QUASI-COMPACTNESS OF FROBENIUS-PERRON OPERATOR FOR PIECEWISE CONVEX MAPS WITH COUNTABLE BRANCHES

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ABSTRACT. In this paper, we prove the quasi-compactness of the Frobenius-Perron operator for a piecewise convex map τ with a countably infinite number of branches on the interval $I = [0, 1]$. We establish that for high enough n iterates of τ, τ^n are piecewise expanding. Using the Lasota-Yorke Inequality derived from references [4] and [14], adapted to meet the assumptions of the Ionescu-Tulcea and Marinescu ergodic theorem [7], we demonstrate the existence of absolutely continuous invariant measure (ACIM) μ for τ, the exactness of the dynamical system (I, τ, μ) and the quasi-compactness of Frobenius-Perron operator P_τ induced by τ. The last fact implies a multitude of strong ergodic properties of τ.

1. INTRODUCTION

This paper investigates the existence of absolutely continuous invariant measures (ACIMs) for a class of dynamical systems: piecewise convex maps with a countably infinite number of branches defined on the unit interval $[0, 1]$ denoted by $\mathcal{T}$. Understanding ACIMs is crucial for analyzing the long-term behaviour and chaotic nature of deterministic dynamical systems.

Let $I = [0, 1]$, $\mathcal{B}$ denote the Borel σ-algebra of subsets of I and let m be the normalized Lebesgue measure on I. Let $\tau : I \to I$ be a piecewise monotonic and hence non-singular transformation. A measure μ on $\mathcal{B}$ is τ-invariant if it remains unchanged under the action of τ, i.e., $\mu(\tau^{-1}A) = \mu(A)$, for all $A \in \mathcal{B}$. To examine ACIMs, we use the Perron-Frobenius operator induced by τ, P_τ from L^1_m to L^1_m:

$$P_\tau f(x) = \sum_{y \in \tau^{-1}(x)} f(y) g(y).$$

P_τ key properties are linearity, positivity, contractivity and preservation of integrals. The fact that a measure $h \cdot m$ is τ-invariant if and only if $P_\tau h = h$, i.e., h is a fixed point of P_τ, makes P_τ essential for studying ACIMs. For more information about ACIMs, the Frobenius-Perron operator and their mutual interconnections we refer the reader to [2] or [17].

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The piecewise convex maps considered until recently had a finite number of branches. We recall the standard assumptions. Let $I = [0, 1]$. A piecewise convex map τ defined on I satisfy the following conditions:

(1) There exist a finite partition $0 = a_0 < a_1 < ..... < a_n = 1$ such that $\tau|_{[a_{i-1},a_i)}$ is continuous, strictly increasing and convex for $i = 1, 2...n$.

(2) $\tau(a_{i-1}) = 0$ and $\tau'(a_{i-1}) > 0$ for $i = 1, 2, ...n$.

(3) $\tau'(0) = 1/\alpha > 1$.

The first to study such maps were Lasota and Yorke [15]. They discovered the three important properties, restated in Proposition 2.3, and used them to prove the existence of the ACIM μ and the exactness of the system (τ, μ). Additional properties of piecewise convex maps were shown in [9–13]. Generalizations, weakening the assumptions or random maps were studied in [1, 5, 6].

Recently, in [3, 8, 18] Lasota and Yorke's results were generalized to the case when a piecewise convex map has a countably infinite number of branches. In this paper, we prove a further result, the quasi-compactness of the operator P_τ, induced by τ. This fully describes the behavior of the system (τ, μ) and implies several strong ergodic properties. This result could not be obtained employing the previously used methods.

In Section 2, we explore the dynamics of piecewise convex maps with a countably infinite branches. These maps are defined on the partition of $I = [0, 1]$ into disjoint open subintervals $I_i = (a_i, b_i)_{i=1}^{\infty}$, whose complement has Lebesgue measure zero. Each restriction τ_i to I_i is an increasing, convex, differentiable function with $\sum_{i\geq 1} \frac{1}{\tau_i'(a_i)} < +\infty$ and $\tau'(0) > 1$, if 0 is not a limit point of partition endpoints. We denote the class of such maps by $\mathcal{T}$. We focus on the Frobenius-Perron operator P_τ associated with these maps, which acts on integrable functions f defined on $[0, 1]$. This operator is central to understanding the distribution of iterates of f under τ. We prove several key properties of P_τ, including its effect on non-increasing functions and bounds on its norm. We show that if τ belongs to the class $\mathcal{T}$, its iterates τ^n retains the piecewise convex structure and summability condition on derivatives. We demonstrate that the set of preimages of partition points is dense in $[0, 1]$, and the derivatives of iterates are uniformly bounded below by a constant greater than one, indicating piecewise expanding behavior.

In Section 3, we study piecewise expanding maps with a countably infinite number of branches. These maps are defined on a partition of I into disjoint subintervals $I_i = (a_i, b_i)_{i=1}^{\infty}$, each homeomorphically mapped by τ onto its image. The expansion behavior is quantified by $g(x)$, the reciprocal of the derivative τ', with $|g(x)| \leq \beta < 1$, defining the class $\mathcal{T}_E$. We prove that if τ is in $\mathcal{T}$, for high enough n, its iterates τ^n are in $\mathcal{T}_E$. We study the Frobenius-Perron operator P_τ

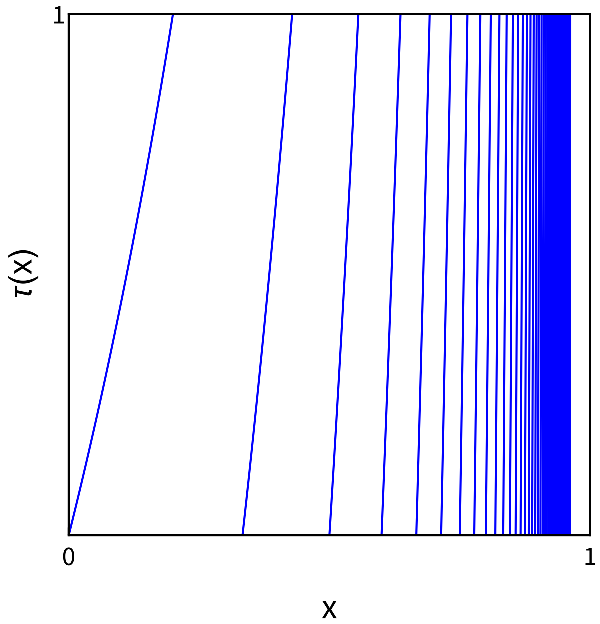

FIGURE 1. A piecewise convex maps with countable number of branches.

and its action on functions in BV, where $BV = \{f \in L^1_m : v(f) < +\infty\}$. We follow [14, 19] to establish Lasota-Yorke inequality for the functions in BV and apply the ergodic theorem of Ionescu-Tulcea and Marinescu [7] to prove the quasi-compactness of P_τ and its implications for ACIMs.

2. Piecewise convex map with countably infinite number of branches

Definition 2.1. Let $I = [0, 1]$ and let $\mathcal{P} = \{I_i = (a_i, b_i)\}_{i=1}^{\infty}$ be a countably infinite family of open disjoint subintervals of I such that Lebesgue measure of $I \setminus \bigcup_{i\geq 1} I_i$ is zero. We define a piecewise convex map τ on partition $\mathcal{I}$ as follows:

(1) For $i = 1, 2, 3..., \tau_i = \tau_{|I_i}$ is an increasing convex differentiable function with $\lim_{x\to a_i^+} \tau_i(x) = 0$. We define $\tau_i(a_i) = 0$ and $\tau_i(b_i) = \lim_{x\to b_i^-} \tau_i(x)$. The values $\tau_i'(a_i)$ are also defined by continuity.

(2) We assume

$$\sum_{i\geq 1} \frac{1}{\tau_i'(a_i)} < +\infty.$$

(3) If $x = 0$ is not a limit point of the partition points, then we have $\tau'(0) = 1/\alpha > 1$, for some $0 < \alpha < 1$. By $\mathcal{T}$ we will denote the set of maps satisfying conditions (1)-(3).

Lemma 2.2. *If $x = 0$ is the limit point of the sequence of left endpoints of the partition intervals, then we have $lim_{x\to 0^+}\tau'(x) = +\infty$, where at points where $\tau'(x)$ is not defined we take*

$\tau'(x) = \tau'_+(x)$. *Then, for some* $0 < r < 1$ *and some* $\alpha < 1$, *we have*

$$\sum_{a_i<r} \frac{1}{\tau'(a_i)} = \alpha < 1.$$

Proof. Condition (3) implies that for any $M > 0$ the inequality $\tau'(a_i) \leq M$ can be satisfied only for a finite number of points a_i. This implies the first claim of the lemma. The second claim follows by the fact that the sums of the tails of a convergent series converge to 0. □

The Frobenius-Perron operator induced on L^1_m by the map $\tau \in \mathcal{T}$ is

$$P_\tau f(x) = \sum_{i\geq 1} \frac{f(\tau_i^{-1}(x))}{\tau_i'(\tau_i^{-1}(x))} \chi_{|\tau_i(I_i)}(x). \tag{1}$$

The proposition below summarizes the properties of maps in $\mathcal{T}$ which were before used to show the existence of acim. We apply different methods but use these properties as well.

Proposition 2.3. *Let* $\tau \in \mathcal{T}$ *and* $f : [0,1] \to \mathbb{R}^+$ *be a non-increasing function. Then,*

(1) $P_\tau f$ *is also non-increasing function;*

(2) For any $x \in [0,1]$ *we have* $f(x) \leq \frac{1}{x}\|f\|_1$*;*

(3) $\|P_\tau f\|_\infty \leq \alpha\|f\|_\infty + D\|f\|_1$, *where* $\alpha < 1$ *is the number specified in Definition 2.1 or Lemma 2.2 and* $D > 0$ *is a constant.*

Proof. (1) Since τ_i is increasing on I_i, τ_i^{-1} is also increasing, for all $i \geq 1$. Let $f \in L^1_m$ be non-increasing. Then for any $x \leq y$ we have $f(x) \geq f(y)$. Hence, $f(\tau_i^{-1}(x)) \geq f(\tau_i^{-1}(y))$. Also, τ_i' is non-decreasing being the derivative of convex function which gives $1/\tau_i'(\tau_i^{-1}(x))$ is non-increasing and $\chi_{|\tau_i(a_i,b_i)}(x)$ is also non-increasing since $\tau_i(a_i) = 0$. The product of non-increasing functions is non-increasing, thus $P_\tau f(x)$ is non-increasing.

(2) For any $0 < x \leq 1$, we have

$$\|f\|_1 = \int_0^1 f(x)dm(x) \geq \int_0^x f(x)dm(x) \geq x \cdot f(x).$$

(3) **Case I**: Let $a_1 = 0$ be a limit point of partition endpoints a_i. Since $P_\tau f$ is non-increasing, $\|P_\tau f\|_\infty \leq P_\tau f(0)$, and we have

$$P_\tau f(0) = \sum_{i\geq 1} \frac{f(\tau_i^{-1}(0))}{\tau_i'(\tau_i^{-1}(0))} = \sum_{i\geq 1} \frac{f(\tau_i^{-1}(\tau_i(a_i))}{\tau_i'(\tau_i^{-1}(\tau(a_i))} = \sum_{i\geq 1} \frac{f(a_i)}{\tau_i'(a_i)},$$

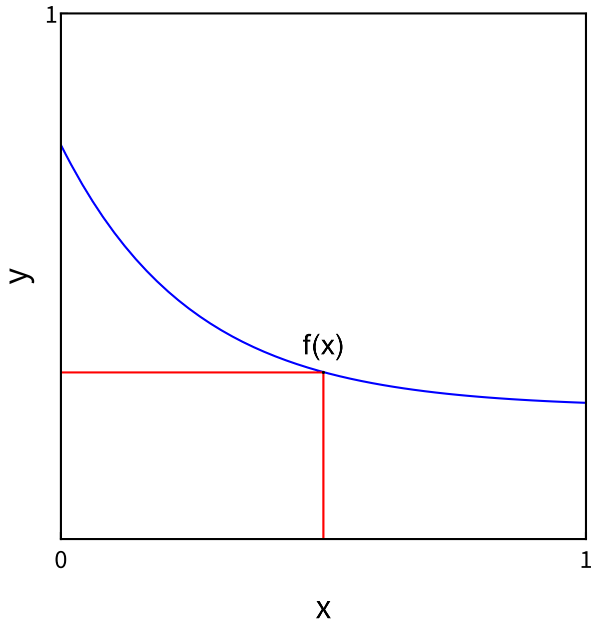

FIGURE 2. Illustration for the proof of part(2) of Proposition 2.3.

$$\leq \sum_{i:a_i<r} \frac{f(a_i)}{\tau_i'(a_i)} + \sum_{i:a_i>r} \frac{f(a_i)}{\tau_i'(a_i)} \leq \alpha \cdot \|f\|_\infty + \sum_{i:a_i>r} \frac{f(a_i)}{\tau_i'(a_i)},$$

$$\leq \alpha \cdot \|f\|_\infty + \sum_{i:a_i>r} \frac{1}{a_i \cdot \tau_i'(a_i)} \cdot \|f\|_1.$$

For this case we define D as $D = \sum_{i:a_i>r} \frac{1}{a_i \cdot \tau_i'(a_i)}$.

Case II: Let $a_1 = 0$ be not a limit point of partition endpoints a_i. Again, since $P_\tau f$ is non-increasing, $\|P_\tau f\|_\infty \leq P_\tau f(0)$, and we have

$$P_\tau f(0) = \sum_{i\geq 1} \frac{f(\tau_i^{-1}(0))}{\tau_i'(\tau_i^{-1}(0))} = \sum_{i\geq 1} \frac{f(\tau_i^{-1}(\tau_i(a_i)))}{\tau_i'(\tau_i^{-1}(\tau(a_i)))} = \sum_{i\geq 1} \frac{f(a_i)}{\tau_i'(a_i)},$$

$$= \frac{f(a_1)}{\tau_1'(a_1)} + \sum_{i\geq 2} \frac{f(a_i)}{\tau_i'(a_i)} \leq \frac{f(0)}{\tau_1'(0)} + \sum_{i\geq 2} \frac{1}{a_i \cdot \tau_i'(a_i)} \cdot \|f\|_1,$$

$$\leq \alpha \cdot \|f\|_\infty + D \cdot \|f\|_1.$$

For this case we define D as $= \sum_{i\geq 2} \frac{1}{a_i \cdot \tau_i'(a_i)}$. □

Let $\mathcal{P}^{(n)} = \mathcal{P} \bigvee \tau^{-1}(\mathcal{P}) \bigvee \cdots \bigvee \tau^{n-1}(\mathcal{P})$. We denote the branches of τ^n by $\tau_i^{(n)}$. Then, $\mathcal{P}^{(n)} = \left\{ I_i^{(n)} = \left(a_i^{(n)}, b_i^{(n)}\right)\right\}_{i=1}^{\infty}$ is a countably infinite family of open disjoint subintervals of I corresponding to τ^n. We have the following results:

Theorem 2.4. *Let $\mathcal{P}$ be a partition for τ and $\mathcal{P}^{(n)}$ denote the partition for τ^n. If $\tau \in \mathcal{T}$ then $\tau^n \in \mathcal{T}$ as well, i.e. ,*

(a) *If $\tau \in \mathcal{T}$ then τ^n is piecewise increasing on $\mathcal{P}^{(n)}$.*

(b) *τ^n is piecewise convex on $\mathcal{P}^{(n)}$.*

(c) *τ^n is piecewise differentiable on $\mathcal{P}^{(n)}$.*

(d) $\lim_{x \to \left(a_i^{(n)}\right)^+} \tau_i^{(n)}(x) = 0$ *for τ^n on $\mathcal{P}^{(n)}$.*

(e)*The condition (2) holds for τ^n. i.e.,*

$$\sum_{i \geq 1} \frac{1}{\left(\tau_i^{(n)}\right)' \left(a_i^{(n)}\right)} < +\infty.$$

(f) *If $x = 0$ is not a limit point of the partition points and condition (3) holds for τ then it holds for τ^n.*

Proof. Proofs of (a), (b) and (c) are simple, since τ is piecewise increasing, convex and differentiable on $\mathcal{P}$ and we know the composition of increasing, convex and differentiable functions is also increasing, convex and differentiable. Hence (a), (b) and (c) hold for τ^n on $\mathcal{P}^{(n)}$. To prove **(d)** we will use induction. From (1) we have, when x approaches a_i from the right hand side $\lim_{x \to a_i^+} \tau_i(x) = 0$. We consider for $n = 2$ one branch of τ^2. The branch $\tau_k^{(2)} = \tau_j \circ \tau_i$ is defined on $I_{i,j} = I_i \cap \tau_i^{-1}(I_j) = \tau_i^{-1}(I_j)$. Since the left endpoint of $\tau_i(I_i)$ is 0, if the interval $\tau_i^{-1}(I_j)$ is not empty then it contains $\tau_i^{-1}(a_j) = a_k^{(2)}$-the left endpoint of $I_k^{(2)}$.
We have,

$$\lim_{x \to a_k^{(2)+}} (\tau_j \circ \tau_i)(x) = \tau_j \left(\tau_i \left(a_k^{(2)}\right)\right) = \tau_j(\tau_i(\tau_i^{-1}(a_j))) = \tau_j(a_j) = 0.$$

Now, we use induction. We assume that the result holds for τ^n. The map τ^n has infinitely many branches and on each branch, it satisfies the property, (d) i.e.,

$$\lim_{x \to a_i^{(n)+}} \tau_i^{(n)}(x) = 0.$$

For $n+1$, if we consider a k^{th} branch of τ^{n+1}, $\tau_k^{(n+1)} = \tau_j \circ \tau_i^{(n)}$. We have,

$$a_k^{(n+1)} = \left(\tau_i^{(n)}\right)^{-1} (a_j).$$

and

$$\lim_{x \to a_k^{(n+1)^+}} \tau_j \left(\tau_i^{(n)}(x)\right) = \tau_j \left(\tau_i^{(n)} \left(a_k^{(n+1)}\right)\right) = \tau_j \left(\tau_i^{(n)} \left(\left(\tau_i^{(n)}\right)^{-1} (a_j)\right)\right) = \tau_j(a_j) = 0.$$

(e) A branch of τ^2 is $\tau_j(\tau_i)$ defined on $I_{i,j} = I_i \cap \tau_i^{-1}(I_j) = \tau_i^{-1}(I_j)$. Let us assume

$\sum_{i\geq 1} \frac{1}{\tau_i'(a_i)} = K$. Then,

$$\sum_{j\geq 1}\sum_{i\geq 1} \frac{1}{(\tau_j \circ \tau_i)'(a_i)} = \sum_{j\geq 1}\sum_{i\geq 1} \frac{1}{\tau_j'(\tau_i(a_i))\tau_i'(a_i)}.$$

$$\leq \sum_{j\geq 1}\sum_{i\geq 1} \frac{1}{\tau_j'(a_j)\tau_i'(a_i)} \leq \sum_{j\geq 1} \frac{1}{\tau_j'(a_j)} \sum_{i\geq 1} \frac{1}{\tau_i'(a_i)} = K \cdot K < +\infty.$$

By induction the result holds for any n.

(f) Let $a_1 = 0$ be not a limit point of partition endpoints a_i. Then, $\tau'(0) = \frac{1}{\alpha} > 1$ and $(\tau^n)'(0) = \frac{1}{\alpha^n} > 1$. □

Lemma 2.5. *Let* $\tau : [0,1] \to [0,1]$ *satisfies the condition (1)-(3). Then, the set* $S = \bigcup_{n=0}^{\infty} \tau^{-n}(\{a_1, b_1, a_2, b_2, ...a_i, b_i...\})$ *is dense in* $[0,1]$.

Proof. **Case I**: We assume $a_1 = 0$ is not a limit point of the partition endpoints. Let

$$S = \bigcup_{n=0}^{\infty} \tau^{-n}(\{a_1, b_1, a_2, b_2, ...a_i, b_i...\}).$$

We want to prove that S is dense in $[0,1]$. Let's assume that it is not true. Then, there exists an interval $[x_0, y_0] \subset [0,1]$ such that,

$$\tau^n([x_0, y_0]) \cap \{a_1, b_1, a_2, b_2, ...a_i, b_i, ...\} = \phi \quad \text{for all } n = 0, 1, 2, 3....$$

Therefore for each n, the points $x_n = \tau^n(x_0)$ and $y_n = \tau^n(y_0)$ belong to the same interval (a_i, b_i). Let $x_n, y_n \in (a_k, b_k)$ and $x_n < y_n$, $k = 1, 2, 3....$ For τ_k defined on (a_k, b_k), $k = 1, 2, 3...$, see Figure 3 for $k = 1$ and Figure 4 for $k > 1$, we have,

$$\tan(\theta_1) = \frac{\tau_k(x_n)}{x_n} \quad \text{and} \quad \tan(\theta_2) = \frac{\tau_k(y_n)}{y_n}.$$

Since τ_k is increasing on (a_k, b_k) we have,

$$\tan(\theta_2) \geq \tan(\theta_1) \implies \frac{\tau_k(y_n)}{y_n} \geq \frac{\tau_k(x_n)}{x_n} \implies \frac{\tau_k(y_n)}{\tau_k(x_n)} \geq \frac{y_n}{x_n}. \tag{2}$$

or,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\tau_k(x_n)}{\tau_k(y_n)} \leq \frac{x_n}{y_n}.$$

Since this holds for $k = 1, 2, 3...$ we obtain for all $n \geq 1$,

$$\frac{x_{n+1}}{y_{n+1}} \leq \frac{x_n}{y_n} \leq ... \leq \frac{x_0}{y_0}. \tag{3}$$

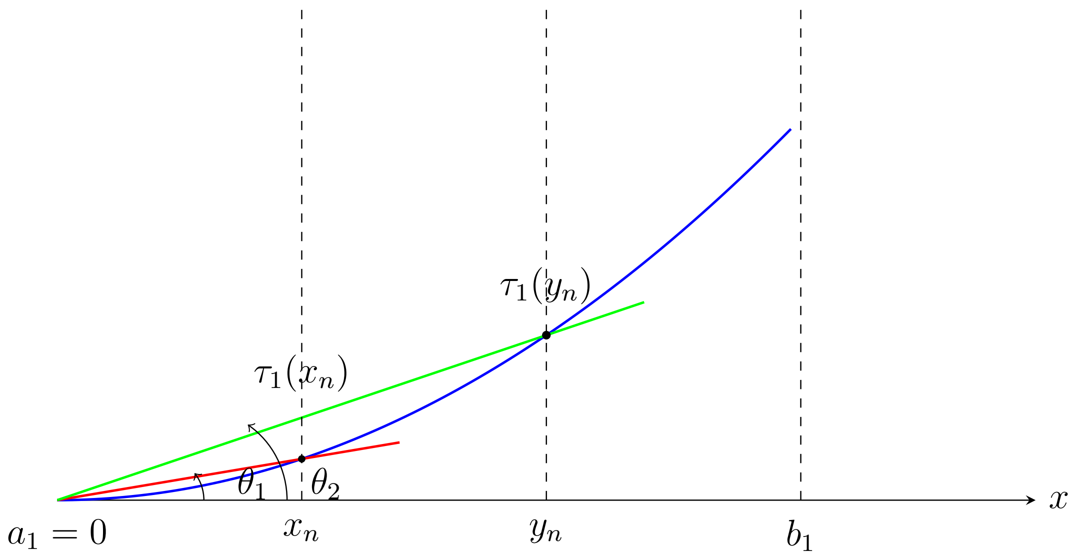

FIGURE 3. When 0 is not a limit point of the partition points and $k = 1$.

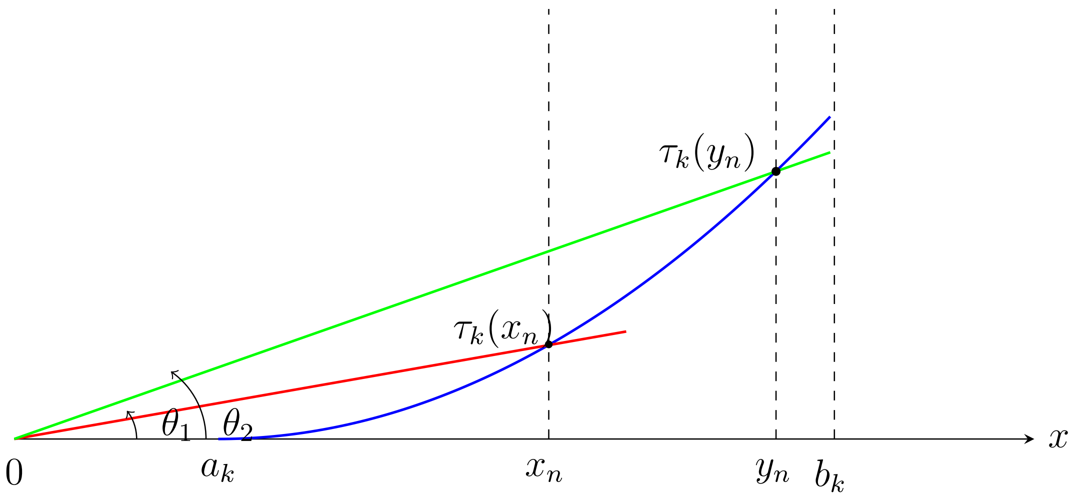

FIGURE 4. When 0 is not a limit point of the partition points and $k > 1$.

For $k > 1$, see Figure 5, we have,

$$\tan(\theta_1) = \frac{\tau_k(x_n)}{x_n - b_1 x_n} \quad \text{and} \quad \tan(\theta_2) = \frac{\tau_k(y_n)}{y_n - b_1 x_n}.$$

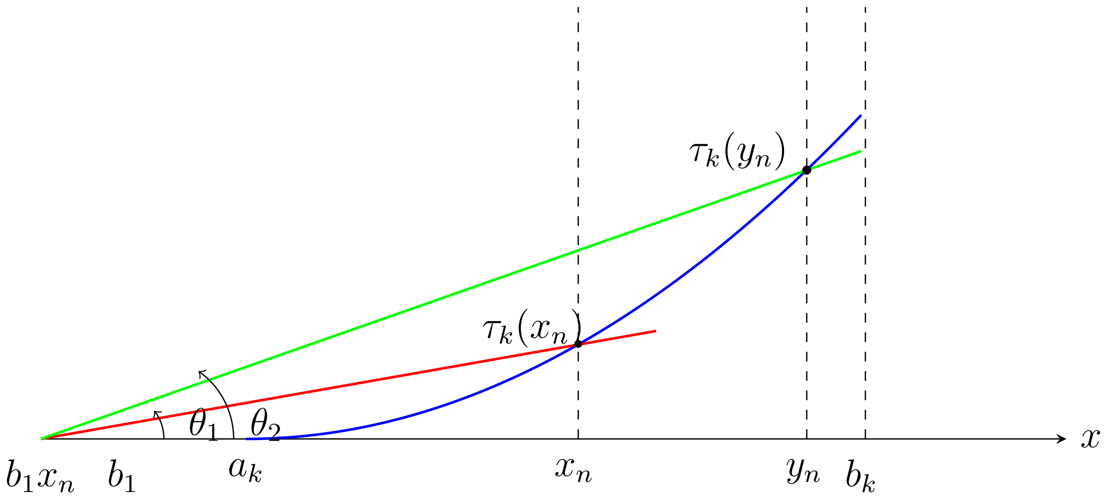

FIGURE 5. When 0 is the limit point of the partition points.

Since τ_k is increasing on (a_k, b_k) we have,

$$\tan(\theta_2) \geq \tan(\theta_1) \implies \frac{\tau_k(y_n)}{y_n - b_1x_n} \geq \frac{\tau_k(x_n)}{x_n - b_1x_n},$$

or

$$\frac{\tau_k(y_n)}{\tau_k(x_n)} \geq \frac{y_n - b_1x_n}{x_n - b_1x_n} = \frac{y_n\left(1 - b_1\frac{x_n}{y_n}\right)}{x_n(1-b_1)}.$$

By (3) we obtain,

$$\frac{1 - b_1\frac{x_n}{y_n}}{1-b_1} \geq \frac{1 - b_1\frac{x_0}{y_0}}{1-b_1}.$$

Thus, for $x_n, y_n \in (a_k, b_k)$ with $k > 1$, we obtain

$$\frac{y_{n+1}}{x_{n+1}} \geq q\frac{y_n}{x_n}, \tag{4}$$

where $q = \left(\dfrac{1 - b_1x_0/y_0}{1-b_1}\right) > 1$.
Since $\tau_1'(x) \geq \tau_1'(0) > 1$, the interval (x_n, y_n) is stretched by τ_1 as long as it stays in (a_1, b_1). Thus, it has to go above b_1 after a finite number of steps. Equation (4) implies that $\lim_{n\to\infty} \frac{y_n}{x_n} = \infty$. Since $\limsup_n x_n \geq b_1$ we have $\limsup_n y_n = \infty$, which is impossible as it contradicts the fact that y_n remain bounded within $[0, 1]$.

Case II: If 0 is the limit point of the partition points, the point 0 is not a left end of any interval (a_i, b_i). We choose an interval (a_j, b_j) such that $b_j < r$. Then $\tau'(x) > 1$ for all $x < b_j$, where

$\tau'(x)$ is not defined we use $\tau'_+(x)$. Again, we will show that the set

$$S = \bigcup_{n=0}^{\infty} \tau^{-n}(\{a_1, b_1, a_2, b_2, ...a_i, b_i...\}),$$

is dense in $[0, 1]$. Suppose it's not true. Then there exist an interval $[x_0, y_0] \subset [0, 1]$ such that

$$\tau^n([x_0, y_0]) \cap \{a_1, b_1, a_2, b_2, ...a_i, b_i, ...\} = \phi \quad \text{for all } n = 0, 1, 2, 3...$$

This means that for each n the points $x_n = \tau^n(x_0)$ and $y_n = \tau^n(y_0)$ belong to the same interval $(a_i, b_i), i = 1, 2, 3....$. For any $x_n, y_n \in (a_k, b_k), k = 1, 2, 3...$, using Figure 4, we obtain

$$\frac{y_{n+1}}{x_{n+1}} = \frac{\tau_k(y_n)}{\tau_k(x_n)} \geq \frac{y_n}{x_n}. \tag{5}$$

Thus, the formula (3) is valid also in this case.

Now, let $x_n, y_n \in (a_k, b_k)$ with $a_k > b_j$, i.e., the interval (a_k, b_k) is on the right hand side of the interval (a_j, b_j). Using Figure 5 with a_1, b_1 replaced by a_j and b_j , correspondingly, we obtain

$$\frac{\tau_k(y_n)}{\tau_k(x_n)} \geq \frac{y_n - b_j x_n}{x_n - b_j x_n} = \frac{y_n\left(1 - b_j \frac{x_n}{y_n}\right)}{x_n(1 - b_j)}.$$

Similarly as Case I for $x_n, y_n \in (a_k, b_k)$ with $b_j < a_k$, we obtain

$$\frac{y_{n+1}}{x_{n+1}} \geq q \cdot \frac{y_n}{x_n}, \tag{6}$$

where $q = \left(\dfrac{1 - b_j x_0/y_0}{1 - b_j}\right) > 1$.

Since $\tau'(x) \geq 1/\alpha > 1$ for all $x \leq b_j$, the subsequent images of any interval $(x_n, y_n) \subset (0, b_j)$ get larger and larger as long as they stay in $(0, b_j)$. At the same time, the points x_{n+i}, y_{n+i} are never separated by the points of the partition. Thus, after a finite number of steps interval (x_n, y_n) moves to the right of the interval $(0, b_j)$. Thus, for infinitely many $n's$ we have $x_n, y_n > b_j$ and according to (6), $\lim_n \dfrac{y_n}{x_n} = \infty$. Also, as we know $\limsup_n x_n \geq b_j$ and we obtain $\limsup_n y_n = \infty$, which is impossible. Hence S is dense in $[0, 1]$.

□

Lemma 2.6. *There exist a natural number n_0 such that for $n > n_0$, $\inf(\tau^n)' \geq \gamma$, for some $\gamma > 1$.*

Proof. Recall, that $\mathcal{P}^{(n)} = \left\{I_i = \left(a_i^{(n)}, b_i^{(n)}\right)\right\}_{i=1}^{\infty}$ is a partition corresponding to τ^n, and the branch of τ^n defined on the interval $\left(a_i^{(n)}, b_i^{(n)}\right)$ is $\tau_i^{(n)}$. We also know that τ^n satisfies conditions

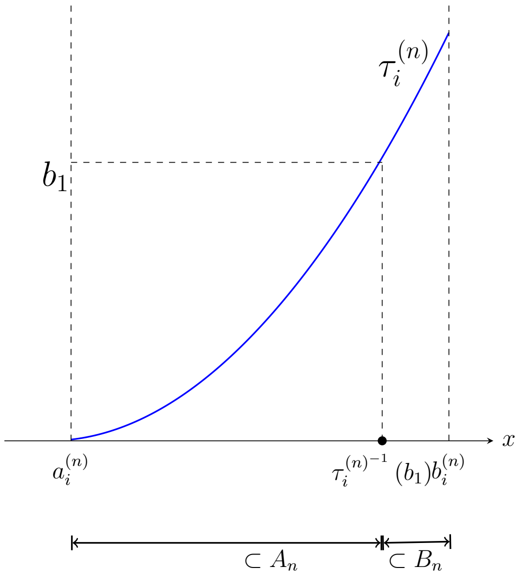

FIGURE 6. n^{th} iterate of τ on I_i.

(1)-(3) with respect to the partition $\mathcal{P}^{(n)}$. Consider the set,

$$S = \bigcup_{n=0}^{\infty} \tau^{-n}(\{a_1, b_1, a_2, b_2, ...a_i, b_i...\}).$$

In Lemma 2.5 we proved that S is dense in $[0,1]$. We consider two cases.

Case I: $a_1 = 0$ is not a limit point of partition endpoints a_i. Since $\bar{S} = [0,1]$, then there exist $\bar{n} \in \mathbf{N}$ such that for any $n > \bar{n}$,

$$\max_i \left(b_i^{(n)} - a_i^{(n)}\right) < \eta = \frac{b_1}{2}\inf(\tau'). \tag{7}$$

Note that, this condition ensures that the length of the longest interval in the partition of I for the n^{th} iterate of τ must be less than η.

Let $n > \bar{n}$. Define,

$$A_n = \tau^{-n}(a_1, b_1) = \bigcup_{i=1}^{\infty} (\tau_i^n)^{-1}((a_1, b_1)),$$

and $B_n = [0,1] \setminus A_n$. We have

$$\tau^n(x) < b_1 \quad \text{if} \quad x \in A_n, \tag{8}$$

and

$$\tau^n(x) \geq b_1 \quad \text{if} \quad x \in B_n. \tag{9}$$

The map τ^n is increasing on each interval $\left(a_i^{(n)}, b_i^{(n)}\right)$ and $\left(\tau_i^{(n)}\right)'$ represents the rate of change of τ^n on this interval. By (7), we know that the length of this interval is less than η. Then, the length of the interval $\left(a_i^{(n)}, \left(\tau_i^{(n)}\right)^{-1}(b_1)\right)$ is also less than η. Therefore, $\left(\tau_i^{(n)}\right)'$ must be sufficiently large to ensure that τ^n increases by at least b_1 over an interval of length less than η which gives,

$$\left(\tau_i^{(n)}\right)'(x) \geq \frac{b_1}{\eta},$$

for some $x \in \left(a_i^{(n)}, \left(\tau_i^{(n)}\right)^{-1}(b_1)\right) = A_n \cap \left(a_i^{(n)}, b_i^{(n)}\right)$. Since $\left(\tau_i^{(n)}\right)'$ is increasing, we have the same inequality for all $x \in B_n \cap \left(a_i^{(n)}, b_i^{(n)}\right)$. Hence by (7),

$$\begin{aligned} \left(\tau \circ \tau_i^{(n)}\right)'(x) &= \tau'\left(\tau_i^{(n)}(x)\right) \cdot \left(\tau_i^{(n)}\right)'(x), \\ &\geq \frac{b_1}{\eta} \inf(\tau') \geq 2, \end{aligned} \tag{10}$$

Whenever $x \in B_n, i = 1, 2, 3....$. For $x \in A_n$ we have,

$$\begin{aligned} \left(\tau \circ \tau_i^{(n)}\right)'(x) = \tau'\left(\tau_i^{(n)}(x)\right) \cdot \left(\tau_i^{(n)}\right)'(x) &\geq \tau'(0)\left(\tau_i^{(n)}\right)'\left(a_i^{(n)}\right), \\ &\geq \tau'(0)\inf(\tau^n)', \end{aligned} \tag{11}$$

Inequalities (10) and (11) give us,

$$\inf\left(\tau^{n+1}\right)' \geq \min\left(2, \tau'(0)\inf(\tau^n)'\right),$$

and consequently, by induction we have

$$\inf(\tau^n)' \geq \min(2, [\tau'(0)]^{n-\bar{n}}\inf(\tau^{\bar{n}})'),$$

For $n > \bar{n}$. This implies that for sufficiently large n we have $\inf(\tau^n)' \geq \gamma$.

Case II: $a_1 = 0$ is a limit point of partition endpoints a_i. We choose an interval (a_j, b_j) such that $b_j < r$. Since S is dense in $[0, 1]$, then there exist $\bar{n} \in \mathbf{N}$ such that for any $n > \bar{n}$,

$$\max_i\left(b_i^{(n)} - a_i^{(n)}\right) < \eta = \frac{b_j}{2}\inf(\tau'). \tag{12}$$

Let $n > \bar{n}$. Define,

$$A_n = \tau^{-n}(0, b_j) = \bigcup_{i=1}^{\infty}(\tau_i^n)^{-1}((0, b_j)),$$

and $B_n = [0, 1] \setminus A_n$. We have

$$\tau^n(x) < b_j \quad \text{if} \quad x \in A_n, \tag{13}$$

and

$$\tau^n(x) \geq b_j \quad \text{if} \quad x \in B_n. \tag{14}$$

Note that for $x \in A_n$ we have $\tau'(\tau^n(x)) \geq \frac{1}{\alpha} > 1$. The map τ^n is increasing on each interval $\left(a_i^{(n)}, b_i^{(n)}\right)$ and $\left(\tau_i^{(n)}\right)'$ represents the rate of change of τ^n on this interval. By (12), we know that the length of this interval is less than η. Then, the length of the interval $\left(a_i^{(n)}, \left(\tau_i^{(n)}\right)^{-1}(b_j)\right)$ is also less than η. Therefore, $\left(\tau_i^{(n)}\right)'$ must be sufficiently large to ensure that τ^n increases by at least b_j over an interval of length less than η which gives,

$$\left(\tau_i^{(n)}\right)'(x) \geq \frac{b_j}{\eta},$$

for some $x \in \left(a_i^{(n)}, \left(\tau_i^{(n)}\right)^{-1}(b_j)\right) = A_n \cap \left(a_i^{(n)}, b_i^{(n)}\right)$. Since $\left(\tau_i^{(n)}\right)'$ is increasing, we have the same inequality for all $x \in B_n \cap \left(a_i^{(n)}, b_i^{(n)}\right)$. Hence by (12),

$$\begin{aligned}\left(\tau \circ \tau_i^{(n)}\right)'(x) &= \tau'\left(\tau_i^{(n)}(x)\right) \cdot \left(\tau_i^{(n)}\right)'(x), \\ &\geq \frac{b_j}{\eta} \inf(\tau') \geq 2,\end{aligned} \tag{15}$$

Whenever $x \in B_n, i = 1, 2, 3....$. For $x \in A_n$ we have,

$$\begin{aligned}\left(\tau \circ \tau_i^{(n)}\right)'(x) = \tau'\left(\tau_i^{(n)}(x)\right) \cdot \left(\tau_i^{(n)}\right)'(x) &\geq \frac{1}{\alpha} \cdot \left(\tau_i^{(n)}\right)'\left(a_i^{(n)}\right), \\ &\geq \frac{1}{\alpha} \cdot \inf(\tau^n)',\end{aligned} \tag{16}$$

Inequalities (15) and (16) give us,

$$\inf\left(\tau^{n+1}\right)' \geq \min\left(2, \frac{\inf(\tau^n)'}{\alpha}\right),$$

and consequently, by induction we have

$$\inf(\tau^n)' \geq \min\left(2, \left(\frac{1}{\alpha}\right)^{n-\bar{n}} \inf\left(\tau^{\bar{n}}\right)'\right),$$

For $n > \bar{n}$. This implies that for sufficiently large n we have $\inf(\tau^n)' \geq \gamma$.

□

3. Piecewise expanding map with countable number of branches

Definition 3.1. Let $I = [0, 1]$ and let $\mathcal{P} = \{I_i = (a_i, b_i)\}_{i=1}^{\infty}$ be a countably infinite family of open disjoint subintervals of I such that Lebesgue measure of $I \setminus \bigcup_{i\geq 1} I_i$ is zero. Let τ be a map from $\cup_{i\geq 1} I_i$ to the interval I, such that for each $i \geq 1$, $\tau_{|I_i}$ extends to a homeomorphism τ_i of $[a_i, b_i]$ onto its image.

Let

$$g(x) = \begin{cases} \frac{1}{|\tau_i'(x)|}, & \text{for } x \in I_i, i = 1, 2, \dots \\ 0, & \text{elsewhere} \end{cases}.$$

We assume $\sup_{x\in I} |g(x)| \leq \beta < 1$. Then, we say τ is a piecewise expanding map with countably many branches and denote this class by $\mathcal{T}_E$.

Lemma 3.2. *If $\tau \in \mathcal{T}$ in the sense of Definition 2.1, then some iterate of $\tau^n \in \mathcal{T}_E$ in the sense of Definition 3.1.*

Proof. Proof of this lemma is a direct consequence of Lemma 2.6 and the condition (2) of Definition 2.1. □

A piecewise expanding map τ is non-singular and the Frobenious-Perron operator corresponding to τ is,

$$P_\tau f(x) = \sum_{i=1}^{\infty} \frac{f\left(\tau_i^{-1}(x)\right)}{\left|\tau'\left(\tau_i^{-1}(x)\right)\right|} \chi_{\tau(I_i)}(x) = \sum_{y\in\tau^{-1}(x)} f(y)g(y). \tag{17}$$

Given $f : I \to \mathbb{R}$ we define variation of f on a subset J of I by

$$V_J(f) = \sup\{\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|\}.$$

where the supremum is taken over all sequence $(x_1, x_2, ...x_k), x_1 \leq x_2 \leq ... \leq x_k, x_i \in J$. We need a variation $v(f)$ for $f \in L^1_m$, the set of all equivalence classes of real-valued, m-integrable functions on I.

Let $BV = \{f \in L^1_m : v(f) < +\infty\}$, where $v(f) = \inf\{V_I f^* : f^* \text{ is a version of } f\}$. We define for $f \in BV$,

$$\|f\|_v = \int |f| dm + v(f).$$

BV is a Banach space with norm $\|.\|_v$.

Note : Every $f \in$ BV has a version f^* with minimal variation. This holds iff for every $x_0 \in I$,

$$f^*(x_0) \in [\lim_{x\to x_0^-} f^*, \lim_{x\to x_0^+} f^*],$$

One-sided limit always exists for f^*. In particular, we choose f^* which is right-hand side continuous.

Proposition 3.3. *For every $f \in BV$ we have,*

$$V_I P_{\tau^n} f \leq A_n \cdot V_I f + B_n \cdot \|f\|_1, \tag{18}$$

where $A_n = \|g_n\|_\infty + \max_{K\in\mathcal{Q}} V_K g_n < 1$, *for n sufficiently large, and* $B_n = \dfrac{\max_{K\in\mathcal{Q}} V_K g_n}{m(K)}$.

Proof. We follow [19]. For $f \in BV$ we have,

$$P_\tau^n f(x) = \sum_{y\in\tau^{-n}(x)} f(y)\cdot g_n(y), \tag{19}$$

where,

$$g_n = \begin{cases} \frac{1}{|(\tau^n)'|}, & \text{on } \cup_{J\in\mathcal{P}^{(n)}} J \\ 0, & \text{elsewhere} \end{cases}.$$

Let $\mathcal{P}^{(n)}$ be a partition of I corresponding to τ^n. Then,

$$P_{\tau^n} f = \sum_{J\in\mathcal{P}^{(n)}} P_{\tau^n}(f\cdot\chi_J),$$

which gives,

$$V_I P_{\tau^n} f \leq \sum_{J\in\mathcal{P}^{(n)}} V_I P_{\tau^n}(f\cdot\chi_J).$$

We notice that for $J \in \mathcal{P}^{(n)}$ we have,

$$\begin{aligned} P_{\tau^n}(f\cdot\chi_J)\circ\tau_J^n(x) &= \sum_{J\in\mathcal{P}^{(n)}} f(\tau_J^{-n}(\tau_J^n(x)))\cdot g_n(\tau_J^{-n}(\tau_J^n(x)))\cdot\chi_J(\tau_J^{-n}(\tau_J^n(x))) \\ &= f(x)\cdot g_n(x)\cdot\chi_J(x), \end{aligned}$$

since $\tau^n|_J$ is monotonic. We have,

$$V_I P_{\tau^n}(f\cdot\chi_J) = V_I(f\cdot g_n\cdot\chi_J) = V_J(f\cdot g_n).$$

Taking summation on both sides we get,

$$\sum_{J\in\mathcal{P}^{(n)}} V_I P_{\tau^n}(f\cdot\chi_J) = \sum_{J\in\mathcal{P}^{(n)}} V_J(f\cdot g_n) = V_I(f\cdot g_n).$$

Let $\mathcal{Q}$ be a finite partition of I. Then we know,

$$V_I(f\cdot g_n) = \sum_{K\in\mathcal{Q}} V_K(f\cdot g_n), \tag{20}$$

and

$$V_K(f \cdot g_n) \leq V_K f \cdot \|g_n\|_\infty + \|f \cdot \chi_K\|_\infty \cdot V_K g, \tag{21}$$

$$\|f \cdot \chi_K\|_\infty \leq \frac{1}{m(K)}\left|\int_K f dm\right| + V_K f. \tag{22}$$

Using (21) and (22) in (20) we get,

$$\begin{aligned} V_I\left(f \cdot g_n\right) &\leq \sum_{K \in \mathcal{Q}}\left(V_K f \cdot \|g_n\|_\infty + \frac{1}{m(K)}\left|\int_K f dm\right| \cdot V_K g_n + V_K f \cdot V_K g_n\right), \\ &\leq V_I f \cdot \|g_n\|_\infty + \frac{\max_{K \in \mathcal{Q}} V_K g_n}{m(K)} \cdot \|f\|_1 + \max_{K \in \mathcal{Q}} V_K g_n \cdot V_I f, \\ &= \left(\|g_n\|_\infty + \max_{K \in \mathcal{Q}} V_K g_n\right) V_I f + \frac{\max_{K \in \mathcal{Q}} V_K g_n}{m(K)} \cdot \|f\|_1. \end{aligned} \tag{23}$$

We know from Richlik's paper [19], that for every $\epsilon > 0$ there exist a finite partition say $\mathcal{Q}$ such that

$$\max_{K \in \mathcal{Q}} V_J g \leq \|g\|_\infty + \epsilon.$$

The result will still be true if we replace $\|g\|_\infty < 1$ by $\|g_n\|_\infty < 1$ for some $n \geq 1$. For $0 < \epsilon < 1$ we can find $n \geq 1$ such that $2 \cdot \|g_n\|_\infty + \epsilon < 1$. Hence,

$$\|g_n\|_\infty + \max_{K \in \mathcal{Q}} V_K g_n \leq 2 \cdot \|g_n\|_\infty + \epsilon < 1.$$

Finally we have,

$$V_I P_{\tau^n} f \leq V_I(f \cdot g_n) \leq A_n \cdot V_I f + B_n \cdot \|f\|_1,$$

where $A_n = \left(\|g_n\|_\infty + \max_{K \in \mathcal{Q}} V_K g_n\right) < 1$ and $B_n = \dfrac{\max_{K \in \mathcal{Q}} V_K g_n}{m(K)}$. □

Lemma 3.4. *(1) For every $c > 0$, the set $F = \{f \in L^1_m : \|f\|_v \leq c\}$ is compact in L^1_m.*
(2) $(BV, \|.\|_v)$ is a Banach space.
(3) BV is dense in L^1_m.

Proof. This is proved in Keller's paper [14]. □

Corollary 3.5. *If τ is piecewise convex then for some $n > 1$ and $f \in BV$, we have*

$$\|P_{\tau^n} f\|_v \leq r \cdot \|f\|_v + C \cdot \|f\|_1,$$

where $r \in (0, 1)$ and $C > 0$.

Proof. We know for $f \in BV$,

$$\|f\|_v = \int |f| dm + v(f) = \|f\|_1 + v(f).$$

So,

$$\|P_{\tau^n} f\|_v = \|P_{\tau^n} f\|_1 + v(P_{\tau^n} f) \leq \|f\|_1 + v(P_{\tau^n} f). \tag{24}$$

Since f^* is a version of $f \in L^1_m$, for $\epsilon > 0$, Proposition 3.3 holds for f^* as well. Hence,

$$V_I P_{\tau^n} f^* \leq A_n \cdot V_I f^* + B_n \cdot \|f^*\|_1,$$

and

$$V_I f^* \leq v(f) + \epsilon.$$

Since $P_{\tau^n} f^*$ is a version of $P_{\tau^n} f$ we have,

$$\begin{aligned} v(P_{\tau^n} f) \leq V_I(P_{\tau^n} f^*) &\leq A_n \cdot V_I f^* + B_n \cdot \|f^*\|_1, \\ &\leq A_n(v(f) + \epsilon) + B_n \cdot \|f\|_1. \end{aligned}$$

From (24) we get,

$$\begin{aligned} \|P_{\tau^n} f\|_v &\leq \|f\|_1 + A_n \cdot v(f) + B_n \cdot \|f\|_1 + A_n \cdot \epsilon, \\ &\leq A_n \cdot \|f\|_v + (1 + B_n)\|f\|_1 + A_n \cdot \epsilon, \end{aligned}$$

since $\epsilon > 0$ is arbitrary, by choosing $r = A_n$ and $1 + B_n = C$, we get the desired result. □

The properties of the operator P_{τ^n} and of the space BV which we proved in (3.3),(3.4) and (3.5) allow us to use an ergodic theorem of Ionescu-Tulcea and Marinescu [7].

Theorem 3.6. *Let $(X, \|.\|_X)$ be a Banach space which is a linear subspace of $(Y, \|.\|_Y)$ such that if $f_n \in X$, $\|f_n\|_X \leq K$ is such that $f_n \to f$ in $Y \implies f \in X$ and $\|f\|_X \leq K$. Let $\mathcal{C}(X)$ be the class of bounded linear operators with image in X satisfies the following conditions: (1) There exists $H > 0$ such that $\|P^n\|_X < H, \forall n \in \mathbf{N}$.*
(2) There exists two positive constants $0 < r < 1$ and $R > 0$ such that,

$$\|Pf\|_X \leq r \cdot \|f\|_X + R \cdot \|f\|_Y,$$

whenever $f \in X$.
(3) $P(B)$ is compact in Y for every bounded $B \in (X, \|.\|_X)$.
Then every $P \in \mathcal{C}(X)$ has a finite number of eigenvalues $\{c_1, c_2, c_3 ... c_p\}$ of modulus 1 with finite

dimensional eigenspaces $\{E_1, E_2, ...E_p\}$*, and*

$$P^n = \sum_{i=1}^{p} c_i^n P_i + S^n,$$

where, if $\{\Psi_i\}_{i=1,2,..,p}$, Ψ_0 *are projections relative to the splitting,*

$$X = \bigoplus_{i=1}^{p} E_i \oplus E_0,$$

$P_i = P \circ \Psi_i$ *and* $\|S^n\|_X = O(q^n)$ *for some* $q \in (0,1)$.

Above theorem helps us to understand the behaviour of P_τ. For the conclusion of the theorem to hold it is enough that some iterate of P satisfies conditions $(1)-(3)$. The spaces $BV = X, L^1_m = Y$ and operator $P_\tau = P$ satisfy the assumptions of Ionescu-Tulcea and Marinescu [7]. Hofbauer and Keller [4, 14] were the first to use this theorem for proving the quasi-compactness of P_τ and the existence of ACIM for τ. Before we use Theorem 3.6, we prove exactness of τ with ACIM.

Lemma 3.7. *Let* $\tau : [0,1] \to [0,1]$ *satisfies Definition 2.1. Then there exists the unique normalized absolutely continuous* τ *invariant measure* μ*. The dynamical system* $([0,1], \mathcal{B}, \mu; \tau)$ *is exact and the density* $h = \dfrac{d\mu}{dx}$ *is bounded and decreasing.*

Proof. We follow [16] closely. The proof is based on Theorem 2 of [16], which states that the existence of a lower function is sufficient for the existence of ACIM and the exactness of the system. The map τ satisfies conditions (1), (2) and (3) of Definition 2.1. We have proved in Lemma 2.5 that S is dense in $[0,1]$. Let $\mathbf{1}_\Delta$ be the characteristic function of an interval $\Delta = [d_0, d_1]$ whose end points belong to the set S. We claim that for sufficiently large n, $P^n_\tau \mathbf{1}_\Delta$ is a decreasing function. We have proved that any iteration of τ satisfies the properties (1), (2) and (3) of Definition 2.1, in particular, we have proved that $\tau_i^{(n)}$ is piecewise convex on $I_i^{(n)} = \left(a_i^{(n)}, b_i^{(n)}\right)$ an element the partition $\mathcal{P}^{(n)}$ corresponding to τ^n and $\tau^n\left(a_i^{(n)}\right) = 0$. This implies that $[\chi_{|\left(a_i^{(n)}, b_i^{(n)}\right)} \cdot g_n] \left(\tau_i^{(n)}\right)^{-1}(x) \cdot \chi_{|\tau_i^{(n)}\left(a_i^{(n)}, b_i^{(n)}\right)}(x)$, is a non-increasing function on $[0,1]$, since g_n is non-increasing as the reciprocal of the derivative of a convex function. We can see that,

$$\left\{a_1^{(n)}, b_1^{(n)}, a_2^{(n)}, b_2^{(n)}, ...a_i^{(n)}, b_i^{(n)}....\right\} = \tau^{-n+1}\{a_1, b_1, a_2, b_2, ...a_i, b_i, ...\}.$$

Because by the definition of the points $a_i^{(n)}, b_i^{(n)}$ they are the preimages of the original partition points. This shows that every next partition is a partition of the previous one, i.e., they are finer and finer. Since $d_1, d_2 \in S$ there is an integer n_0 sufficiently large such that d_i belongs to the partition $\left\{a_1^{(n)}, b_1^{(n)}, a_2^{(n)}, b_2^{(n)}, ...a_i^{(n)}, b_i^{(n)}...\right\}$ for $n \geq n_0$. The Frobenius-Perron operator for τ^n is

$$P_\tau^n f(x) = \sum_{y \in \tau^{-n}(x)} f(y) g_n(y).$$

In particular, for $f = \mathbf{1}_\Delta$ and $n \geq n_0$ we have,

$$P_\tau^n \mathbf{1}_\Delta(x) = \sum_{y \in \tau^{-n}(x)} g_n(y) \cdot \chi_{\tau^n\left(I_i^{(n)}\right)}(y).$$

Since $\tau^n\left(I_i^{(n)}\right)$ is of the form $\left(0, \tau^n\left(b_i^{(n)}\right)\right)$, $P_\tau^n \mathbf{1}_\Delta$ is non-increasing as a sum of non-increasing functions. Now, let D_0 be a subset of L_m^1 consisting of all functions of the form

$$f(x) = \sum_{i=1}^{\infty} c_i \mathbf{1}_{\Delta_i}, c_i \geq 0,$$

where the endpoints of the intervals Δ_i belong to S. Since S is dense in $[0, 1]$, the set D_0 is dense in L_m^1. Now, we construct a lower function for P_τ. Let $f \in D_0$. There exists $n_0 = n_0(f)$ such that $P_\tau^n f$ is non-increasing for $n \geq n_0$. By part (2) of Proposition 2.3 for any $\tau \in \mathcal{T}$, P_τ preserves the cone of non-increasing functions [2]. In particular we have $P_\tau^n f(x) \leq 1/x$ for $n \geq n_0$. Now, using this estimate and Proposition 2.3 we get,

$$P_\tau^{n+1} f(0) = P_\tau(P_\tau^n f(0)) \leq \alpha \cdot P_\tau^n f(0) + D.$$

where $\alpha < 1$ and D are defined as in Proposition 2.3 for both cases. Using an induction argument we get,

$$P_\tau^{n+n_0} f(0) \leq \alpha^n \cdot P_\tau^{n_0} f(0) + \frac{D}{1-\alpha}.$$

Let $K = 1 + \frac{D}{1-\alpha}$. For sufficiently large n, say $n \geq n_1$, We have $P_\tau^n f(0) \leq K$.
Define $h = \frac{1}{2}\mathbf{1}_{[0,1/(2K)]}$. We will prove,

$$P_\tau^n f(x) \geq h(x) \text{ for } n \geq n_1.$$

By contradiction, if there exist $x_0 \in [0, 1/(2K)]$ such that $P_\tau^n f(x_0) < h(x_0) = \frac{1}{2}$ then,

$$1 = \int_0^{x_0} P_\tau^n f dx + \int_{x_0}^1 P_\tau^n f dx < x_0 P_\tau^n f(0) + (1 - x_0) P_\tau^n f(x_0) \leq \frac{1}{2K} \cdot K + \frac{1}{2} \ = 1,$$

Which is not possible. Hence $P_\tau^n f(x) \geq h(x)$ for $n \geq n_1$. □

Lemma 3.7 implies that the only eigenvalue of P_τ of modulus 1 is 1 and that it's eigenspace is one dimensional. With Theorem 3.6 this gives the following :

Theorem 3.8. *For a piecewise convex map τ with countable number of branches, it's Frobenius-Perron operator P_τ is quasi-compact on the space BV. More precisely, we have*
(1) $P_\tau : L^1_m \to L^1_m$ has 1 as the only eigenvalue of modulus 1.
(2) Set $E_1 = \{f \in L^1_m \mid P_\tau f = f\} \subseteq BV$ and E_1 is one dimensional.
(3) $P_\tau = \Psi + Q$, where Ψ represents the projection on eigenspace E_1, $\|\Psi\|_1 \leq 1$ and Q is a linear operator on L^1_m with $Q(BV) \subseteq BV$, $\sup_{n\in\mathbf{N}} \|Q^n\|_1 < \infty$ and $Q \cdot \Psi = 0$.
(4) $Q(BV) \subset BV$ and, considered as a linear operator on $(BV, \|.\|_v)$, Q satisfies $\|Q^n\|_v \leq H \cdot q^n$ $(n \geq 1)$ for some constants $H > 0$ and $0 < q < 1$.

Proof. The results 1 to 4 are direct consequences of Ionescu-Tulcea and Marinescu ergodic Theorem [7] and Lemma 3.7. □

Quasi-compactnes of P_τ implies several important ergodic properties for the system (τ, μ) such as exponential decay of correlation, Central Limit Theorem and many other problestic consequences, see [4, 14].

- **Weak Mixing:** Since 1 is the only eigenvalue of P_τ with modulus 1 and the corresponding eigenspace is one-dimensional, the system (τ, μ) does not have any non-trivial periodic components. This implies that (τ, μ) is weakly mixing and has several important statistical and ergodic properties, including:
- **Exponential Decay of Correlations:** For functions of bounded variation, the correlation function decays exponentially fast. This means that for any two observables $f, g \in BV$, there exist constants $C > 0$ and $\rho < 1$ such that:

$$\left| \int f \cdot (g \circ \tau^n)\, d\mu - \int f\, d\mu \int g\, d\mu \right| \leq C \|f\|_{BV} \|g\|_{BV} q^n.$$

- **Central Limit Theorem:** The system satisfies the Central Limit Theorem, meaning the sum of observations (properly normalized) converges in distribution to a normal distribution. Specifically, for a function $f \in BV$ with $\int f d\mu = 0$, the sequence of partial sums $S_n = \sum_{i=0}^{n-1} f \circ \tau^i$ satisfies:

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}\big(0, \sigma^2\big),$$

where σ^2 is the variance given by:

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int (S_n)^2\, d\mu.$$

The CLT states that the normalized partial sums $\frac{S_n}{\sqrt{n}}$ converges in distribution to a normal distribution $\mathcal{N}\big(0,\sigma^2\big)$ as $n \to \infty$.

- **Almost Sure Invariance Principle (ASIP):** Let $\tau \in \mathcal{T}$, let μ be its ACIM, and let $f \in BV$ be a real-valued function such that $\int f\, d\mu = 0$. For some $1 \leq s < \infty$, define the sequence of partial sums:

$$S(t) = \sum_{0\leq n<t}^{s} f \circ \tau^n.$$

The variance σ^2 is given by the absolutely convergent series:

$$\sigma^2 = \int f^2 d\mu + 2\sum_{k=1}^{\infty} \int f \cdot \big(f \circ \tau^k\big)\, d\mu.$$

Assume $\sigma^2 \neq 0$. Then the following holds:

(1) The integral of $S(t)^2$ satisfies:

$$\int S(t)^2 d\mu = t \cdot \sigma^2 + O(1).$$

(2) The normalized partial sums satisfy a central limit type approximation:

$$\sup_{z\in\mathbb{R}} \left| \mu\left(\frac{S(t)}{\sigma\sqrt{t}} \leq z\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx \right| = O\big(t^{-\theta}\big),$$

for some $\theta > 0$.

(3) Without changing the distribution, one can redefine the process $(S(t))_{t\geq 0}$ on a richer probability space together with a standard Brownian motion $(B(t))_{t\geq 0}$ such that:

$$\big|\sigma^{-1} \cdot S(t) - B(t)\big| = O\big(t^{-1/2}\big) \quad \mu\text{-almost everywhere.}$$

The ASIP indicates that the process $S(t)$ can be coupled with a standard Brownian motion $B(t)$ in such a way that their paths remain close almost surely, with an error term that decays as $t^{-1/2}$. This result leverages the mixing properties and the structure of the Frobenius-Perron operator to establish a strong approximation.

- **Other Probabilistic Properties:** μ is the equilibrium state for $\log g$ on I, i.e.,

$$h(\mu) + \int \log g d\mu = \sup\{h(\nu) + \int \log g d\nu |\ \nu \text{ is a } \tau\text{-invariant probability on } I\ \},$$

where $h(\nu)$ is the entropy of (τ, ν).

Note 3.9. *No result in this paper implies that the invariant measure is supported on the whole interval* $[0, 1]$*, even when map* τ *is onto. We can see this on the example of the map*

$$\tau(x) = \begin{cases} 2x & \text{if } x \in [0, 1/4); \\ 2x - 1/2 & \text{if } x \in [1/4, 1/2); \\ 2x - 1 & \text{if } x \in [1/2, 1]. \end{cases}$$

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