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A Family of Piecewise Expanding Maps having Singular Measure as a limit of ACIM's

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Abstract. Keller [**9**] introduced families of W–shaped maps that can have a great variety of behavior. As a family approaches a limit W map, he observed behavior that was either described by a probabilty density function (pdf) or by a singular point measure. Based on this, Keller conjectured that instability of the absolutely continuous invariant measure (acim) can result only from the existence of small invariant neighbourhoods of the fixed critical point of the limit map. In this note we show that the conjecture is not true. We construct a very simple family of W maps with acim's supported on the whole interval, whose limiting dynamical behavior is captured by a singular measure. Key to the analysis is the use of a general formula for invariant densities of piecewise linear and expanding maps [**6**].

1. *Introduction*

Usually, the absolutely continuous invariant measure of a piecewise expanding map of an interval is stable under deterministic or even random perturbations. This means that if we consider a family of piecewise expanding maps τ_a, $a > 0$ with acims μ_a, converging to a piecewise expanding map τ_0 with acim μ_0, then under general assumptions μ_as converge to μ_0. One such assumption is that for some positive ε, $|\tau_a'| > 2 + \varepsilon$ for all $a \geq 0$. Actually, much more is proven in [**10**] and [**1**]. This stability property is often useful, for example, in establishing results about metastable systems [**14**].

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Keller [**9**] introduced the family of W-maps that are piecewise expanding and exhibit a wide variety of behaviour. This was done to understand whether in dimension one the expanding constant ensuring stability is really 2 rather than 1 as for zero-dimensional systems [**7**]. This regularity was later confirmed in [**8**] by showing that this constant for a piecewise expanding n-dimensional system with rectangular partition is $n + 1$.

Key to the complexity of Keller's families is the fact that, as the parameter approaches 0, say, the behavior near a folded critical point plays a crucial role. This critical point has slope 2 on one side and -2 on the other. Thus, the entire family is uniformly piecewise expanding and each member has a unique absolutely continuous invariant measure. However, the stability of probability density functions that one might expect in families of uniformly piecewise expanding maps does not occur. Keller provided an example of a family for which the limit of acims is a singular measure. This occurred because of the existence of diminishing invariant neighbourhoods of the critical point. Keller conjectured that this is the only mechanism which can cause such limiting behaviour.

In this paper we construct a family of simple W-maps which disproves Keller's conjecture. All our maps are piecewise expanding with slopes strictly greater than 2 in magnitude and are exact with their acim's supported on all of $[0, 1]$.

Standard bounded variation methods cannot be applied in this setting as the slopes of maps in our family are not uniformly bounded away from 2. The Lasota-Yorke BV inequality [**2**] cannot be invoked for families of maps whose slopes are not bounded away from 2 in magnitude. Other research related to the objectives of this note can be found in [**3**], [**13**] and [**11**]. However, none of the methods used in these papers apply to the family of maps we consider.

In a recent paper [**5**], a 3-parameter countable family of transitive maps was constructed which converges to the W-map for which the acim's approach any given convex combination of the point measure at $1/2$ and the W-map acim. All maps in this family are Markov, which makes the analysis easier.

In this note we shall utilize the main result of [**6**] which proves that the invariant pdf for any piecewise linear map which is eventually expanding has a convenient infinite series expansion. The estimates on the family of pdfs derived from this representation allows us to prove our main result, that the acim's of the family of W-maps approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map.

In Section 2 we introduce our family and state the main theorem, which is proved in Section 3. In Section 4, we show computational results for some pdfs of the W_a maps when a is small.

2. *Family of W_a maps and the main result*

We consider the family $\{W_a : 0 \le a\}$ of maps of $[0,1]$ onto itself defined by

$$W_a(x) = \begin{cases} 1-4x\ , & \text{for } 0 \le x < 1/4\ ; \\ (2+a)(x-1/4)\ , & \text{for } 1/4 \le x < 1/2\ ; \\ 1/2 + a/4 - (2+a)(x-1/2)\ , & \text{for } 1/2 \le x < 3/4\ ; \\ 4(x-3/4)\ , & \text{for } 3/4 \le x \le 1\ . \end{cases} \tag{1}$$

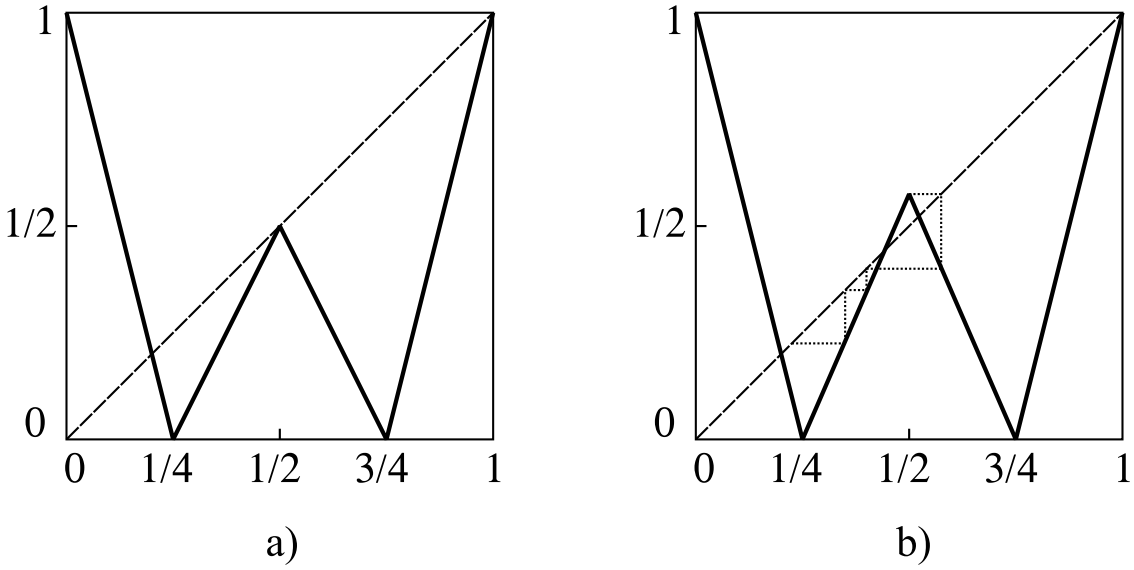

FIGURE 1. a) map W_0, b) map W_a, $a > 0$, with a few first points of the trajectory of $1/2$.

The map W_0 is Keller's W-map [**9**]. We consider only small $a > 0$ as we are interested in the limiting behaviour of the W_a's as $a \to 0$. Fig. 1 shows the graphs of W_a for $a = 0$ and $a > 0$. Every W_a is a piecewise linear, piecewise expanding map with minimal modulus of the slope equal to $2+a$. Every W_a has a unique acim μ_a supported on $[0,1]$ and is exact with respect to this measure. The transitivity of such maps is proven in [**5**], and the uniqueness of acim and exactness follow directly from the Li-Yorke paper [**12**].

Let h_a denote the normalized density of μ_a, $a \ge 0$. It is easy to check that for W_0, μ_0 has density

$$h_0 = \begin{cases} \frac{3}{2}\ , & \text{for } 0 \le x < 1/2\ ; \\ \frac{1}{2}\ , & \text{for } 1/2 \le x \le 1\ . \end{cases} \tag{2}$$

Our goal is to prove the following theorem:

THEOREM 2.1. *As $a \to 0$ the measures μ_a converge $*$-weakly to the measure*

$$\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}\ ,$$

where $\delta_{(\frac{1}{2})}$ is the Dirac measure at point $1/2$.

The proof relies on the general formula for invariant densities of piecewise linear maps [**6**] and direct calculations. The calculations depend on the parameter a, but we suppress it whenever there is no confusion.

3. *Proofs*

This section contains the proof of Theorem 2.1, divided into a number of steps.

3.1. *Formula for non-normalized invariant density of W_a*

We adapt the general formulas of [**6**] to our case and obtain the following formula for f_a:

LEMMA 3.1. *For small $a > 0$ there exists $A < -1$ such that*

$$f_a = 1 + 2A \left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right). \tag{3}$$

is a W_a-invariant non-normalized density.

Here,

$$\chi^s(t, y) = \begin{cases} \chi_{[0,y]} & \text{for } t > 0 \ ; \\ \chi_{[y,1]} & \text{for } t < 0 \ , \end{cases}$$

and $\beta(1/2, n)$ is the cumulative slope along the n steps of the trajectory of $1/2$ defined by:

$$\beta(1/2, 1) = 2 + a, \text{ and}$$

$$\beta(1/2, n) = (2 + a) \cdot W_a'(W_a(1/2)) \cdot W_a'(W_a^2(1/2)) \cdots W_a'(W_a^{n-1}(1/2)), \text{ for } n \geq 2.$$

The detailed justification of formula (3) is in Subsection 3.2.

For small positive a, the first image of $1/2$ is $W_a(1/2) = 1/2 + a/4$ and the next image lands just below the fixed point slightly less than $1/2$. The following forward images of $1/2$ form a decreasing sequence until they go below $1/4$. Let k be the first iterate j when $W_a^j(1/2)$ is less than $1/4$. That is, $k = \min\{j \geq 1 : W_a^j(1/2) \leq 1/4\}$. Then, the consecutive cumulative slopes of $1/2$, namely $\beta(1/2, j), 1 \leq j \leq k$, are

$$(2 + a), -(2 + a)^2, -(2 + a)^3, \ldots, -(2 + a)^k \ ,$$

and

$$f_a = 1 + 2A \left(\frac{\chi_{[0,W_a(1/2)]}}{(2 + a)} + \sum_{j=2}^{k} \frac{\chi_{[W_a^j(1/2),1]}}{(2 + a)^j} + \ldots \right). \tag{4}$$

3.2. *Justification of the formula for f_a*

Using the notation of [**6**], we have the following lemma:

LEMMA 3.2. *(a) N=4, K=2, L=0;*

(b) $\alpha = (1, 1/2 + a/4, 1/2 + a/4, 1)$, $\beta = (-4, 2 + a, -(2 + a), 4)$, $\gamma = (0, 0, 0, 0)$;

(c) The digits $A = (a_1, a_2, a_3, a_4)$, where $a_1 = -1, a_2 = 1/2 + a/4, a_3 = -3/2 - 3a/4, a_4 = 3$;

(d) There are two c_i's, which are $c_1 = (1/2, 2)$ and $c_2 = (1/2, 3)$, and $j(c_1) = 2$, $j(c_2) = 3$. Then, $W_u = \{c_1, c_2\}$,$W_l = \emptyset$, $U_l = \{c_2\}$,$U_r = \{c_1\}$.

(e) $\beta(c_1, 1) = 2 + a$ since $j(c_1) = 2$, then $\beta(c_1, 2) = -(2 + a)^2$ and $\beta(c_1, k) = -(2 + a)^k$ up to k defined in Subsection 3.1, $k = \min\{j \geq 1 : W_a^j(1/2) \leq 1/4\}$;

(f) $\beta(c_2, 1) = -(2 + a)$ since $j(c_2) = 3$, then $\beta(c_2, 2) = (2 + a)^2$ and $\beta(c_2, k) = (2 + a)^k$ up to the same k in part (e), $W_a^n(c_1) = W_a^n(c_2)$ for all n;

(g) Based on (f), we have the following for the matrix $S = S_{i,j}, i, j = 1, 2$:

For $c_1 \in U_r$

$$S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1, n) > 0))\delta(W_a^n(c_1) > 1/2) + \delta(\beta((c_1, n) < 0))\delta(W_a^n(c_1) < 1/2)}{|\beta(c_1, n)|},$$

$$S_{1,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1, n) > 0))\delta(W_a^n(c_1) > 1/2) + \delta(\beta((c_1, n) < 0))\delta(W_a^n(c_1) < 1/2)}{|\beta(c_1, n)|}.$$

For $c_2 \in U_l$

$$S_{2,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_a^n(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_a^n(c_2) < 1/2)}{|\beta(c_2, n)|},$$

$$S_{2,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_a^n(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_a^n(c_2) < 1/2)}{|\beta(c_2, n)|},$$

where $\delta($"condition"$)$ is equal to 1 if the "condition" holds and to 0 if it does not.

Remark: It follows from (e, f) of Lemma 3.2 that $S_{i,j}$ are equal for $i, j = 1, 2$. Let Id be the 2×2 identity matrix, $V = [1, 1]$. Then, for the solution, $D = [D_1, D_2]$, of the following system :

$$\left(-S^T + Id\right) D^T = V^T \ , \tag{5}$$

we have $D_1 = D_2$. Let us denote them by A.

Let I_1, I_2, I_3, I_4 be the partition of $I = [0, 1]$, where $I_1 = [0, 1/4), I_2 = (1/4, 1/2), I_3 = (1/2, 3/4)$ and $I_4 = (3/4, 1]$. Let $\beta_1 = -4, \beta_2 = 2+a, \beta_3 = -(2+a)$, and $\beta_4 = 4$.We define the following index:

$$j(x) = j \text{ for } x \in I_j, j = 1, 2, 3, 4,$$

and

$$j(c_1) = 2, j(c_2) = 3.$$

Already defined for Lemma 3.1 we have cumulative slopes for iterates of points:

$$\beta(x, 1) = \beta_{j(x)}, \quad \text{and } \beta(x, n) = \beta(x, n-1) \cdot \beta_{j(W_a^{n-1}(x))}, \qquad n \geq 2 \ ,$$

and

$$\chi^s(t, y) = \begin{cases} \chi_{[0,y]} & \text{for } \ t > 0 \ ; \\ \chi_{[y,1]} & \text{for } \ t < 0 \ . \end{cases}$$

Using Theorem 2 in [**6**] directly, we obtain Lemma 3.2. Now, we can prove Lemma 3.1:

Proof: First, by part (g) of Lemma 3.2, since the first and fourth branches of W_a have slope of modulus $4 > 2 + a$,

$$S_{i,j} \leq \sum_{n=1}^{\infty} \frac{1}{(2+a)^n} = \frac{1}{1+a} < 1 \ .$$

On the other hand, for small a

$$S_{i,j} \geq \frac{1}{2+a} + \frac{1}{(2+a)^2} > 1/2.$$

Now, the solution of the system (5) will be $D_1 = D_2 = \frac{1}{1-2S_{1,1}} < -1$. By Theorem 2 in [**6**], it follows from (d, e, f) of Lemma 3.2 that:

$$\begin{aligned} f_a &= 1 + D_1 \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1,n), W_a^n(c_1))}{|\beta(c_1,n)|} + D_2 \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2,n), W_a^n(c_2))}{|\beta(c_2,n)|} \\ &= 1 + A \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1,n), W_a^n(1/2))}{|\beta(c_1,n)|} + A \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2,n), W_a^n(1/2))}{|\beta(c_2,n)|} \\ &= 1 + 2A \left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2,n), W_a^n(1/2))}{|\beta(1/2,n)|} \right), \end{aligned}$$

which completes the proof. □

3.3. *Estimates on f_a* Recall that $k = \min\{j \geq 1 : W_a^j(1/2) \leq 1/4\}$. Clearly, $k > 1$. Furthermore, we have the following lemma:

LEMMA 3.3. *(I) for $2 \leq m \leq k$, $W_a^m(1/2) = -\frac{1}{4}a\frac{a(2+a)^{m-1}+1}{1+a} + \frac{1}{2}$;*
(II) $\lim_{a\to 0} ak = 0$;
(III) $\lim_{a\to 0} \frac{1}{a(2+a)^k} = 0$.
Moreover, if we let $k_1 = [\frac{2}{3}k]$ (integer part of $2k/3$), we have
(IV) $\lim_{a\to 0} \frac{1}{a(2+a)^{k_1}} = 0$;
(V) $\lim_{a\to 0} a^2(2+a)^{k_1} = 0$;
(VI) $\lim_{a\to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{2}$.

Proof: Suppose (I) is true. By the definition of k, $0 \leq W_a^{k-1}(1/2) \leq 1/4$. That is,

$$0 \leq -\frac{1}{4}a\frac{a(2+a)^{k-1}+1}{1+a} + \frac{1}{2} \leq \frac{1}{4}. \tag{6}$$

The first inequality of (6) implies

$$a^2(2+a)^{k-2} \leq 1, \tag{7}$$

and the second implies

$$a^2(2+a)^{k-1} \geq 1, \tag{8}$$

and so

$$2 \leq \liminf_{a\to 0} a^2(2+a)^k \leq \limsup_{a\to 0} a^2(2+a)^k \leq 4 \ .$$

Assertions (II), (III) and (V) follow immediately. To prove (IV) we have from (7)

$$a \le \frac{2+a}{(2+a)^{k/2}} \ ,$$

and so using also (8):

$$\begin{aligned}\frac{1}{a(2+a)^{k_1}} &= \frac{(2+a)^{k-k_1}}{a(2+a)^k} \le \frac{a(2+a)^{k-k_1}}{2+a} \\ &\le \frac{2+a}{(2+a)^{k/2}}\frac{1}{2+a}(2+a)^{k-k_1} = \frac{1}{(2+a)^{k_1-k/2}} \to 0 \ .\end{aligned}$$

Finally (VI) follows from (I) and (V).

Now, let us prove (I). For $m=2$, it is easy to check that $W_a^2(1/2) = \frac{2-a-a^2}{4}$ which is the same as $-\frac{1}{4}a\frac{a(2+a)+1}{1+a} + \frac{1}{2}$. Suppose (I) holds for $m = i < k$, that is

$$W_a^i(1/2) = -\frac{1}{4}a\frac{a(2+a)^{i-1}+1}{1+a} + \frac{1}{2}.$$

Then for $m = i+1$,

$$\begin{aligned}W_a^{i+1}(1/2) &= (2+a)(-\frac{1}{4}a\frac{a(2+a)^{i-1}+1}{1+a} + \frac{1}{2} - \frac{1}{4}) \\ &= -\frac{1}{4}a\frac{a(2+a)^i+2+a}{1+a} + \frac{1}{2} + \frac{a}{4} \\ &= -\frac{1}{4}a\frac{a(2+a)^i+1}{1+a} + \frac{1}{2} \ .\end{aligned}$$

This completes the proof. □

Let $\delta(\text{“condition”})$ be equal to 1 if the “condition” holds and to 0 if it does not. Lemma 3.2 implies that

$$S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\beta((1/2,n)>0))\delta(W_a^n(1/2)>1/2) + \delta(\beta((1/2,n)<0))\delta(W_a^n(1/2)<1/2)}{|\beta(1/2,n)|} \ .$$

Also, it was shown there that $A = \frac{1}{1-2S_{1,1}}$. Since

$$S_{1,1} \ge \sum_{n=1}^{k_1} \frac{1}{(2+a)^n} = \frac{\frac{1}{2+a} - \frac{1}{(2+a)^{k_1+1}}}{1-\frac{1}{2+a}},$$

and

$$S_{1,1} \le \sum_{n=1}^{\infty} \frac{1}{(2+a)^n} = \frac{1}{1+a},$$

we have

$$A_l = \frac{1+a}{a-1+\frac{2}{(2+a)^{k_1}}} \le A \le \frac{1+a}{a-1} = A_h \ . \tag{9}$$

Note that, for small a, both estimates A_l and A_h are smaller that -1.

Let us define,

$$g_l = \frac{\chi_{[0,W_a(1/2)]}}{(2+a)} + \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(2+a)^j} \ ,$$

and

$$g_h = g_l + \sum_{j=k_1+1}^{\infty} \frac{1}{(2+a)^j} = g_l + \frac{1}{(1+a)(2+a)^{k_1}} \ .$$

Let us further define $f_l = 1 + 2A_l g_h$ and $f_h = 1 + 2A_h g_l$. It follows from (4) and (9) that

$$f_l \le f_a \le f_h \ . \tag{10}$$

Let $\chi_1 = \chi_{[0,1/2+a/4]}$, $\chi_j = \chi_{[W_a^j(1/2),1/2+a/4]}, j = 2, 3, \ldots, k_1$, $\chi_c = \chi_{(1/2+a/4,1]}$. Now we will represent the functions f_l and f_h as combinations of functions χ_j, $j = 1, \ldots, k_1$ and χ_c. After some calculations, we obtain:

$$\begin{aligned} f_l &= \left(\frac{2}{2+a}A_l + 1\right)\chi_1 + 2A_l \sum_{n=2}^{k_1} \frac{\chi_n}{(2+a)^n} + \\ &\quad + \left(2A_l \frac{\frac{1}{2+a} - \frac{1}{(2+a)^{k_1}}}{1+a} + 1\right)\chi_c + 2A_l \frac{1}{(1+a)(2+a)^{k_1}}; \end{aligned}$$

$$f_h = \left[A_h \frac{2}{2+a} + 1\right]\chi_1 + 2A_h \sum_{n=2}^{k_1} \frac{\chi_n}{(2+a)^n} + \left(2\frac{\frac{1}{2+a} - \frac{1}{(2+a)^{k_1}}}{a-1} + 1\right)\chi_c.$$

Note that (9) implies that both A_l, A_h are smaller than $-(1+2a)$. Using this we can show that all coefficients in the representation of f_l and f_h are negative for sufficiently small a.

3.4. *Normalization* Let us define $J_1 = [0, W_a^{k_1}(1/2)]$, $J_2 = (W_a^{k_1}(1/2), 1/2+a/4]$, $J_3 = (1/2 + a/4, 1]$. We will calculate integrals of f_h over each of these intervals and use them to normalize f_h. We have

$$\begin{aligned} C_1 = \int_{J_1} f_h \ d\lambda &= \int_{J_1} \left[2\left(\frac{1+a}{a-1}\frac{1}{2+a}\right) + 1\right]\chi_1 \ d\lambda \\ &= \left[2\left(\frac{1+a}{a-1}\frac{1}{2+a}\right) + 1\right] W_a^{k_1}(\frac{1}{2}) = \frac{a^2+3a}{(a-1)(2+a)} W_a^{k_1}(\frac{1}{2}) \ . \end{aligned}$$

Using Lemma 3.3, we have $\lim\limits_{a\to 0} \frac{C_1}{a} = -\frac{3}{4}$. In the same way we can see that for any $0 < \alpha < 1/2$, we obtain

$$\lim_{a\to 0} \frac{1}{a} \int_0^{\alpha} f_h d\lambda = -\frac{3}{2}\alpha \ . \tag{11}$$

On the interval J_2, the integral of f_h is:

$$\begin{aligned} C_2 = \int_{J_2} f_h \, d\lambda &= \int_{J_2} \left[2\left(\frac{1+a}{a-1} \frac{1}{2+a} \right) + 1 \right] \chi_1 \, d\lambda + 2\frac{1+a}{a-1} \sum_{j=2}^{k_1} \int_{J_2} \frac{\chi_j}{(2+a)^j} \, d\lambda \\ &= \frac{a^2+3a}{(a-1)(2+a)} \left(\frac{1}{2} + \frac{a}{4} - W_a^{k_1}(\frac{1}{2}) \right) \\ &\quad + 2\frac{1+a}{a-1} \left[\frac{(k_1-1)a^2}{4(2+a)(1+a)} + \frac{a}{4(1+a)} \frac{1 - \frac{1}{(2+a)^{k_1-1}}}{1+a} \right] . \end{aligned}$$

Using Lemma 3.3, we have

$$\lim_{a \to 0} \frac{C_2}{a} = -\frac{1}{2}. \tag{12}$$

On the interval J_3, the integral of f_h is:

$$\begin{aligned} C_3 = \int_{J_3} f_h \, d\lambda &= \int_{J_3} \left(2\frac{\frac{1}{2+a} - \frac{1}{(2+a)^{k_1}}}{a-1} + 1 \right) \chi_c \, d\lambda \\ &= \left(2\frac{\frac{1}{2+a} - \frac{1}{(2+a)^{k_1}}}{a-1} + 1 \right) (\frac{1}{2} - \frac{a}{4}) . \end{aligned}$$

Using Lemma 3.3, we have

$$\lim_{a \to 0} \frac{C_3}{a} = -\frac{1}{4} .$$

In the same way we can see that for any $0 < \alpha < 1/2$, we obtain

$$\lim_{a \to 0} \frac{1}{a} \int_{1/2+\alpha}^{1} f_h d\lambda = -\frac{1}{2} \left(\frac{1}{2} - \alpha \right) . \tag{13}$$

If we define $B = C_1 + C_2 + C_3$, then $\frac{f_h}{B}$ is a normalized density. We see that

$$\lim_{a \to 0} \frac{B}{a} = -\frac{3}{2} .$$

3.5. *Conclusion of the proof* Now, we will use our foregoing calculations to show that the normalized measures $(f_h/B) \cdot \lambda$ converge $*$-weakly to the measure $\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}$, as $a \to 0$.

For any interval $[0, \alpha]$, $0 < \alpha < 1/2$ as $a \to 0$, formula (11) implies

$$\lim_{a \to 0} \int_0^{\alpha} \frac{f_h}{B} d\lambda = \frac{-\frac{3}{2}\alpha}{-\frac{3}{2}} = \alpha . \tag{14}$$

For J_2, which converges to the point $1/2$, formula (12) implies

$$\lim_{a \to 0} \int_{J_2} \frac{f_h}{B} d\lambda = \frac{-\frac{1}{2}}{-\frac{3}{2}} = \frac{1}{3} . \tag{15}$$

For any interval $[1/2 + \alpha, 1]$, $0 < \alpha < 1/2$, formula (13) implies

$$\lim_{a \to 0} \int_{1/2+\alpha}^{1} \frac{f_h}{B} d\lambda = \frac{-\frac{1}{2}\left(\frac{1}{2} - \alpha\right)}{-\frac{3}{2}} = \frac{1}{3} \left(\frac{1}{2} - \alpha \right) . \tag{16}$$

Formulas (14), (15) and (16) together show that measures $(f_h/B)\cdot\lambda$ converge $*$-weakly to the sum of the measure with density $\chi_{[0,1/2]}+\frac{1}{3}\chi_{[1/2,1]}$ and $\frac{1}{3}$ of a unit point mass at $1/2$, i.e., to the measure $\frac{2}{3}\mu_0+\frac{1}{3}\delta_{(\frac{1}{2})}$.

Now, we will show the same for the normalized measure defined using f_l. To this end, let us note that

$$\begin{aligned} f_h-f_l &= 2A_hg_l-2A_lg_h=2(A_h-A_l)g_l-2A_l\frac{1}{(1+a)(2+a)^{k_1}} \\ &= 2\frac{1+a}{a-1}\frac{-2/(2+a)^{k_1}}{a-1+2/(2+a)^{k_1}}g_l-2A_l\frac{1}{(1+a)(2+a)^{k_1}}\ , \end{aligned}$$

where $|g_l|\le 1$ and $\lim_{a\to 0}A_l=-1$. Using Lemma 3.3 once again, we can show that, for any subinterval $J\subset[0,1]$, we have

$$\lim_{a\to 0}\frac{1}{a}\int_J(f_h-f_l)d\lambda=0\ .$$

For $J=[0,1]$ this means that the normalizations of f_l and f_h are asymptotically the same. Thus, the limit for a general J implies that the $*$-weak limit of normalized measures defined using f_l is the same as for those defined using f_h. Together with inequality (10) this proves Theorem 2.1.

4. *Computational Results*

We present in Fig. 2 graphs of W_a normalized invariant densities for a): $a=0.1$, b): $a=0.05$ and c): $a=0.01$. They were obtained using Maple 13. Note that the vertical scales of the graphs are very different.

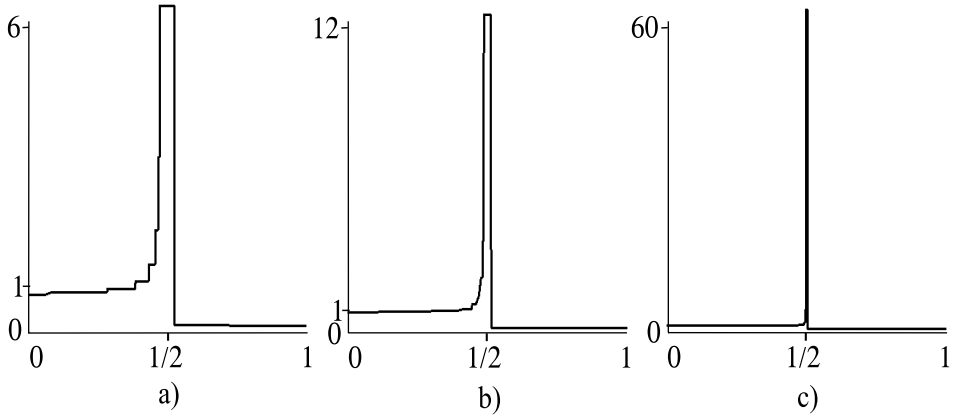

FIGURE 2. W_a-invariant pdf's for a): $a=0.1$, b): $a=0.05$ and c): $a=0.01$.

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