A Random Map Description for Quantum Superposition

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Abstract

Chaotic maps are deterministic yet asymptotically in time behave in a statistical manner. In this note we present a chaotic dynamical model that is consistent with observable quantum mechanics and succeeds in presenting a physical process for superposition of wave functions, namely chaotic random maps. In place of the wavefuction we shall use real chaotic maps as the underlying mechanism for the observed probability density functions.

Let $\psi_i(x,t)$, $i=1,2$ be two eigenfunctions of a quantum mechanical particle system. We associate with each $\psi_i(x,t)$ a deterministic nonlinear point transformation $\tau_i(x)$ whose unique invariant probability density function is the observed density $\rho_i(x,t) = \psi_i^*(x,t)\psi_i(x,t)$. We consider the superposed wavefunction $\psi(x,t) = a\psi_1(x,t) + b\psi_2(x,t)$ and show that we can associate with $\psi(x,t)$, a random chaoitic map related to $\tau_1(x)$and $\tau_2(x)$, whose invariant probability density function $f_t(x)$ is equal to $\psi^*(x,t)\psi(x,t)$, where t denotes time. This description allows for a physical interpretation of quantum superposition. Numerical simulations of a two-slit experiment is done

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which show that the random map dynamics achieves the interference superposition pattern very accurately.

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1. **Introduction**

Let $\psi(x,t)$ be a wavefunction of a quantum particle system. The stochastic mechanics of Nelson [2] shows that for a normalized solution ψ of the time-dependent Schrödinger equation, there exists an associated diffusion process satisfying the stochastic differential equation

$$dx_t = a_t(x_t)dt + \sqrt{2\nu}dw_t, \tag{1}$$

where the diffusion coefficient $\nu = \hbar/2m$ ($\hbar$ is the Planck constant and m the mass of the particle), and the forward drift coefficient is

$$a_t(x) = \frac{\hbar}{m}\left[\mathrm{Re}\left[\frac{\nabla\psi}{\psi}\right] + \mathrm{Im}\left[\frac{\nabla\psi}{\psi}\right]\right] \tag{2}$$

and the associated probability density function is

$$\rho_t(x) = \psi^*(x,t)\psi(x,t). \tag{3}$$

We note, however, that the attempt to attribute meaning to a superposition of wavefunctions using stochastic differential equations fails [3, Section 4.8] as it amounts to merely defining a certain difussion to be the sum of the diffusion processes associated with the two wavefunctions, but there is no real identification of the superposed process with the individual ones. In the random map approach described in the sequel we will make a clear and reversible identification.

In Section 2 we present the notation and a review of our deterministic description of quantum mechanics [1] using chaotic maps. In Section 3, we review the motivation for our discrete time model for quantum mechanics. In Section 4, we use a result from [5] to associate a position dependent random map to the superposition of wavefunctions and identify the relationship with the individual wavefunctions. This identification is reversible so that given any random map that describes a superposed state, we can identify the wavefunctions which would produce the superposed state using conventional quantum mechanics. An example is worked in Section 5.

2. **Notation and Review**

Let R = (-∞, ∞) and let $T : \mathbb{R} \to \mathbb{R}$ possess a unique absolutely continuous invariant measure μ which has the probability density function (pdf) f, that is

$$\int_A f dx = \int_{T^{-1}A} f dx$$

for any measurable set $A \subset \mathbb{R}$. The Frobenius-Perron $P_T f$ operator acting on the space of integrable functions is defined by

$$\int_{T^{-1}A} f dx = \int_A P_T f dx.$$

The operator P_T transforms probability density functions into probability density functions under the transformation T, where T is assumed to be nonsingular.

Let $h : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Then $\tau = h \circ T \circ h^{-1}$ is a transformation from $\mathbb{R}$ into $\mathbb{R}$ which is said to be differentially conjugate to T and whose probability density function is given by

$$k = (f \circ h^{-1}) \cdot \mid (h^{-1})' \mid \tag{4}$$

Let the transformation T possess the probability density function f. Suppose we are given a probability density function g on $\mathbb{R}$, can we find a transformation τ, derived from T, such that g is the unique probability density function invariant under τ? The answer is yes. Using (4), we must find h^{-1} such that

$$(f \circ h^{-1}) \cdot (h^{-1})' = g, \tag{5}$$

where we have assumed, without loss of generality, that h^{-1} is an increasing function on $\mathbb{R}$. Now let

$$F(x) = \int_{-\infty}^{x} f(y) dy$$

be the distribution function associated with f. Then, from (5) and the change of variable formula, we have

$$F(h^{-1}(x)) = \int_{-\infty}^{x} g(y) dy.$$

Since F is a monotonically increasing function, it has a unique inverse and

$$h^{-1}(x) = F^{-1}\left(\int_{-\infty}^{x} g(y)dy\right) \tag{6}$$

Thus, we have found $h^{-1}(x)$ and hence $h(x)$ such that $\tau = h \circ T \circ h^{-1}$ has the unique probability density function $g(x)$. Summarizing, given any probability density function $g(x)$, we have shown the existence of a point transformation τ whose unique probability density function is $g(x)$.

3. Chaotic Dynamical System Model for Quantum Mechanics

We make two general assumptions for our model:

1) Time is discrete, as implied by string theory and quantum loop gravity. This, of course, implies that all continuous time theories such as quantum mechanics and general relativity are at best only good approximations to a discrete time reality.

2) Observables such as position and velocity are described by a probability density function, such as is the case in quantum mechanics.In classical mechanics and relativity theory, we can view the flows as point measures rather than points, which may be approximations to pdfs with very narrow support on R^3.

Quantum mechanics formalism stipulates that the square of the wavefunction, ψ, is the observable pdf, f. In our model of the underlying process for particle motion, we use a real chaotic map, τ, that generates f,via the Birkhoff Ergodic Theorem. We postulate that the iteration time is of the order of the Planck time, 10^{-44} seconds, while the observation time is of the order of 10^{-10} seconds or more. The gap in these times allows for many iterations on τ between observations and so, by the Birkhoff ErgodicTheorem, can reveal the observable pdf.

Now let $\{\tau_1, \tau_{2,\ldots\ldots,}\tau_K\}$ be a collection of 1-dimensional maps and define a random map to be a discrete-time dynamical system in which one of the maps is randomly selected and applied at each iteration with constant probability $p_k, .p_k > 0, \sum_{k=1}^{K} p_k = 1$,.A measure μ on $[0, 1]$ is called invariant under τ if

$$\mu(A) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1} A) \tag{7}$$

for each measurable set A. In [4] it is shown that the following sufficient condition is sufficient for the existence of an absolutely continuous invariant measure for such a random map:

$$\sum_{k=1}^{K} \frac{p_k}{|\tau_k'|} \leq \gamma < 1 \tag{8}$$

for some constant γ.

Although such dynamical systems have application in the study of fractals [6] they are not rich enough for our purposes because they do not generate a sufficiently large class of invariant densities. To enlarge the class of pdfs that are attainable from random maps we allow the probabilities of selecting the maps to be functions of position. The main result of [5] provides a sufficient condition for the existence of an absolutely continuous invariant measure for position dependent random maps. The pdf $f(x)$ of this measure is the solution of the equation:

$$\sum_{k=1}^{K} P_{\tau_k}(p_k(x)f(x)) = f(x) \tag{9}$$

where P_{τ_k} is the Frobenius-Perron operator associated with τ_k. If $\Gamma = \{\tau_1, \ldots, \tau_K\}$ is a set of maps, we denote by A_Γ the set of all attainable densities, i.e., the set of densities f which satisfy (9), for all possible choices of probability weight $\{p_1(x), \ldots, p_K(x)\}$. We now state the result of [5] that is needed in the sequel.

Proposition 1 *If the set of maps $\Gamma = \{\tau_1, \ldots, \tau_K\}$ contains the identity map, then the set A_Γ of attainable densities is equal to the set of all densities.*

Proof. The identity map preserves any density. ∎

We will use the following theorem from [5].

Theorem 2 *Let $\{\tau_1, \ldots, \tau_K\}$ be a collection of maps. Let f_k be an invariant density of τ_k, $k = 1, \ldots, K$. For any positive constants a_k, $k = 1, \ldots, K$, there exists a system of probability functions $p_1, \ldots, p_K$ such that the density $f = a_1 f_1 + \cdots + a_K f_K$ is invariant under the random map $T = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$. It is enough to set*

$$p_k = \frac{a_k f_k}{a_1 f_1 + \cdots + a_K f_K} \quad , \quad k = 1, 2, \ldots, K,$$

where we assume that $0/0 = 0$.

4. Superposition of Wavefunctions

Let ψ_1 and ψ_2 be two wavefunctions which are eigenstates of a quantum particle system. By the foregoing method, we know there exist point transformations $\tau_1$and τ_2 whose invariant pdf's are $\psi_1^*\psi_1$ and $\psi_2^*\psi_2$, respectively. We consider the wavefunction $\psi = a\psi_1 + b\psi_2$, which is a superposition of two eigenfunctions. Clearly $\psi^*\psi$ is a probability density function which, in general, is time dependent.

By Proposition 1, we know that $\psi^*\psi$ can be realized at any time t as a position dependent random map consisting of the three maps $T = \{\tau_1,\tau_2, Id; p_1(x), p_2(x), 1-p_1(x)-p_2(x)\}$. We now ask: can this procedure be reversed. That is, given pdf's $f_1(x), f_2(x), f_t(x)$, can we find wavefunctions $\psi_1$and ψ_2 such that $f_1 = \psi_1^*\psi_1$, $f_2 = \psi_2^*\psi_2$ and $f_t(x) = (\psi_1+\psi_2)^*(\psi_1+\psi_2)$. The answer is yes. Let $\psi_1 = \sqrt{f_1}e^{is_1t}$ and $\psi_2 = \sqrt{f_2}e^{is_2t}$. Then

$$(\psi_1 + \psi_2)^*(\psi_1 + \psi_2) = f_1 + f_2 + 2\sqrt{f_1 f_2}[\cos(s_1 - s_2)t] = f_t \qquad (10)$$

Since f_1, f_2 and f_t are known, $(s_1 - s_2)$ can be computed from (10). Thus, we have determined ψ_1 and ψ_2 up to a constant phase.

5. Example: Two-slit experiment

Let us consider the two-slit experiment, with slit size .01 and slit centers located at positions $x = -1$ and $x = 1$. We assume that the pdf's at the slits are Gaussian densities with variance .005 as shown in Figures 1a) and 1b), that is,

$$\begin{aligned} f_1(x) &= \frac{1}{10\sqrt{\pi}}\exp(-(x+1)^2/100)\ , \\ f_2(x) &= \frac{1}{10\sqrt{\pi}}\exp(-(x-1)^2/100)\ , \end{aligned} \qquad (11)$$

and their superposition is given by equation (24) of [7] which, for $t = 1$, becomes

$$\begin{aligned} f(x) = \frac{1}{2+\exp(-1/100-100)}\big[f_1(x) + f_2(x) \\ + \frac{1}{10\sqrt{\pi}}\cos(2x)\exp(-(x^2+1)/100)\big]\ . \end{aligned} \qquad (12)$$

The random map we now construct consists of three maps, τ_i, $i = 1, 2$, constructed as in Section 2 which have pdfs f_1 and f_2, respectively, and the identity map $\tau_3(x) = x$. We can view this random map as giving the particle a choice of moving to the left (under the influence of the left slit), moving to the right (under the influence of the right slit) or remaining in the same place, as reflected in the identity map τ_3. If the iteration time is some multiple of say the Planck time, then within an iteration time a particle can move left and right a number of times and still at the iteration time end up close to its original position.

It can be easily shown that if we set $a_1 = a_2 = 0.1$, $a_3 = 0.8$, then

$$f_3(x) = \frac{1}{a_3}\big[f(x) - f_1(x) - f_2(x)\big] \ ,$$

is a density and the superposition density can be written as

$$f = a_1 f_1 + a_2 f_2 + a_3 f_3.$$

They are shown in Figures 1 and 2. Figure 1 shows f_1 in part a) and f_2 in b). Figure 2 shows f in a) and f_3 in b).

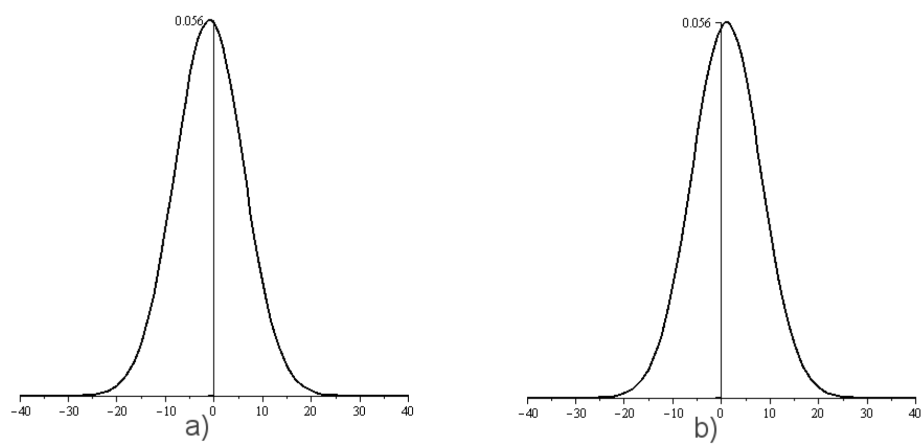

Figure 1: a) shows density f_1, b) shows density f_2.

According to Theorem 2 we construct the probabilities

$$p_i(x) = \frac{a_i f_i(x)}{f(x)} \ , \quad i = 1, 2, 3 \ .$$

They are shown in Figure 3.

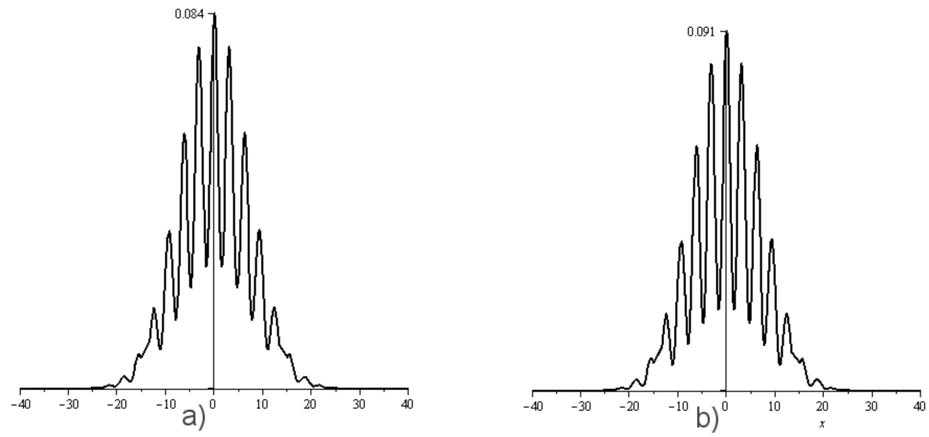

Figure 2: a) shows density f, b) shows density f_3 .

Using the "recipe" from Section 2 we construct maps τ_1 and τ_2 corresponding to the densities f_1 and f_2. We have

$$h_1^{-1}(x) = \int_{-\infty}^{x} f_1(t)dt = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\frac{1}{10}x + \frac{1}{10})$$

and

$$h_2^{-1}(x) = \int_{-\infty}^{x} f_2(t)dt = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\frac{1}{10}x - \frac{1}{10}) \ ,$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt \ .$$

We define $\tau_i = h_i \circ T \circ h_i^{-1}$, $i = 1, 2$, where $T(x) = 1 - 2|x - 1/2|$ is the tent map. We used $\tau_3(x) = x$ as the third map. Figure 4 shows the maps τ_1 (left) and τ_2 (right) in part a) and in part b) the results of numerical simulation of 2,000,000 iterations of random map $\{\tau_1, \tau_2, \tau_3; p_1, p_2, p_3\}$.

6. Observations:

1) In the foregoing model of the 2-slit experiment we are not compelled to say that the particle goes through both slits. Rather, we say that the particle is a particle and passes through one slit or the other. But between the slit screen and the detecting screen the particles motion is governed by the spacetime geometry determined by the physical structure of the experiment (size of 2 open slits, their separation and distance between slit screen and

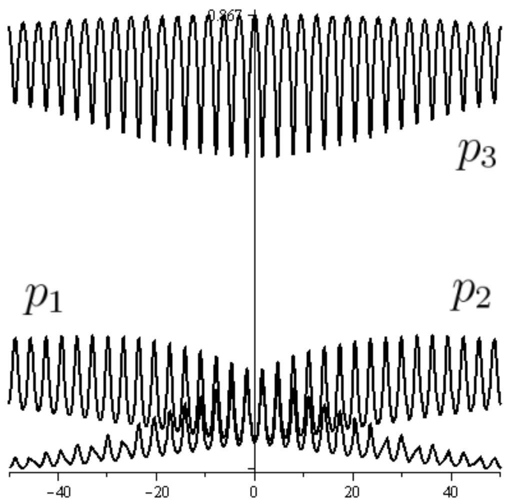

Figure 3: The position dependent probabilities p_1, p_2, p_3.

detecting screen) and which is described mathematically by the weighting probabilities for the 2 maps in the random map. From this perspective spacetime in the quantum setting is a complex structure which can only be described probabilistically.

2) The random transformation model for quantum mechanics lends itself to an interpretation of nonlocality since a jump from one transformation to another is a discontinuous effect that can propel a quantum particle across the universe in the time span of one iteration of the process.

3) Since the map τ_i is piecewise onto all of the real line, a few, iterations amounts to a very small duration of time, but the particle orbit may traverse a large part of R. Also, the switching from one map to another can cause the particle to be pushed far out. This process, iterating at the Planck time, may explain nonlocality since the particle may appear to be in two distant locations at once during the observation of time which is many times as large as the Planck time, and hence during such an observation the particle may have the time to travel back and forth between the two positions numerous times, at (finite) speeds far greater than that of light.

4) The choice of the maps τ_i may not be unique. However this would not change the foregoing theory since all we need is a dynamical mechanism that generates the desired pdfs. How fast or slow this is accomplished is not important since the physical process is assumed to be iterated at a very small

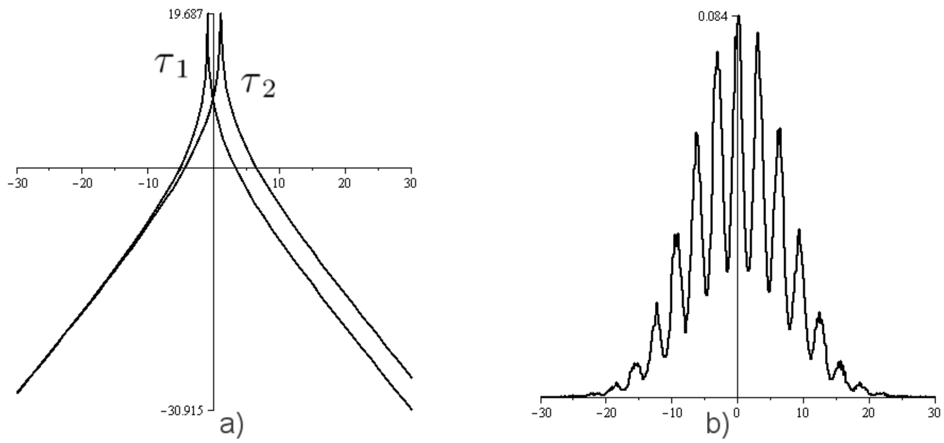

Figure 4: a) maps τ_1 and τ_2 b) results of numerical simulation.

time.

5) The maps, which are real observable transformations, have taken on the role of the complex wave function which on its own does not have a satisfactory physical interpretation.

References

[1] A. Boyarsky and P. Góra *A Description of Stochastic Systems Using Chaotic Maps*, J. Appl. Math. Stoch. Anal. **2004:2** (2004), 137–141.

[2] E. Nelson, Phys. Rev. **150**: 1079, (1966).

[3] M. Nagasawa, *Schrödinger Equation and Diffusion Theory*, Birkhaüser, New York 1993.

[4] S. Pelikan, *Invariant densities for random maps of the interval*, Trans. Amer. Math Soc. **281** (1984), 813–824.

[5] P. Góra, and A. Boyarsky, *Absolutely continuous invariant measures for random maps with position dependent probabilities*, J. Math. Anal. Appl. **278** (2003), 225–242.

[6] M. Barnsley, *Fractals Everywhere*, Academic Press, London, 1988.

[7] M. McClendon and H. Rabitz, *Numerical simulations in stochastic mechanics*, Physical Review A **37**, No. 9, (1998), 3479–3492.