contraction if

$$d(f(x), f(y)) \leq \lambda \cdot d(x, y) \quad , \quad 0 \leq \lambda < 1 \ ,$$

for all $x, y \in X$.
Example: $f_1(x) = \frac{1}{3}x$, $x \in \mathbb{R}$.
Another: $f_2(x) = \frac{1}{3}x + \frac{2}{3}$, $x \in \mathbb{R}$.
Theorem (Banach): A contraction in a complete metric space has exactly one fixed point:

$$f(c) = c \ .$$

IFS

We use more than one function, for example f_1 and f_2.
How to use them both ?
Two methods:
1) Random map: with probability p apply f_1 and with probability $1-p$ apply f_2.
2) Define function on sets, not on points.

Hausdorff space

If X is our space, we define Hausdorff space $\mathcal{H}(X)$ as space of compact nonempty subsets of X. We define distance d_H between subsets. It can be proved that $\mathcal{H}(X)$ is complete if X is complete.

IFS as a function on sets

For $f : X \to X$ we naturally define $F : \mathcal{H}(X) \to \mathcal{H}(X)$ as

$$F(A) = \{f(x) : x \in A\} \ .$$

Once we have to functions $F_1, F_2 : \mathcal{H}(X) \to \mathcal{H}(X)$ we define IFS $\mathcal{F} : \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$\mathcal{F}(A) = F_1(A) \cup F_2(A) \ .$$

Attractor of the IFS

Theorem: If f_1, f_2 are contractions on X, then $\mathcal{F} = (F_1, F_2)$ is a contraction on $\mathcal{H}(X)$. Thus, it has the unique "fixed point", a compact set called the **attractor** of the IFS $\mathcal{F}$, satisfying

$$\mathcal{F}(A) = A \ .$$

It can be approximated by iteration starting from any compact set

$$A = \lim_{n\to\infty} \mathcal{F}^n(B) \ ,$$

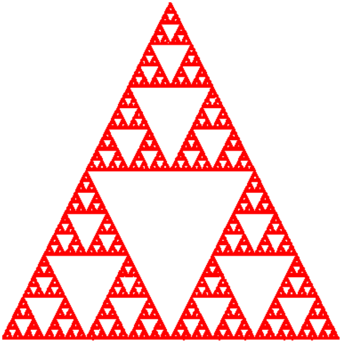
for any $B \in \mathcal{H}(X)$.

Cantor set

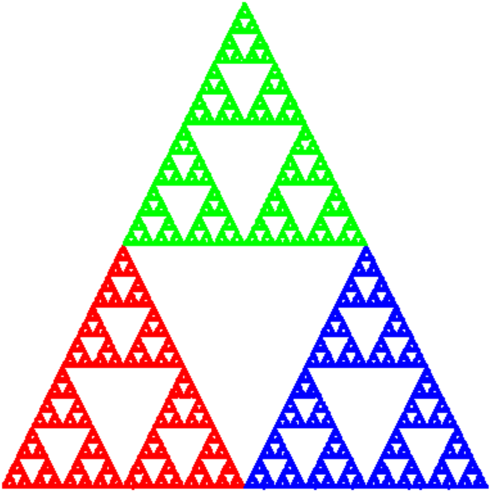
$f_1(x) = \frac{1}{3}x$, $f_2(x) = \frac{1}{3}x + \frac{2}{3}$, $x \in \mathbb{R}$.

Sierpinski triangle

$$f_1([x,y]) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;,\; f_2([x,y]) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \;,$$
$$f_3([x,y]) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} \;.$$



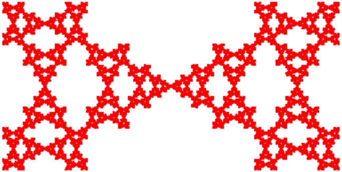
Sierpinski triangle



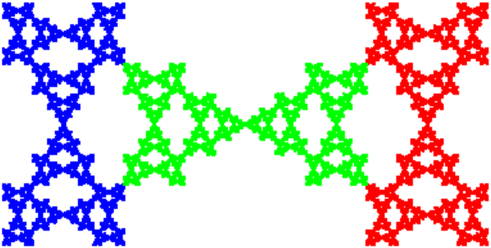
Two arrows

$$f_1([x,y]) = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} \ , \ f_2([x,y]) = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{3}{4} \\ 0 \end{bmatrix} \ ,$$

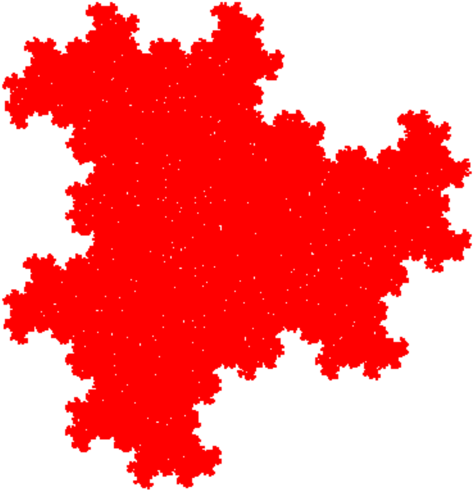
$$f_3([x,y]) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ .$$



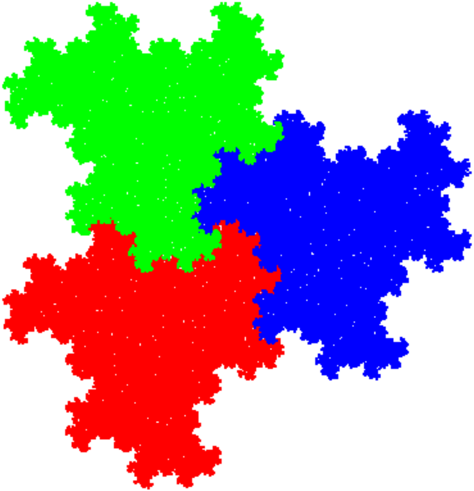
Two arrows



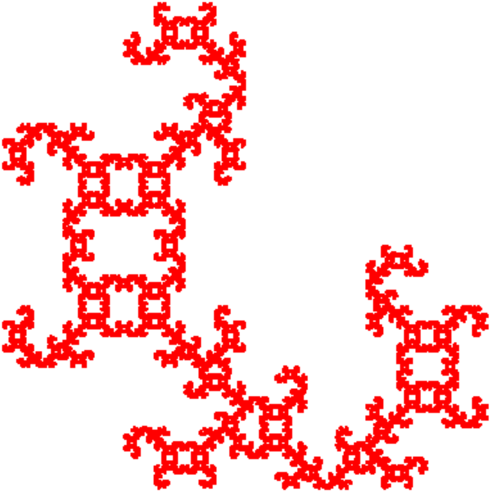
Solid Dragon



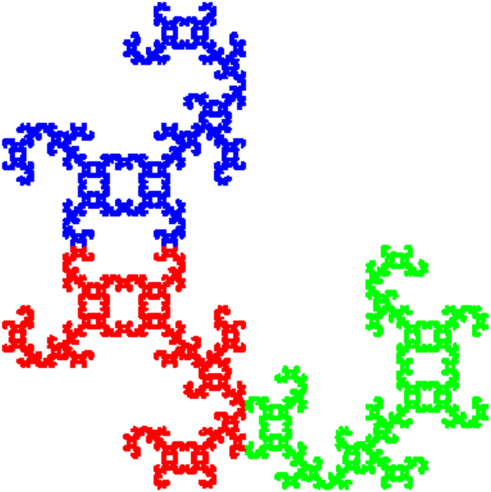
Solid Dragon



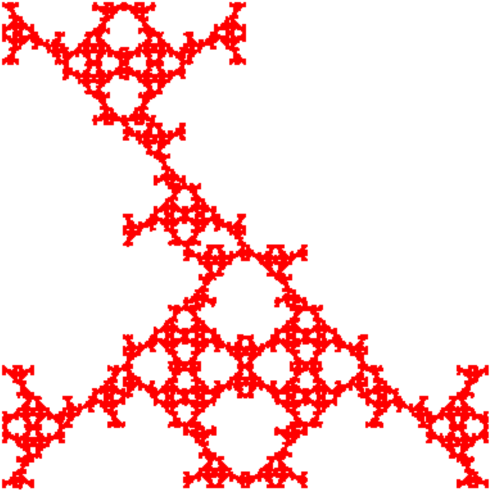
Dragon



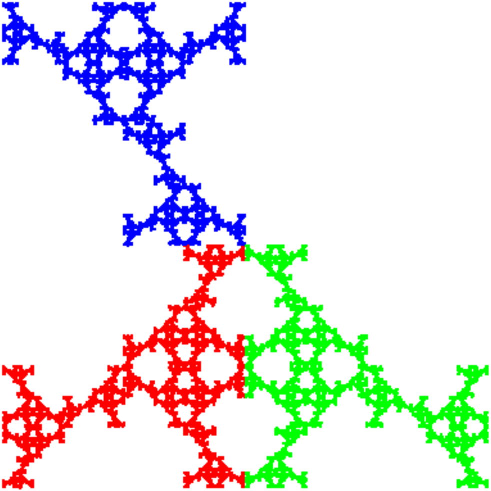
Dragon



Giraffe



Giraffe



Fern

Lena

Fractal Image Compression

Introduced by Barnsley and his "Iterated Systems Inc." Used by Microsoft, Encyclopedia Britannica, Spectrum Holobyte and some other companies.

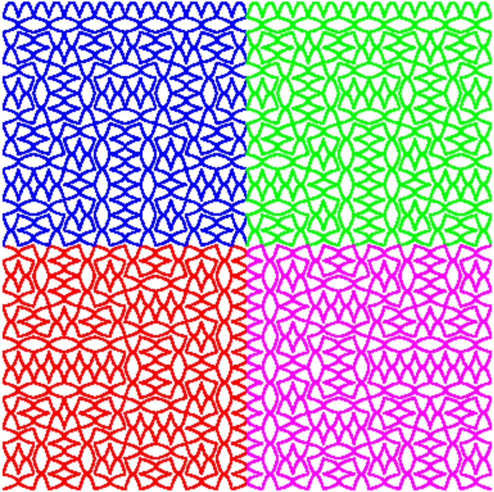
Gives compression of 50-100 times while JPEG gives 3-6 times.

Moreover the fractaly compressed image has infinite resolution, i.e., never shows pixelization like JPEG images.

Out of fashion, due to marketing errors and development of wavelets techniques.

Peano curve

Peano curve is a continuos function $f : [0, 1] \to [0, 1] \times [0, 1]$ which is onto,i.e., the graph covers the whole square.



Julia sets in Complex Dynamics

$$f(z) = z^2 + c \quad , \quad c \in \mathbb{C} \quad , \quad z \in \mathbb{C} \; .$$

This map is not a contraction but has two inverse branches: solutions of the equation $w = z^2 + c$

$$f_1(w) \; , \; f_2(w) \; ,$$

which are local contractions. IFS build of them produces so called Julia sets dependent on the parameter c.

Julia sets in Complex Dynamics

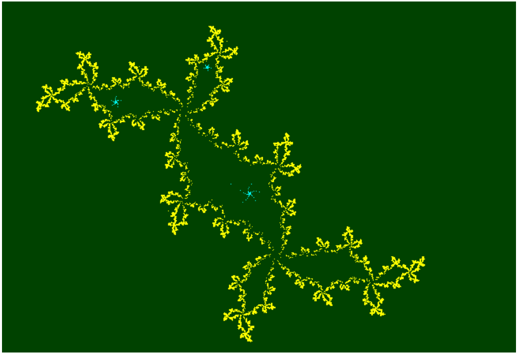


Figure: Julia set for $c = -0.1708 + 0.8179i$

References

- Barnsley, Michael F., **Fractals everywhere**. Second edition. Academic Press Professional, Boston, MA, 1993. xiv+534 pp. ISBN: 0-12-079061-0
- Barnsley, Michael F., **Fractal image compression**. Notices Amer. Math. Soc. 43 (1996), no. 6, 657–662. on-line: http://www.ams.org/notices/199606/barnsley.pdf
- Barnsley web page: http://www.superfractals.com/