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Invariant densities for piecewise linear and increasing maps of constant slope

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Abstract. We find an explicit formula for the invariant density h of piecewise linear, piecewise increasing map τ of an interval $[0,1]$ with constant slope $\beta > 1$, at least for β large enough. The construction involves matrix $\mathbf{S}$ which is defined in a way somewhat similar to the defining of kneading matrix of a continuous piecewise monotonic map. We prove that if $\frac{1}{\beta}$ is not an eigenvalue of $\mathbf{S}$, then dynamical system $(\tau, h \cdot m)$ is ergodic.

1. *Introduction*

In this paper we continue the investigations of invariant densities (with respect to Lebesgue measure m) for piecewise linear maps with a constant slope $\beta > 1$. The first results about the classical β-maps were obtained by Rényi [15], Parry [12] and Gelfond [5]. Later, Parry generalized [13] them further. These maps have all the branches increasing.

The maps with both increasing and decreasing branches were investigated in [6].

Both just mentioned classes of maps allowed the shorter (i.e., not onto) branch only as the last or/and the first branch of the map. In this paper we consider piecewise linear maps with increasing branches and allow middle branches to be not onto as well. We consider four increasingly general classes of maps. First, we assume that images of shorter branches touch 0 (Sections 2–5). Such maps are related to the so called "greedy" expansions with deleted digits. The notion was introduces by Pedicini [14]. We recommend [4] for further information and references. Next, we investigate a similar class of maps with images of shorter branches touching 1 (Section 6) . These are related to so called "lazy" expansions with deleted digits [4]. Maps of the next class have shorter branches of both kinds (Section 7). Finally, in Section 8 we consider the general case of piecewise linear, piecewise increasing maps of constant slope $\beta > 1$, with images of shorter branches touching 0, or touching 1, or hanging in between.

The construction of τ-invariant density h involves a matrix $\mathbf{S}$ defined in a way somewhat similar to defining of kneading matrix of a continuous piecewise monotonic map [1, 10]. We proved that if $\frac{1}{\beta}$ $(= e^{-\text{entropy}})$ is not an eigenvalue of $\mathbf{S}$, then dynamical system $(\tau, h\cdot m)$ is ergodic. We conjecture that the inverse of this statement also holds. Note that for our class of maps ergodicity implies topological transitivity.

If τ has all branches onto, then Lebesgue measure is τ-invariant. Therefore, we consider only maps with at least one shorter branch. Since $\beta > 1$, our τ always admits an absolutely continuous invariant measure. We will denote it by μ.

We are mainly interested in absolutely continuous τ-invariant measure. The general theory of such measures for piecewise expanding maps of an interval is well developed and we often refer to its results. The classical papers are [8] and [9] among many others. There is a number of books on the subject, see, e. g., [2] or [7].

While working on this project the author used extensively the computer program Maple 11. The programs with examples and illustrations, as well as their pdf printouts, are available at http://www.mathstat.concordia.ca/faculty/pgora/deleted .

2. *Maps related to the greedy expansion with deleted digits*

In this section we describe maps related to so called greedy expansion with deleted digits [4, 14]. Throughout the paper δ(condition) will denote 1 when the condition is satisfied and 0 otherwise.

Let τ be a piecewise linear, piecewise increasing map interval $[0,1]$ onto itself, such that the image of each branch touches 0 and with constant slope $\beta > 1$. Let $N \geq 2$ be the number of branches of τ and $K < N$ the number of shorter branches, i.e., not onto. Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_K$ be the heights of the shorter branches and let $1 \leq k_1, k_2, \ldots, k_K \leq N$ denote the numbers of these branches correspondingly. We do not assume any order of k_j's. Then, we have

$$\beta = N - K + \alpha_1 + \alpha_2 + \cdots + \alpha_K \ .$$

The endpoints of the maximal intervals of monotonicity of τ are $0 = b_1 < b_2 < \cdots < b_N < b_{N+1} = 1$ and

$$b_j = \frac{1}{\beta}\left(j - 1 - \sum_{i=1}^{K} \delta(j > k_i)(1-\alpha_{k_i}) \right) \ , \quad j = 1, \ldots, N \ .$$

The map τ is defined on the partition $\mathcal{P}_\tau = \{I_1, I_2, \ldots, I_{N-1}, I_N\}$, of the interval $[0,1]$, where

$$\begin{aligned} I_1 &= [0, b_2] \ ; \\ I_j &= (b_j, b_{j+1}] \quad \text{for} \quad 2 \leq j \leq N-1 \ ; \\ I_N &= (b_N, 1] \ . \end{aligned} \tag{1}$$

The points

$$c_i = b_{k_i+1}, \quad i = 1, \ldots, K \ ,$$

the right hand side endpoints of the domains of the shorter branches of τ, play special role in further study. Note,

$$\tau(c_i) = \alpha_i \ .$$

We define the set of "digits" $A = \{a_1, a_2, \dots, a_N\}$, where

$$a_j = b_j \cdot \beta \ , \quad j = 1, 2, \dots, N \ .$$

Note,

$$M_a = \max_{1 \le j \le N-1} (a_{j+1} - a_j) = 1 \qquad \text{and} \qquad \beta \le 1 + \frac{a_N}{M_a} \ , \tag{2}$$

as $\beta = a_N + 1$ if $k_i < N$ for $i = 1, 2, \dots, K$, or $\beta = a_N + \alpha_i$ if $k_i = N$.

Map τ (occasionally denoted also by τ_A) is defined as

$$\tau(x) = \beta \cdot x - a_j, \qquad \text{for} \qquad x \in I_j \ , \ j = 1, 2, \dots, N \ .$$

For any $x \in [0, 1]$ we define its "index" $j(x)$ and its "digit" $a(x)$:

$$j(x) = j \qquad \text{for} \quad x \in I_j \ , \ j = 1, 2, \dots, N \ ,$$

and

$$a(x) = a_{j(x)}.$$

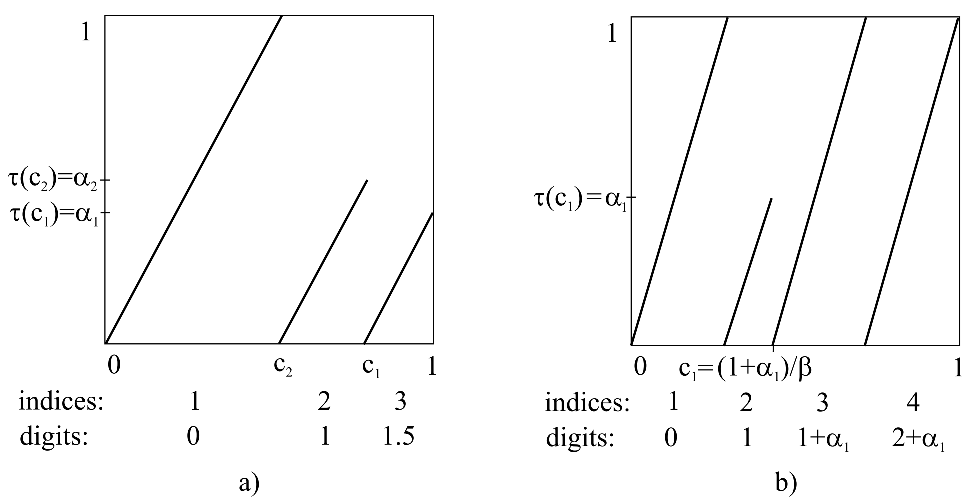

FIGURE 1. Graphs of maps of a) Example 1 and b) Example 2.

The following proposition describes the "greedy expansion with deleted digits". It is called like this since if we allow different scaling of the set $\{a_1, \dots, a_N\}$, then the numbers can be represented using non-consecutive digits. For example $a_1 = 0, a_2 = 2, a_3 = 3$ gives expansion of numbers in $[0, 2]$ using digits $0, 2, 3$. Proposition 1 is a special case of a theorem proved in [14].

Proposition 1. *For any $x \in [0,1]$ we have*

$$x = \sum_{n=0}^{\infty} \frac{a(\tau^n(x))}{\beta^{n+1}} \ .$$

Moreover,

$$\tau^k(x) = \beta^k \cdot \sum_{n=k}^{\infty} \frac{a(\tau^n(x))}{\beta^{n+1}} \ ,$$

for any $k \geq 0$.

Proof: We have $\tau(x) = \beta x - a(x)$ or

$$x = \frac{a(x)}{\beta} + \frac{\tau(x)}{\beta} \ .$$

Using this equality inductively n-times we obtain

$$x = \frac{a(x)}{\beta} + \frac{a(\tau(x))}{\beta^2} + \cdots + \frac{a(\tau^{n-1}(x))}{\beta^n} + \frac{\tau^n(x)}{\beta^n} \ ,$$

which proves both statements. □

We will call the representation defined in Proposition 1 the τ-expansion of x. Under our assumptions, the only number with finite τ-expansion is $x = 0$. It holds since $\tau(x) > 0$ for $x > 0$ and the only fixed point of τ in I_1 is 0. Thus, all other numbers have infinite τ-expansions. When $\tau(1) = 1$ and such an expansion, starting from some place $M+1$ (assuming $a_{j_M} < a_N$), contains only a_N's, then we can write

$$x = \sum_{n=0}^{M} \frac{a_{j_n}}{\beta^{n+1}} + \sum_{n=M+1}^{\infty} \frac{a_N}{\beta^{n+1}} = \sum_{n=0}^{M} \frac{a_{j_n}}{\beta^{n+1}} + \frac{a_N}{\beta^{M+2}} \frac{\beta}{\beta-1} = \sum_{n=0}^{M-1} \frac{a_{j_n}}{\beta^{n+1}} + \frac{a_{j_M}+1}{\beta^{M+1}} \ ,$$

and consider it a finite expansion.

3. *Invariant density*

In this section we find an invariant density of τ in a special case of $K = 1$. The case of larger K is considered in the next sections. We denote Lebesgue measure on $[0,1]$ by m.

An integrable nonnegative function h is a density of an m-absolutely continuous τ-invariant measure if and only if it satisfies Perron-Frobenius equation:

$$h(x) = \sum_{y:\tau(y)=x} h(y)/|\tau'(y)| = (P_\tau(h))(x),$$

for almost all $x \in [0,1]$. Operator P_τ is called Perron-Frobenius operator [2]. The preimages of x are $x(j) = (x + a_j)/\beta$, $j = 1, \ldots, N$, with the remark that the preimage $x(k_1)$ corresponding to the shorter branch of τ exists only for $x \in [0, \alpha_1]$. In our case the Perron-Frobenius equation becomes

$$\beta \cdot h(x) = \sum_j h(x(j)). \tag{3}$$

THEOREM 2. *Let $\tau = \tau_A$ satisfy the assumption of Section 2 and $K = 1$. If the first branch is onto, then the non-normalized τ-invariant density h is given by the formula*

$$h(x) = \frac{1}{\beta} + D_1 \cdot \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]}(x) \frac{1}{\beta^{n+1}} \ ,$$

where

$$D_1 = \frac{1}{1 - \beta S_{1,1}} \ ,$$

and $S_{1,1}$ is defined in (4). If the first branch is not onto, the support of τ-invariant absolutely continuous measure is the interval $[0, \alpha_1]$. The restricted map $\tau_{|_{[0,\alpha_1]}}$ is a classical β-map. The formula for the density applies after necessary changes.

The proof of Theorem 1 follows the method of Parry [12].

Proof: We want to calculate $\sum_j h(x(j))$ for any $x \in [0, 1]$. Recall that $\delta(\text{condition})$ is 1 if the condition is satisfied and 0 otherwise. We have

$$\sum_j \delta(x(j) \le \tau^n(c_1)) = j(\tau^n(c_1)) - 1 + \delta(x \le \tau^{n+1}(c_1)) \\ - \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_1) > c_1) \ .$$

We obtain:

$$\sum_j h(x(j)) = \frac{1}{\beta} [N - 1 + \delta(x \le \tau(c_1))] \\ + D_1 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \left[j(\tau^n(c_1)) - 1 + \delta(x \le \tau^{n+1}(c_1)) - \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_1) > c_1) \right] \\ = \frac{1}{\beta} [N - 1 + \delta(x \le \tau(c_1))] \\ + D_1 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} [j(\tau^n(c_1)) - 1 - \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_1) > c_1)] \\ + D_1 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \delta(x \le \tau^{n+1}(c_1)) + D_1 \frac{\delta(x \le \tau(c_1))}{\beta} + 1 - D_1 \frac{\delta(x \le \tau(c_1))}{\beta} - 1 \ .$$

Since

$$D_1 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \delta(x \le \tau^{n+1}(c_1)) + D_1 \frac{\delta(x \le \tau(c_1))}{\beta} + 1 = \beta h(x) \ ,$$

we only need to find a constant D_1 such that

$$0 = \frac{1}{\beta} [N - 1 + \delta(x \le \tau(c_1))] - D_1 \frac{\delta(x \le \tau(c_1))}{\beta} - 1 \\ + D_1 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} [j(\tau^n(c_1)) - 1 - \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_1) > c_1)] \ .$$

Let us define

$$S_1 = \sum_{n=1}^{\infty} \frac{j(\tau^n(c_1)) - 1}{\beta^{n+1}} \quad \text{and} \quad S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\tau^n(c_1) > c_1)}{\beta^{n+1}} \ . \tag{4}$$

For $x \le \tau(c_1)$ we obtain

$$0 = \frac{N}{\beta} - \frac{D_1}{\beta} - 1 + D_1 S_1 \ .$$

For $x > \tau(c)$ we obtain

$$0 = \frac{N-1}{\beta} - 1 + D_1 S_1 - D_1 S_{1,1} \ .$$

We need to solve the system of equations

$$\begin{aligned} D_1\left(S_1 - \frac{1}{\beta}\right) &= 1 - \frac{N}{\beta} \ ; \\ D_1\left(S_1 - S_{1,1}\right) &= 1 - \frac{N-1}{\beta} \ , \end{aligned}$$

or an equivalent system

$$\begin{aligned} D_1\left(S_1 - \frac{1}{\beta}\right) &= 1 - \frac{N}{\beta} \ ; \\ D_1\left(-S_{1,1} + \frac{1}{\beta}\right) &= \frac{1}{\beta} \ . \end{aligned}$$

Since

$$\begin{aligned} S_1 - S_{1,1}(1-\alpha_1) &= \sum_{n=1}^{\infty} \frac{j(\tau^n(c_1)) - 1 - \delta(\tau^n(c_1) > c_1)(1-\alpha_1)}{\beta^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{a(\tau^n(c_1))}{\beta^{n+1}} = \frac{\tau(c_1)}{\beta} = \frac{\alpha_1}{\beta} \ . \end{aligned} \tag{5}$$

and $\beta - N + 1 = \alpha_1$ adding the first equation to the second multiplied by $(1-\alpha_1)$ we obtain $0 = 0$. Thus, the equations are dependent and D_1 can be calculated from any of them. We will use the second one. We have

$$S_{1,1} = \sum_{n=1}^{\infty} \frac{\delta(\tau^n(c_1) > c_1)}{\beta^{n+1}} \le \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} = \frac{1}{\beta(\beta-1)} \ .$$

If $\beta > 2$, we get $S_{1,1} < \frac{1}{\beta}$ and $D_1 = 1/(1 - \beta S_{1,1})$.

If $\beta \le 2$, then τ has two branches (as $K = 1$) and we consider two cases:

a) $k_1 = 2$: Then, $c_1 = 1$ and $\delta(\tau^n(c_1) > c_1) = 0$ for all $n \ge 1$. Thus, $S_{1,1} = 0$ and $D_1 = 1$. We obtained classical Parry's formula [12].

b) $k_1 = 1$: The τ-invariant absolutely continuous measure is supported on $[0, \alpha_1]$. τ restricted to this interval is a map from case a). □

Example 1: Let $N = 3$, $K = 2$ and $\alpha_1 = 0.4$, $k_1 = 3$, $\alpha_2 = 0.5$, $k_2 = 2$. Then, $\beta = 1.9$. The digits are $A = \{0, 1, 1.5\}$. The graph of map τ is shown in Figure 1 a). Using Maple 11 we calculated $D_1 = 1.00$ and $D_2 \simeq 1.48988$. The normalizing constant of the density is $\simeq 1.32347$. The graph of τ is shown in Figure 1 a) and its normalized density h in Figure 2 a).

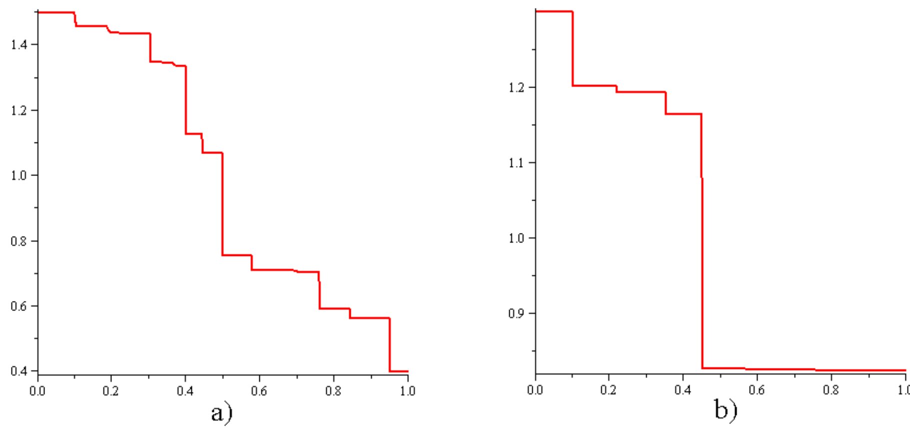

FIGURE 2. Invariant densities for maps of Examples 1 and 2.

Example 2: Let $N = 4$, $K = 1$ and $\alpha_1 = 0.45$, $k_1 = 2$. The digits are $A = \{0, 1, 1.45, 2.45\}$, $\beta = 3.45$. The graph of map τ is shown in Figure 1 b). Using Maple 11 we calculated and $D \simeq 1.41271$. The normalizing constant of the density is $\simeq 0.35169$. The graph of τ is shown in Figure 1 b) and its normalized density h in Figure 2 b).

The following proposition describes the ergodic properties of τ.

PROPOSITION 3. *Let us consider a dynamical system $\{\tau, h \cdot m\}$, where τ is a map of Theorem 1 and h its invariant density. Then, τ is exact on the support of h.*

Proof: It follows from the general theory, e.g. [2, Chapter 8], that the support of h consists of a finite number of intervals. To show exactness it is enough to prove that iterates of some interval in the support grow to cover the whole $[0, 1]$. It is also known that this support contains a neighborhood J of some inner endpoint of the partition $\mathcal{P}_\tau$. Then, the image $\tau(J)$ covers a neighborhood of 0. If the first branch is onto, then the consecutive iterates grow to cover the whole $[0, 1]$. If the first branch is not onto, we consider two cases: If $\beta > 2$, then the longest connected component of iterates of J grows until it covers two consecutive inner endpoints of the partition $\mathcal{P}_\tau$. Since the first branch is the only non-onto one, the next image covers $[0, 1]$. If $\beta \leq 2$, we are in situation of case b) of the previous proof. τ restricted to $[0, \alpha_1]$ has the first branch onto. $\square$

4. *Maps with two or more shorter branches*

In this section we generalize the result of Section 3 to maps related to greedy expansion with deleted digits which have more than one shorter (not onto) branches.

We will use the following fragment of Perron-Frobenius theorem for non-negative matrices [11].

THEOREM A. *If* $\mathbf{S} = (S_{i,j})_{1\le i,j\le M}$ *is a matrix with non-negative entries, then all eigenvalues* λ *of* $\mathbf{S}$ *satisfy*

$$|\lambda| \le \max_{1\le i\le M} \sum_{j=1}^{M} S_{i,j} \ . \tag{6}$$

THEOREM 4. *Let* $\tau = \tau_A$ *will be the map defined in Section 2 and* $K > 1$*. Let*

$$h(x) = \frac{1}{\beta} + \sum_{i=1}^{K} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta^{n+1}} \ , \tag{7}$$

where constants D_i*,* $i = 1, \dots, K$*, satisfy the system (11). If the system (11) is solvable, then* h *is* τ*-invariant and the dynamical system* $\{\tau, h\cdot m\}$ *is exact. In particular, this holds if* $\beta > K + 1$*. If the last branch is shorter, then condition* $\beta > K$ *is sufficient.*

Remark 1: If the system (11) is solvable, then it is uniquely solvable. The existence of a solution implies the existence of invariant density h with full support and exactness. Then, the invariant density h is unique up to a multiplicative constant. Existence of another solution would create a different invariant density, which is not a multiple of h because of $\frac{1}{\beta}$ summand and since $\tau(c_i) = \alpha_i < 1$.

We will prove Theorem 4 in a special case $K = 2$ first and then we will present the general proof. We will discuss the examples with $\beta \le K + 1$ afterwards.

Proof of the case $K = 2$*:* We have to show that the invariant density is of the form

$$h(x) = \frac{1}{\beta} + D_1 \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]} \frac{1}{\beta^{n+1}} + D_2 \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_2)]} \frac{1}{\beta^{n+1}} \ ,$$

where constants D_1, D_2 can be found from the system (9). Again we need to calculate $\sum_j h(x(j))$ for any $x \in [0,1]$. For $i = 1, 2$ have

$$\sum_j \delta(x(j) \le \tau^n(c_i)) = j(\tau^n(c_i)) - 1 + \delta(x \le \tau^{n+1}(c_i))$$
$$- \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_i) > c_1) - \delta(x > \tau(c_2)) \cdot \delta(\tau^n(c_i) > c_2) \ .$$

We obtain:

$$
\begin{aligned}
&\sum_j h(x(j)) = \frac{1}{\beta}\big[N - \delta(x > \tau(c_1)) - \delta(x > \tau(c_2))\big] \\
&+ D_1 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}}\big[j(\tau^n(c_1)) - 1 + \delta(x \le \tau^{n+1}(c_1)) - \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_1) > c_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \delta(x > \tau(c_2)) \cdot \delta(\tau^n(c_1) > c_2)\big] \\
&+ D_2 \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}}\big[j(\tau^n(c_2)) - 1 + \delta(x \le \tau^{n+1}(c_2)) - \delta(x > \tau(c_1)) \cdot \delta(\tau^n(c_2) > c_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \delta(x > \tau(c_2)) \cdot \delta(\tau^n(c_2) > c_2)\big] \\
&= \beta \cdot h(x) \\
&+ \frac{1}{\beta}\big[N - \delta(x > \tau(c_1)) - \delta(x > \tau(c_2))\big] \\
&+ D_1 S_1 - D_1 \delta(x > \tau(c_1)) S_{1,1} - D_1 \delta(x > \tau(c_2)) S_{1,2} - \frac{D_1}{\beta}\delta(x \le \tau(c_1)) \\
&+ D_2 S_2 - D_2 \delta(x > \tau(c_1)) S_{2,1} - D_2 \delta(x > \tau(c_2)) S_{2,2} - \frac{D_2}{\beta}\delta(x \le \tau(c_2)) - 1 \ ,
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
S_i &= \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}}\big(j(\tau^n(c_i)) - 1\big) \quad \text{and} \\
S_{i,j} &= \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \delta(\tau^n(c_i) > c_j) \ , \quad i, j = 1, 2 \ .
\end{aligned}
$$

We need to find the constants D_1, D_2 such that the last three lines of (8) sum up to 0. Substituting $x < \tau(c_1)$, $\tau(c_1) \le x < \tau(c_2)$ and $x > \tau(c_2)$ we obtain system of equations:

$$
\begin{aligned}
&D_1\left(S_1 - \frac{1}{\beta}\right) + D_2\left(S_2 - \frac{1}{\beta}\right) = 1 - \frac{N}{\beta} \ ; \\
&D_1\left(S_1 - S_{1,1}\right) + D_2\left(S_2 - S_{2,1} - \frac{1}{\beta}\right) = 1 - \frac{N-1}{\beta} \ ; \\
&D_1\left(S_1 - S_{1,1} - S_{1,2}\right) + D_2\left(S_2 - S_{2,1} - S_{2,2}\right) = 1 - \frac{N-2}{\beta} \ .
\end{aligned}
$$

Subtracting the second equation from the third and the first from the second we obtain a nicer equivalent system:

$$
\begin{aligned}
D_1\left(S_1 - \frac{1}{\beta}\right) &+ D_2\left(S_2 - \frac{1}{\beta}\right) &&= 1 - \frac{N}{\beta} \ ; \\
D_1\left(-S_{1,1} + \frac{1}{\beta}\right) &+ D_2\left(-S_{2,1}\right) &&= \frac{1}{\beta} \ ; \\
D_1\left(-S_{1,2}\right) &+ D_2\left(-S_{2,2} + \frac{1}{\beta}\right) &&= \frac{1}{\beta} \ .
\end{aligned}
\tag{9}
$$

The rank of the system is 2. We can show this multiplying the second equation by $(1-\alpha 1)$, the third equation by $1-\alpha_2$ and summing up all three equations. The sum of coefficients in the first column is

$$S_1 - S_{1,1}(1-\alpha_1) - S_{1,2}(1-\alpha_2) - \frac{\alpha_1}{\beta} = \sum_{n=1}^{\infty} \frac{a(\tau^n(c_1))}{\beta^{n+1}} - \frac{\alpha_1}{\beta} = \frac{\tau(c_1)}{\beta} - \frac{\alpha_1}{\beta} = 0 \ ,$$

and similarly the sum of coefficients in the first column is

$$S_2 - S_{2,1}(1-\alpha_1) - S_{2,2}(1-\alpha_2) - \frac{\alpha_2}{\beta} = \sum_{n=1}^{\infty} \frac{a(\tau^n(c_2))}{\beta^{n+1}} - \frac{\alpha_2}{\beta} = \frac{\tau(c_2)}{\beta} - \frac{\alpha_2}{\beta} = 0 \ .$$

The third column sums up to

$$1 - \frac{N}{\beta} + \frac{1-\alpha_1}{\beta} + \frac{1-\alpha_2}{\beta} = 1 - \frac{N-2+\alpha_1+\alpha_2}{\beta} = 0 \ ,$$

since $\beta = N-2+\alpha_1+\alpha_2$. Let us define matrix

$$\mathbf{S} = \begin{pmatrix} S_{1,1} & S_{2,1} \\ S_{1,2} & S_{2,2} \end{pmatrix} \ .$$

The system of the last two equations in (9) has unique solution if and only if $\frac{1}{\beta}$ is not an eigenvalue of $\mathbf{S}$. We have $S_{i,j} \le \frac{1}{\beta(\beta-1)}$, for $i,j=1,2$. Thus, by Theorem A all the eigenvalues of $\mathbf{S}$ have modulus smaller than $\frac{2}{\beta(\beta-1)}$. The condition $\frac{2}{\beta(\beta-1)} < \frac{1}{\beta}$ is equivalent to $\beta > 3$. Thus, at least for $\beta > 3$ the constants D_1, D_2 exist and the formula for the τ-invariant density is valid.

If the last branch is shorter, then one of c_i's, say $c_{i_0} = 1$. We have $S_{i_0,i} = 0$ for $i=1,2$ and Perron-Frobenius estimate on the modulus of eigenvalues of $\mathbf{S}$ is $\frac{1}{\beta(\beta-1)}$. Thus, $\beta > 2$ is sufficient in this case. Note, then $D_{i_0} = 1$.

To prove exactness, as in the proof of Proposition 3 it is enough to show that an interval J in the support of an ergodic absolutely invariant measure grows under iteration to cover the whole $[0,1]$. Since h is supported on $[0,1]$ we can find either J covering the fixed point in an onto branch or a pair J_1, J_2 touching the fixed point x_0 in an inner onto branch. In the first case images $\tau^n(J)$ grow to cover the whole $[0,1]$. In the second case images $\tau^n(J_1)$ grow to cover $[0,x_0]$ and images $\tau^n(J_2)$ grow to cover $[x_0,1]$. Since it is an inner branch the image of $[x_0,1]$ touches 0, which shows uniqueness of absolutely continuous invariant measure and exactness.
□

Proof of the general case:

$$h(x) = \frac{1}{\beta} + \sum_{i=1}^{K} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta^{n+1}} \ ,$$

where constants D_i, $i=1,\dots,K$ satisfy the system (11). Again we need to calculate $\sum_j h(x(j))$ for any $x \in [0,1]$. For $i=1,\dots,K$ have

$$\sum_j \delta(x(j) \le \tau^n(c_i)) = j(\tau^n(c_i)) - 1 + \delta(x \le \tau^{n+1}(c_i))$$
$$- \sum_{k=1}^{K} \delta(x > \tau(c_k)) \cdot \delta(\tau^n(c_i) > c_k) \ .$$

We obtain:

$$
\begin{aligned}
&\sum_j h(x(j)) = \frac{1}{\beta}\left[N - \sum_{i=1}^{K} \delta(x > \tau(c_i))\right] \\
&+ \sum_{i=1}^{K} D_i \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \Big[j(\tau^n(c_i)) - 1 + \delta(x \le \tau^{n+1}(c_i)) \\
&\qquad\qquad\qquad\qquad - \sum_{k=1}^{K} \delta(x > \tau(c_k)) \cdot \delta(\tau^n(c_i) > c_k) \Big] \\
&= \beta \cdot h(x) \\
&+ \frac{1}{\beta}\left[N - \sum_{i=1}^{K} \delta(x > \tau(c_i))\right] \\
&+ \sum_{i=1}^{K} D_i S_i - \sum_{i=1}^{K} D_i \sum_{k=1}^{K} \delta(x > \tau(c_k)) S_{i,k} - \sum_{i=1}^{K} \frac{D_i}{\beta} \delta(x \le \tau(c_i)) - 1 \ ,
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
S_i &= \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \big(j(\tau^n(c_i)) - 1 \big) \quad \text{and} \\
S_{i,k} &= \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \delta(\tau^n(c_i) > c_k) \ , \quad i, k = 1, 2, \ldots, K \ .
\end{aligned}
$$

We need to find the constants D_i, $i = 1, 2, \ldots, K$, such that the last two lines of (10) sum up to 0. Substituting $x < \tau(c_1)$, $\tau(c_i) \le x < \tau(c_{i+1})$ for $i = 1, 2, \ldots, K-1$ and $x > \tau(c_K)$ we obtain system of $K+1$ equations (below index k numbering the equations changes from 0 to K):

$$
\begin{aligned}
&D_1\left(S_1 - \frac{1}{\beta}\right) + \cdots + D_i\left(S_i - \frac{1}{\beta}\right) + \cdots + D_K\left(S_K - \frac{1}{\beta}\right) = 1 - \frac{N}{\beta} \ ; \\
&D_1\left(S_1 - S_{1,1}\right) + \cdots + D_i\left(S_i - S_{i,1} - \frac{\delta(i \ge 2)}{\beta}\right) + \ldots \\
&\qquad\qquad\qquad + D_K\left(S_K - S_{K,1} - \frac{1}{\beta}\right) = 1 - \frac{N-1}{\beta} \ ; \\
&D_1\left(S_1 - S_{1,1} - S_{1,2}\right) + \cdots + D_i\left(S_i - S_{i,1} - S_{i,2} - \frac{\delta(i \ge 3)}{\beta}\right) + \ldots \\
&\qquad\qquad\qquad + D_K\left(S_K - S_{K,1} - S_{K,2} - \frac{1}{\beta}\right) = 1 - \frac{N-2}{\beta} \ ; \\
&\qquad\qquad\qquad\vdots \\
&D_1\left(S_1 - \sum_{j=1}^{k} S_{1,j}\right) + \cdots + D_i\left(S_i - \sum_{j=1}^{k} S_{i,j} - \frac{\delta(i \ge k+1)}{\beta}\right) + \ldots
\end{aligned}
$$

$$+ D_K \left(S_K - \sum_{j=1}^{k} S_{K,\,j} - \frac{1}{\beta} \right) = 1 - \frac{N-k}{\beta} \ ;$$

$$\vdots$$

$$D_1 \left(S_1 - \sum_{j=1}^{K} S_{1,\,j} \right) + \dots + D_i \left(S_i - \sum_{j=1}^{K} S_{i,\,j} \right) + \dots$$
$$+ D_K \left(S_K - \sum_{j=1}^{K} S_{K,\,j} \right) = 1 - \frac{N-K}{\beta} \ .$$

Subtracting the $(K-1)$'st equation from the K'th one, then $K-2$'nd from the $K-1$'st, etc, we obtain a nicer equivalent system (again $0 \le k \le K$) :

$$\begin{array}{llll}
D_1 \left(S_1 - \frac{1}{\beta} \right) + \dots + & D_i \left(S_i - \frac{1}{\beta} \right) + \dots + & D_K \left(S_K - \frac{1}{\beta} \right) & = 1 - \frac{N}{\beta} \ ; \\
D_1 \left(-S_{1,1} + \frac{1}{\beta} \right) + \dots + & D_i \left(-S_{i,1} \right) + \dots + & D_K \left(-S_{K,1} \right) & = \frac{1}{\beta} \ ; \\
D_1 \left(-S_{1,2} \right) + \dots + & D_i \left(-S_{i,2} + \frac{\delta(i=2)}{\beta} \right) + \dots + & D_K \left(-S_{K,2} \right) & = \frac{1}{\beta} \ ; \\
 & \vdots & & \\
D_1 \left(-S_{1,k} \right) + \dots + & D_i \left(-S_{i,k} + \frac{\delta(i=k)}{\beta} \right) + \dots + & D_K \left(-S_{K,k} \right) & = \frac{1}{\beta} \ ; \\
 & \vdots & & \\
D_1 \left(-S_{1,K} \right) + \dots + & D_i \left(-S_{i,K} \right) + \dots + & D_K \left(-S_{K,K} + \frac{1}{\beta} \right) & = \frac{1}{\beta} \ .
\end{array} \tag{11}$$

The rank of the system is K. We can show this multiplying the k'th equation by $(1-\alpha_k)$, $k = 1, \dots, K$ and summing up all the equations. The sum of coefficients in the i'th column, $i = 1, \dots, K$, is

$$S_i - \sum_{k=1}^{K} S_{i,k}(1-\alpha_k) - \frac{\alpha_i}{\beta} = \sum_{n=1}^{\infty} \frac{a(\tau^n(c_i))}{\beta^{n+1}} - \frac{\alpha_i}{\beta} = \frac{\tau(c_i)}{\beta} - \frac{\alpha_i}{\beta} = 0 \ .$$

The $K+1$'st, right hand side, column sums up to

$$1 - \frac{N}{\beta} + \sum_{k=1}^{K} \frac{1-\alpha_k}{\beta} = 1 - \frac{N - K + \alpha_1 + \dots + \alpha_K}{\beta} = 0 \ ,$$

since $\beta = N - K + \alpha_1 + \dots + \alpha_K$. Let us define $K \times K$ matrix

$$\mathbf{S} = \begin{pmatrix} S_{1,1} & \dots & S_{1,K} \\ \vdots & \dots & \vdots \\ S_{K,1} & \dots & S_{K,K} \end{pmatrix} \ .$$

The matrix of the system of the last K equations in (11) is $\mathbf{S}^T$ (A^T denotes the transpose of A). The system has unique solution if and only if $\frac{1}{\beta}$ is not an eigenvalue of $\mathbf{S}$. We have $S_{i,k} \leq \frac{1}{\beta(\beta-1)}$, for $i, k = 1, \dots, K$. Thus, by Theorem A all the eigenvalues of $\mathbf{S}$ have modulus smaller than $\frac{K}{\beta(\beta-1)}$. The condition $\frac{K}{\beta(\beta-1)} < \frac{1}{\beta}$ is equivalent to $\beta > K+1$. Thus, at least for $\beta > K+1$ the constants D_i, $i = 1, 2, \dots, K$ exist and the formula for the τ-invariant density is valid.

The estimate for β in case of shorter last branch and the exactness of τ is proven in the same way as for $K = 2$. □

The number $\frac{1}{\beta}$ can be written as $\frac{1}{\beta} = \exp(-H)$ where H is the entropy of the system $\{\tau, h \cdot m\}$. We have proved the following

Corollary 5. *If no eigenvalue of the matrix* $\mathbf{S}$ *is equal to* $\exp(-H)$*, then the system* $\{\tau, h \cdot m\}$ *is exact.*

There are matrix methods of detecting topological transitivity of piecewise monotone continuous interval maps [1, 10], which is implied by exactness for our class of maps. Perhaps matrix $\mathbf{S}$ can be used for this purpose in a more general setting. We conjecture that the inverse of the Corollary 5 also holds. We proved this for maps τ_A with one shorter branch.

Conjecture 1: Let τ be piecewise linear, piecewise increasing map of constant slope $\beta > 1$ with shorter branches touching 0. Then, $1/\beta$ is not an eigenvalue of matrix $\mathbf{S} \iff$ dynamical system τ, μ is exact, where μ is absolutely continuous τ-invariant measure supported on $[0, 1]$.

Here, we prove a proposition which we will use below.

Proposition 6. *Let us define additional (and artificial) points* c_i *as the right hand side endpoints of the domains of onto branches. Let us extend system of equations (11) by adding columns and rows corresponding to these added points.Then, the new larger matrix* $\bar{\mathbf{S}}$ *has* $\frac{1}{\beta}$ *as an eigenvalue. This means that the extended system of equations is not solvable.*

Proof: Let us assume that there is only one additional point c_i and put the additional row and column as first in the matrix $\bar{\mathbf{S}}$. The general case can be proved in a similar way.

Let denote the additional unknown by D_0. Let $h_k = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_k)]} \frac{1}{\beta^{n+1}}$, $k = 0, 1, \dots, K$ denote the function next to coefficient D_k in the formula for the density h.

We will prove the proposition by contradiction. Let us assume that $\frac{1}{\beta}$ is not an eigenvalue of $\bar{\mathbf{S}}$. Then, the extended system of equations is uniquely solvable and

$$h = \frac{1}{\beta} + D_0 h_0 + \sum_{i=1}^{K} D_i h_i$$

is the invariant density. We consider two cases:

First, assume that the last branch is onto. We have

$$h_0=\sum_{n=1}^{\infty}\chi_{[0,\tau^n(c_0)]}\frac{1}{\beta^{n+1}}=\frac{1}{\beta(\beta-1)}\ .$$

For any real s we can write a multiple of invariant density h

$$\begin{aligned}h_s&=(1+\frac{s}{\beta-1})\left(\frac{1}{\beta}+D_0h_0+\sum_{i=1}^{K}D_ih_i\right)\\&=\frac{1}{\beta}+(s+(1+\frac{s}{\beta-1})D_0)h_0+\sum_{i=1}^{K}(1+\frac{s}{\beta-1})D_ih_i\ .\end{aligned}$$

Since it is of the form (7) the constants $[s+(1+\frac{s}{\beta-1})D_0,(1+\frac{s}{\beta-1})D_1,\dots,(1+\frac{s}{\beta-1})D_K]$ satisfy the extended system of equations for any s. This is a contradiction.

Now, assume that the last branch is not onto, i.e., $c_{i_0}=1$ for some $1\le i_0\le K$ and $D_{i_0}=1$. We have

$$h_0=\sum_{n=1}^{\infty}\chi_{[0,\tau^n(c_0)]}\frac{1}{\beta^{n+1}}=\frac{1}{\beta^2}+\frac{1}{\beta}h_{i_0}\ .$$

Again, we can write a multiple of invariant density

$$\begin{aligned}h_s&=(1+\frac{s}{\beta})\left(\frac{1}{\beta}+D_0h_0+h_{i_0}+\sum_{\substack{i=1\\i\neq i_0}}^{K}D_ih_i\right)\\&=\frac{1}{\beta}+\frac{s}{\beta^2}+(1+\frac{s}{\beta})D_0h_0+(1+\frac{s}{\beta})h_{i_0}+\sum_{\substack{i=1\\i\neq i_0}}^{K}(1+\frac{s}{\beta})D_ih_i\ .\end{aligned}$$

Since $D_{i_0}=1$ the constants $[(1+\frac{s}{\beta})D_0+s,(1+\frac{s}{\beta})D_1,\dots,D_{i_0},\dots,(1+\frac{s}{\beta})D_K]$ satisfy the extended system of equations for any s. This, again, is a contradiction. □

5. *Special cases*

In this section we consider maps τ_A with $K=2$ shorter branches satisfying $\beta\le 3$, or $\beta\le 2$ if the last branch is shorter.

We will first consider cases when τ has two shorter branches, $\beta\le 2$ and the last branch is shorter. This means that τ has 3 branches.

(A) The first branch is onto: Then, τ is exact, which can be proved as in Proposition 3. Let us assume that $c_1=1$. Then, $S_{1,1}=S_{2,1}=0$ and $D_1=1$. D_2 has to satisfy $D_2(-S_{2,2}+\frac{1}{\beta})=\frac{1}{\beta}+S_{1,2}$. We will show that

$$S_{2,2}<\frac{1}{\beta}\ . \tag{12}$$

We have $\alpha_1+\alpha_2\le 1$ so $\tau(c_2)\le c_2$. Also, whenever $\tau^n(c_2)>c_2$ then $\tau^{n+1}(c_2)\le c_2$. Thus, $S_{2,2}<\frac{1}{\beta^2(\beta^2-1)}$ and (12) is shown at least for $\beta>\beta_1\simeq 1.32472$ such that $\frac{1}{\beta_1^2(\beta_1^2-1)}=\frac{1}{\beta_1}$.

Assume that $\beta \leq \beta_1$. Then, $(\beta+1)(\beta-1) \leq \frac{1}{\beta}$. Since $\alpha_2 < \beta - 1$ this means that $\tau(c_2) < c_2$ and $\tau^2(c_2) < c_2$. Moreover, whenever $\tau^n(c_2) > c_2$ then the next two iterates are smaller then $\frac{1}{\beta}$. Thus, $S_{2,2} < \frac{1}{\beta^3(\beta^3-1)}$ and (12) is shown at least for $\beta > \beta_2 \simeq 1.19385$ such that $\frac{1}{\beta_2^3(\beta_2^3-1)} = \frac{1}{\beta_2}$.

Assume again that $\beta \leq \beta_2$. Then, $\beta(\beta^2+\beta+1)(\beta-1) \leq \frac{1}{\beta}$ which means that $\tau^k(c_2) < c_2$ for $k = 1, 2, 3, 4$. Moreover, whenever $\tau^n(c_2) > c_2$ then the next four iterates are smaller then $\frac{1}{\beta}$. Thus, $S_{2,2} < \frac{1}{\beta^5(\beta^5-1)}$ and (12) is shown at least for $\beta > \beta_3 \simeq 1.10735$ such that $\frac{1}{\beta_3^5(\beta_3^5-1)} = \frac{1}{\beta_3}$.

Since the positive solutions of $x^n(x^n - 1) = x$ converge to 1 as n converges to infinity, repeating the above reasoning inductively we can prove (12) for all $\beta > 1$. The case $c_2 = 1$ can be proven similarly.

Example 3: τ considered in case (A) gives an example of maps for which invariant density h exists although β can be arbitrarily close to 1. On the other hand the set of digits $A = 0, 0.5, 1$ provides an example of a map τ_A with the slope $\beta = 2$ which is not exact (on $[0, 1]$) and for which the formula for h is not valid.

(B) The first branch is shorter. Assume that it corresponds to index k_2. Then, the fixed point in the middle onto branch is $x_0 = \alpha_2/(\beta - 1)$ and $x_0 \geq \alpha_2$. The support of absolutely continuous invariant measure is the interval $[0, \alpha_2]$ and τ restricted to this interval is classical β-map.

Now, we consider situation where the last branch is onto and $\beta \leq 3$. This means that τ has 3 or 4 branches.

3 branches case: Since the last branch of τ is onto, the first and the second branch are shorter. We always have $\alpha 1 \leq \alpha 2$.

(C) $k_1 = 1$, $k_2 = 2$: There are two possibilities:

(Ca) α_1 is below the fixed point on the second branch (or this fix point does not exist). Then, map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_1]$. τ restricted to this interval is a classical β-map and the invariant density can be found by Parry's formula (or our formula after rescaling).

(Cb) The image of the first branch covers the fixed point on the second branch. Then, map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first and the last branches shorter. This situation is considered in (B).

(D) $k_2 = 1$, $k_1 = 2$: Map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first branch onto. This situation is considered in (A).

4 branches case: The last branch of τ is onto.

(E) The first branch is onto. $2 < \beta \leq 3$. τ is exact.

(F) The two first branches are shorter. $2 < \beta \leq 3$.

$k_1 = 1$, $k_2 = 2$: Since the fixed point in the second branch is $x_0 = \frac{\alpha_1}{\beta-1} < \alpha_1$ the image of the first branch covers it. There are two cases:

(Fa) If α_2 is above the fixed point in the third, onto branch, then τ is exact.

(Fb) If α_2 is below the fixed point in the third, onto branch, then map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first and the last branches shorter. This situation is considered in (B).

$k_1 = 2$, $k_2 = 1$: Situation is similar as in cases (Fa), (Fb).

(G) The first and the third branches are shorter. $2 < \beta \le 3$. Since again the image of the first branch covers the fixed point in the second onto branch, map τ is exact.

6. *Maps with the shorter branches "at the top"*

In this section we consider piecewise linear maps of an interval $[0, 1]$ with constant slope $\beta > 1$, all branches increasing and such that the images of shorter branches touch 1. Such maps are related to so called "lazy expansions with deleted digits" [4].

Let $\tilde{\tau}$ be a map described above. Let $N \ge 2$ be the number of branches of $\tilde{\tau}$ and $K < N$ the number of shorter branches, i.e., not onto. Let $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_K$ be the heights of the shorter branches and let $1 \le \tilde{k}_1, \tilde{k}_2, \dots, \tilde{k}_K \le N$ denote the numbers of these branches correspondingly. We do not assume any order of $\tilde{k}_j$'s. Then, we have

$$\beta = N - K + \alpha_1 + \alpha_2 + \cdots + \alpha_K \ .$$

The endpoints of the maximal intervals of monotonicity of $\tilde{\tau}$ are $0 = \tilde{b}_1 < \tilde{b}_2 < \cdots < \tilde{b}_N < \tilde{b}_{N+1} = 1$ and

$$\tilde{b}_j = \frac{1}{\beta}\left(j - 1 - \sum_{i=1}^{K} \delta(j > \tilde{k}_i)(1 - \alpha_{\tilde{k}_i}) \right) \ , \quad j = 1, \dots, N \ .$$

The map $\tilde{\tau}$ is defined on the partition $\mathcal{P}_{\tilde{\tau}} = \{J_1, J_2, \dots, J_{N-1}, J_N\}$, of the interval $[0, 1]$, where

$$\begin{aligned} J_1 &= [0, \tilde{b}_2) \ ; \\ J_j &= [\tilde{b}_j, \tilde{b}_{j+1}) \quad \text{for} \quad 2 \le j \le N-1 \ ; \\ J_N &= [\tilde{b}_N, 1] \ . \end{aligned}$$

Note that intervals J_j are open (closed) on different side than intervals I_j for τ. The points

$$\tilde{c}_i = \tilde{b}_{\tilde{k}_i}, \quad i = 1, \dots, K \ ,$$

the left hand side endpoints of the domains of the shorter branches of $\tilde{\tau}$, play special role in further study. Note,

$$\tilde{\tau}(\tilde{c}_i) = 1 - \alpha_i \ , \ i = 1, \dots, K \ .$$

We define the set of "digits" $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N\}$, where

$$\tilde{a}_j = \left(j - 1 - \sum_{i=1}^{K} \delta(j > \tilde{k}_i - 1)(1 - \alpha_{\tilde{k}_i}) \right) \ , \quad j = 1, \dots, N \ .$$

We also have

$$\tilde{a}_j = \beta(\tilde{b}_{j+1} - \tilde{b}_1) - 1, \quad j = 1, \dots, N \ .$$

Map $\tilde{\tau}$ (occasionally denoted also by $\tilde{\tau}_{\tilde{A}}$) is given by

$$\tilde{\tau}(x) = \beta \cdot x - \tilde{a}_j, \quad \text{for} \quad x \in J_j \ , \ j = 1, 2, \dots, N \ .$$

Note, if the first branch is not onto, then some digits are negative.

For any $x \in [0,1]$ we define its "index" $\tilde{j}(x)$ and its "digit" $\tilde{a}(x)$:

$$\tilde{j}(x) = j \qquad \text{for} \quad x \in J_j \ , \ j = 1, 2, \dots, N \ ,$$

and

$$\tilde{a}(x) = \tilde{a}_{\tilde{j}(x)}.$$

The "lazy expansion with deleted digits " is defined using $\tilde{A}$ and $\tilde{\tau}$ similarly as the "greedy" one and an analog of Proposition 1 holds. See [4] for more information.

We will now show that map $\tilde{\tau}_{\tilde{A}}$ is conjugated to some map τ_A by diffeomorphism $f(x) = 1 - x$ on $[0,1]$. First we define a "conjugated" set of digits $A = \{a_1, a_2, \dots, a_N\}$ with $a_j = \tilde{a}_N - \tilde{a}_{N-j+1}$, $j = 1, 2, \dots, N$. In particular, $a_1 = 0$ and $a_N = \tilde{a}_N - \tilde{a}_1$. This defines the endpoints of maximal monotonicity intervals for τ: $b_j = a_j/\beta$, $j = 1, 2, \dots, N$. Note that $b_1 = 0$ and $b_N = 1$. We define intervals I_j as in (1). The lengths of J_j and I_{N-j+1} are equal, $j = 1, 2, \dots, N$. The shorter intervals are $k_1, \dots, k_K$ where $k_i = N - \tilde{k}_i + 1$, $i = 1, \dots, K$.

THEOREM 7. *The maps $\tilde{\tau}_{\tilde{A}}$ and τ_A are conjugated by the diffeomorphism $f(x) = 1-x$. If h is a τ_A-invariant density, then the density $\tilde{h}(x) = h(1-x)$ is $\tilde{\tau}_{\tilde{A}}$-invariant. We have*

$$\tilde{h}(x) = \frac{1}{\beta} + \sum_{i=1}^{K} \tilde{D}_i \sum_{n=1}^{\infty} \chi_{[\tau^n(\tilde{c}_i),1]} \frac{1}{\beta^{n+1}} \ ,$$

where constants $\tilde{D}_i = D_i$, $i = 1, \dots, K$, satisfy the system (11). $\tilde{D}_i$'s can be also obtained directly from the system similar to (11), where quantities $S_{i,j}$ are replaced by

$$\tilde{S}_{i,j} = \sum_{n=1}^{\infty} \frac{\delta(\tau^n(\tilde{c}_i) < \tilde{c}_j)}{\beta^{n+1}} \ , \ 1 \le i, j \le K \ .$$

Proof: Both τ_A and $f \circ \tilde{\tau}_{\tilde{A}} \circ f^{-1}$ are piecewise linear, piecewise increasing maps with constant slope β and the images of shorter intervals touch 0. The equality of the lengths of the intervals I_j and J_{N-j+1}, $j = 1, 2, \dots, N$, proves that they are identical. Then, the formula for $\tilde{h}$ follows, using the fact that $\tilde{\tau}^n(\tilde{c}_i) = 1 - \tau^n(c_i)$, for all $n \ge 0$ and $i = 1, \dots, K$. $\square$

Example 4: Let $N = 5$, $K = 3$, $\alpha_1 = 0.4$, $\tilde{k}_1 = 1$, $\alpha_2 = 0.5$,$\tilde{k}_2 = 3$, $\alpha_3 = 0.7$, $\tilde{k}_3 = 2$. Then, $\tilde{A} = \{-0.6, 0.1, 0.6, 1.6, 2.6\}$ and $\beta = 3.6$. The conjugated set of digits is $A = \{0, 1, 2, 2.5, 3.2\}$. Using Maple 11 we calculated $D_1 = \widetilde{D}_1 = 1.00$, $D_2 = \widetilde{D}_2 \simeq 1.38693$, $D_3 = \widetilde{D}_3 \simeq 1.00166$. The normalizing constant for both densities is $\simeq 0.45935$. The maps are shown in Figure 3 and the normalized densities $\tilde{h}$ and h are shown in Figure 4.

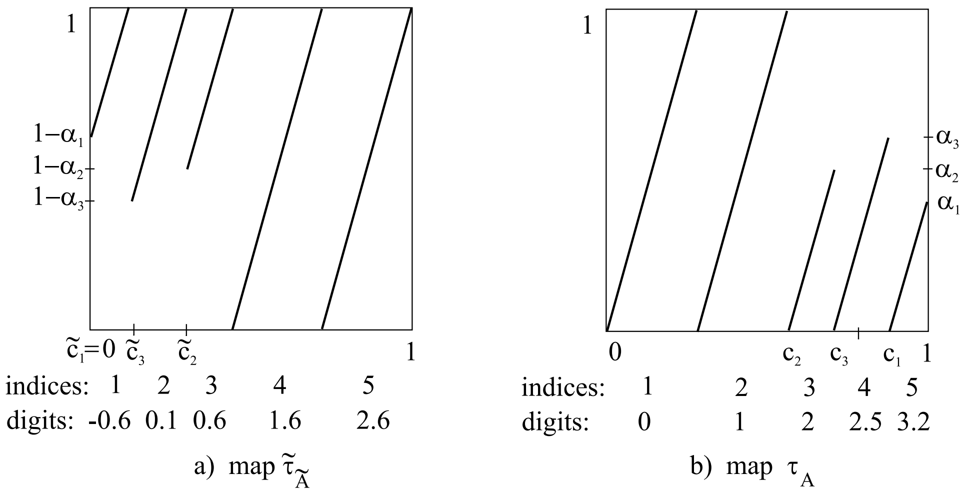

FIGURE 3. Maps $\tilde{\tau}_{\tilde{A}}$ and τ_A of Example 4.

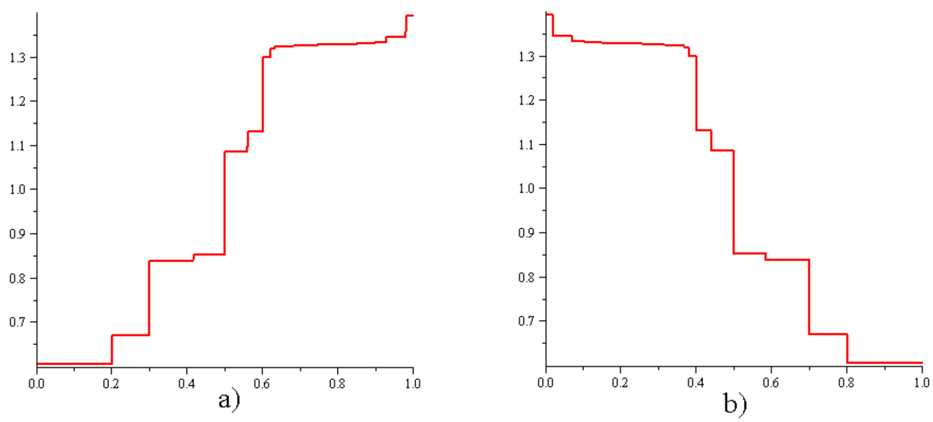

FIGURE 4. Invariant densities for maps of Example 4: a) $\tilde{\tau}_{\tilde{A}}$, b) τ_A.

7. *Maps related to "mixed" expansions.*

In this section we join the results of previous sections and consider maps with some shorter branches touching 0 and others touching 1.

Let τ be a piecewise linear, piecewise increasing map of interval $[0,1]$ onto itself with constant slope $\beta > 1$. Let N denote the number of branches of τ and $K \le N$ the number of shorter, not onto, branches. We removed the requirement of having at least one onto branch but still assume that τ is onto. Let $1 \le k_1 < k_2 < \cdots < k_K \le N$ be the indices of shorter branches. In this section we have changed the convention and assume that these indices are numbered in increasing order. Let α_i denote the hight of k_i'th branch, $i = 1, \dots, K$. We do not assume any order among α_i's. We define also a vector $U = [U_1, \dots, U_N]$. We set $U_j = 0$ if the jth branch is onto or "greedy" and $U_j = 1$ if the jth branch is "lazy".

As in the previous sections, we have

$$\beta = N - K + \alpha_1 + \alpha_2 + \cdots + \alpha_K \ .$$

The endpoints of the maximal intervals of monotonicity of τ are $0 = b_1 < b_2 < \cdots < b_N < b_{N+1} = 1$ and

$$b_j = \frac{1}{\beta}\left(j - 1 - \sum_{i=1}^{K} \delta(j > k_i)(1-\alpha_{k_i})\right) \ , \quad j = 1, \dots, N \ . \tag{13}$$

We assume that map τ is defined on the partition $\mathcal{P}_\tau = \{I_1, I_2, \dots, I_{N-1}, I_N\}$, of the interval $[0,1]$, where

$$\begin{aligned} I_1 &= [0, b_2) \ ; \\ I_j &= (b_j, b_{j+1}) \quad \text{for} \quad 2 \le j \le N-1 \ ; \\ I_N &= (b_N, 1] \ . \end{aligned} \tag{14}$$

It may be not possible to reasonably define τ at some inner b_j's. This problem affects a countable set of points, preimages of inner b_i's. Since we will need to iterate some of these points we define two extensions of τ: τ_g the extension by continuity to partition

$$\mathcal{P}_g = \{[0, b_2], (b_2, b_3], \dots, (b_{N-1}, b_N], (b_N, 1]\},$$

and τ_l the extension by continuity to partition

$$\mathcal{P}_l = \{[0, b_2), [b_2, b_3), \dots, [b_{N-1}, b_N), [b_N, 1]\}.$$

Now, we will define points c_i. It may happen that two of them are equal as numbers but we want to consider them as different so strictly speaking each c_i below should be treated as a pair (c, j), where $c \in [0,1]$ and $1 \le j \le N$. Let us define the points c_i, $i = 1, \dots, K$ as follows

$$c_i = b_{k_i+1} = (b_{k_i+1}, k_i), \ \text{ if } \ U_{k_i} = 0 \quad \text{ and } \quad c_i = b_{k_i} = (b_{k_i}, k_i), \ \text{ if } \ U_{k_i} = 1 \ .$$

These are the right hand side endpoints of the domains of the "greedy" branches and the left hand side endpoints of the domains of the "lazy" branches, respectively. We group points c_i into two disjoint sets: W_g contains c_i's associated with "greedy" branches ($U_{k_i} = 0$) and W_l contains c_i's associated with "lazy" branches ($U_{k_i} = 1$). Note,

$$\begin{aligned} \tau(c_i) &= \alpha_i \qquad &\text{for} \quad c_i \in W_g \ , \\ \tau(c_i) &= 1 - \alpha_i \qquad &\text{for} \quad c_i \in W_l \ . \end{aligned}$$

We define the set of "digits" $A = \{a_1, a_2, \dots, a_N\}$, where

$$a_j = j - 1 - \sum_{i=1}^{K} \delta(j > k_i - U_j)(1 - \alpha_{k_i}) \ , \quad j = 1, \dots, N \ .$$

a_j can be also calculated as

$$a_j = \beta b_j \ , \ \text{ if } \ U_j = 0 \quad \text{ and } \quad a_j = \beta b_{j+1} - 1, \ \text{ if } \ U_j = 1 \ \ .$$

Now, we can write map τ as

$$\tau(x) = \beta \cdot x - a_j, \qquad \text{for} \quad x \in I_j \ , \ j = 1, 2, \dots, N \ .$$

For any $x \in [0,1] \setminus \{b_2, \dots, b_N\}$ we define its "index" $j(x)$ and its "digit" $a(x)$:

$$j(x) = j \qquad \text{for} \quad x \in I_j \ , \ j = 1, 2, \dots, N \ ,$$

and

$$a(x) = a_{j(x)}.$$

We also define (for all $x \in [0,1]$) the indices $j_g(x)$, $j_l(x)$ and digits $a_g(x)$, $a_l(x)$. The expansions analogous to described in Proposition 1 are then defined and analogues of of this proposition hold (for almost all x in case of τ-expansion). These expansions are identical for almost all $x \in [0,1]$. To represent points c_i we will use "greedy" expansion if $c_i \in W_g$ and "lazy" expansion if $c_i \in W_l$.

We now prepare for the description of τ-invariant density h. Let us define

$$\begin{aligned} S_{i,j} &= \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \delta(\tau_g^n(c_i) > c_j) \ , \qquad \text{for} \quad c_i \in W_g \ \text{ and all } \ c_j \ , \\ S_{i,j} &= \sum_{n=1}^{\infty} \frac{1}{\beta^{n+1}} \delta(\tau_l^n(c_i) < c_j) \ , \qquad \text{for} \quad c_i \in W_l \ \text{ and all } \ c_j \ . \end{aligned} \tag{15}$$

Let $\mathbf{S}$ be the matrix $(S_{i,j})_{1 \le i,j \le K}$ and $\mathbf{Id}$ denote $K \times K$ identity matrix. Let $v_\beta = \left[\frac{1}{\beta}, \dots, \frac{1}{\beta}\right]$ be K-dimensional vector and let $D = [D_1, \dots, D_K]$ denote the solution of the system

$$(-\mathbf{S}^T + \frac{1}{\beta}\mathbf{Id})D = v_\beta \ , \tag{16}$$

where A^T denotes the transpose of A.

THEOREM 8. *Let $\tau = \tau_A$ will be the map defined in this section, i.e., any piecewise linear, piecewise increasing map with the images of shorter branches touching 0 or 1. Let*

$$h(x) = \frac{1}{\beta} + \sum_{i \in W_g} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau_g^n(c_i)]} \frac{1}{\beta^{n+1}} + \sum_{i \in W_l} D_i \sum_{n=1}^{\infty} \chi_{[\tau_l^n(c_i),1]} \frac{1}{\beta^{n+1}} \ , \tag{17}$$

where constants D_i, $i = 1, \dots, K$, satisfy the system (16). If the system (16) is solvable, then h is τ-invariant and the dynamical system $\{\tau, h \cdot m\}$ is ergodic.

In particular, system (16) is uniquely solvable if $\beta > K + 1$. If the last branch is "greedy " or if the first branch is "lazy", then condition $\beta > K$ is sufficient. When both possibilities are realized it is enough to have $\beta > K - 1$. Whenever one of these cases happens the corresponding constant $D_i = 1$.

The dynamical system $\{\tau, \mu\}$ can have at most two ergodic components. If there are actually two such components, then $\frac{1}{\beta}$ is an eigenvalue of matrix $\mathbf{S}$ and system (16) is not solvable. If τ has at least two onto branches, then it is exact.

Remark 2: If the system (16) is solvable, then it is uniquely solvable. This can be proved exactly as Remark 1 since ergodicity of $\{\tau, h \cdot m\}$ implies the uniqueness of invariant density (up to a multiplicative constant).

Proof: The proof of Theorem 8 is analogous to that of Theorem 4. h satisfies Perron-Frobenius equation almost everywhere, for all x except possibly preimages of inner endpoints of partition $\mathcal{P}_\tau$ intervals. We only have to prove the ergodic properties part. It follows from the general theory that the support of each ergodic component contains neighborhood of some inner endpoint of the partition. Since image of each branch touches either 0 or 1, there can be at most two ergodic components.

Let us assume that there are two ergodic components. Since h is supported on $[0, 1]$ there exists $x_0 \in [0, 1]$ such that the support of one component is $J_0 = [0, x_0]$ and the support of the other is $J_1 = [x_0, 1]$. Let $\tau_0 = \tau_{|_{J_0}}$ and $\tau_1 = \tau_{|_{J_1}}$.

We have $\tau_g^n(c_k) \le c_j$ for all $n \ge 1$ and all $c_k \in J_0$, $c_j \in J_1$ and $\tau_l^n(c_k) \ge c_j$ for all $n \ge 1$ and all $c_k \in J_1$, $c_j \in J_0$. Thus, matrix $\mathbf{S}$ is a block matrix

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_0 = (S_{i,j})_{1 \le i, j \le L} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 = (S_{i,j})_{L+1 \le i, j \le K} \end{pmatrix},$$

where $c_1, \dots, c_L \in J_0$ and $c_{L+1}, \dots, c_K \in J_1$.

The image of at least one $c_{i_0} \in J_0$ and at least one $c_{i_1} \in J_1$ is equal to x_0 as otherwise there would be a hole in the support of h. Even if x_0 is a fixed point in a common onto branch od τ, there must exist such points.

Thus, the matrix $\mathbf{S}_0$ is the extended matrix for map τ_0 on J_0 described in Proposition 6. Thus, $\frac{1}{\beta}$ is an eigenvalue of $\mathbf{S}_0$. Similarly, it is an eigenvalue of $\mathbf{S}_1$. Thus, the solution for constants D_i is not unique.

Since h has full support, each of the systems $\tau_0, h \cdot m_{|_{J_0}}$, $\tau_1, h \cdot m_{|_{J_1}}$ is exact. Each can be considered separately and the invariant densities can be combined.

If τ has at least two onto branches, then the fixed points in these branches, x_0 and x_1 are different. Each of intervals $[0, x_0]$, $[0, x_1]$, $[x_0, 1]$, $[x_1, 1]$, is completely contained in a support of an ergodic component. Thus, we have at most one ergodic component. Since arbitrary neighborhood of any of these fixed points grows under iteration to cover the whole $[0, 1]$ the system is exact. □

In the case we consider now, when τ can have branches touching 0 or 1 we proved a weaker following statement and we make a weaker conjecture:

Corollary 9. *If $1/\beta$ is not an eigenvalue of matrix $\mathbf{S}$, then system $\tau, h \cdot m$ is ergodic.*

Conjecture 2: Let τ be a map which has shorter branches touching 0 and shorter branches touching 1. If $1/\beta$ is not an eigenvalue of matrix $\mathbf{S} \iff$ dynamical system τ, μ is ergodic, where μ is absolutely continuous τ-invariant measure supported on $[0, 1]$.

Example 8 shows that we cannot always expect exactness when system (16) is solvable.

Example 5: Let $N = 6$, $K = 4$, $k_1 = 1$, $\alpha_1 = 0.4$, $k_2 = 3$, $\alpha_2 = 0.5$, $k_3 = 4$, $\alpha_3 = 0.7$, $k_4 = 5$, $\alpha_4 = 0.7$, $U = [1, 0, 1, 0, 1, 0]$. Then, $A = \{-0.6, 0.4, 0.9, 1.9, 2.3, 3.3\}$ and $\beta = 4.3$. We have $c_1 = 0$, $c_2 \simeq 0.33$, $c_3 = c_4 \simeq 0.60$. $c_3 \in W_g$ and the three others are in W_l. Using Maple 11 we calculated $D_1 \simeq 3.18872$, $D_2 \simeq 5.1468$, $D_3 = D_4 \simeq 7.22277$. The normalizing constant is $\simeq 0.45935$. Map τ is shown in Figure 5 a) and the normalized density h in Figure 6 a).

Example 6: Let $N = 6$, $K = 2$, $k_1 = 1$, $\alpha_1 = 0.4$, $k_2 = 6$, $\alpha_2 = 0.5$, $U = [1, 0, 0, 0, 0, 0]$. Then, $A = \{-0.6, 0.4, 1.4, 2.4, 3.4, 4.4\}$ and $\beta = 4.9$. We have $c_1 = 0 \in W_l$ and $c_2 = 1 \in W_g$. This is example of Parry's (β, α)-map, $\tau_{(\beta,\alpha)} = \beta x + \alpha \pmod 1$, with our β and $\alpha = 1 - \alpha_1$. Since the first branch is "lazy", the last "greedy" and there are no other shorter branches τ is exact for all $\beta > 1$. Using Maple 11 we calculated $D_1 = D_2 \simeq 1.34483$. The normalizing constant is $\simeq 0.26459$. The normalized τ-invariant density h is shown in Figure 6 b). Our h is exactly equal to the density from Parry's formula although represented differently.

Example 7: Let $N = 3$, $K = 2$, $k_1 = 1$, $\alpha_1 = 0.5$, $k_2 = 3$, $\alpha_2 = 0.5$, $U = [0, 0, 1]$. Then, $A = \{0, 1/2, 1\}$ and $\beta = 2$. τ obviously has two exact components. Matrix $\mathbf{S}$ has an eigenvalue $1/2$ and system (16) is not uniquely solvable. Graph of a more complex map with similar properties is shown in Figure 5 b). In this case matrix $\mathbf{S}$ also has $\frac{1}{\beta}$ as an eigenvalue.

Example 8: Let $N = 4$, $K = 4$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$, $U = [1, 1, 0, 0]$. Then, $A = \{-0.5, 0, 1, 1.5\}$ and $\beta = 2$. τ obviously is ergodic and τ^2 has two exact components. System (16) is solvable, $D_1 = D_4 = -0.5$, $D_2 = D_3 = -1$ and normalizing factor is -0.5.

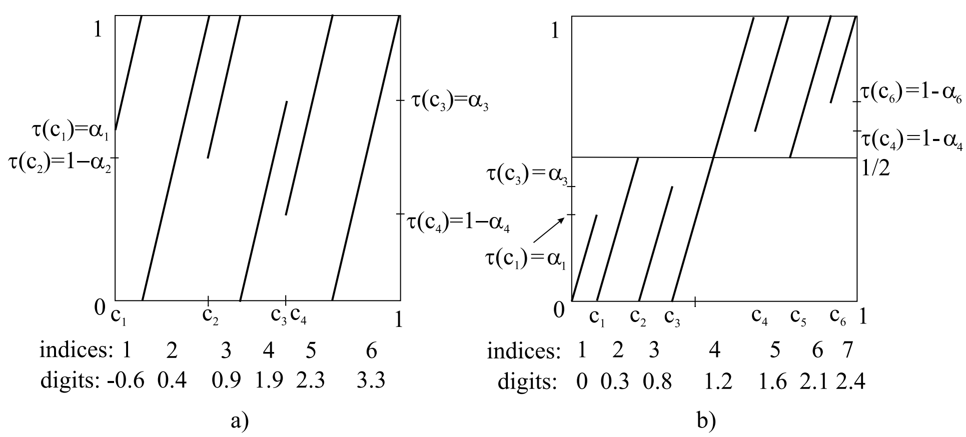

FIGURE 5. Graphs of a) map τ of Example 5 and b) map τ mentioned in Example 7.

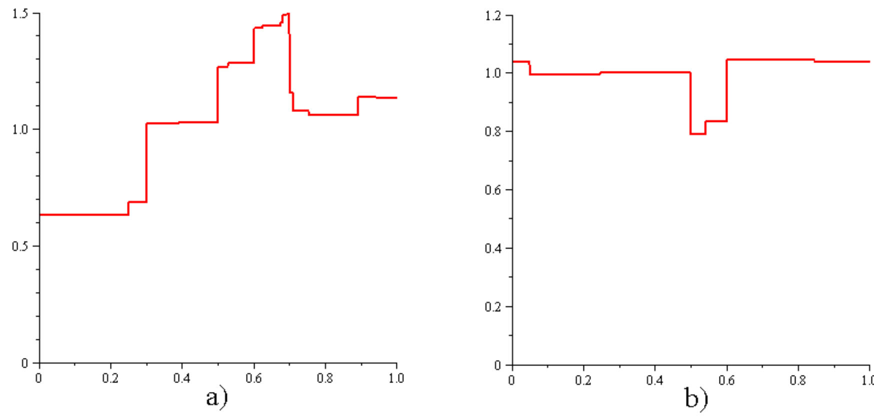

FIGURE 6. Invariant densities for maps of a) Example 5 and b) Example 6 .

8. *General maps with shorter branches touching 0 or 1 or hanging in between.* In this section we further generalize the results of previous sections and consider maps with some shorter branches hanging in between 0 and 1.

Let τ be a piecewise linear, piecewise increasing map of interval $[0, 1]$ onto itself with constant slope $\beta > 1$. Again let N denote the number of branches of τ and $K \leq N$ the number of shorter, not onto, branches. We allow $L \leq K$ shorter branches not to touch 0 or 1. We will call them "hanging" branches.

Let $1 \leq k_1 < k_2 < \cdots < k_K \leq N$ be the indices of shorter branches and $1 \leq l_1 < l_2 < \cdots < l_L \leq N$ the indices of hanging branches. For each l_i we have a k_{j_i} such that $l_i = k_{j_i}$.

As before let α_i denote the hight of k_i'th branch, $i = 1, \ldots, K$. We also denote by γ_i, $i = 1, \ldots, L$ the heights of images of the left hand side endpoints of the domains of hanging branches. We always have $\alpha_{j_i} + \gamma_i < 1$, for $l_i = k_{j_i}$.

We again define vector $U = [U_1, \ldots, U_N]$ and a new vector $UU = [UU_1, \ldots, UU_N]$ to indicate the positions of hanging branches. We set $U_j = 0$ if the jth branch is onto, "greedy" or hanging and $U_j = 1$ if the jth branch is "lazy". We set $UU_j = 1$ if the jth branch is "hanging" and 0 for all other branches.

As in the previous sections, we have

$$\beta = N - K + \alpha_1 + \alpha_2 + \cdots + \alpha_K \ .$$

The endpoints of the maximal intervals of monotonicity of τ are $0 = b_1 < b_2 < \cdots < b_N < b_{N+1} = 1$ and are again expressed by formula (13). We assume that map τ is defined on the partition $\mathcal{P}_\tau$ defined in (14) and again create its two extensions τ_g and τ_l.

The definition of the points c_i is again complicated as in the previous section. Each of them will actually be a pair (c, j) where $c \in [0, 1]$ and $1 \leq j \leq N$ and c is one of the endpoints of interval I_j. We define index function on points c_i: $j(c_i, k) = k$. We define $K + L$ points c_i. They are:

the right hand side endpoints of domains of shorter branches touching 0 ("greedy" branches);

the left hand side endpoints of of domains of shorter branches touching 1 ("lazy" branches);

both endpoints of domains of shorter "hanging" branches.

We number them in such a way that $c_1 < c_2 < \cdots < c_{K+L-1} < c_{K+L}$, where $(c, j) < (d, k)$ if either $c < d$ or $c = d$ and $j < k$. The indices of points c_i no longer correspond directly to indices of α_j's. We group them into two disjoint sets: W_g containing c_i's associated with "greedy" branches and right hand side endpoints of domains of "hanging" branches ; W_l containing c_i's associated with "lazy" branches and left hand side endpoints of domains of "hanging" branches. Note,

$$\begin{aligned}
\tau(c_i) &= \alpha_s && \text{for} \quad c_i \in W_g \ \text{ if } \ k_s = j(c_i),\ U_{j(c_i)} = UU_{j(c_i)} = 0 \ , \\
\tau(c_i) &= \alpha_s + \gamma_z && \text{for} \quad c_i \in W_g \ \text{ if } \ k_s = l_z = j(c_i),\ UU_{j(c_i)} = 1 \ , \\
\tau(c_i) &= 1 - \alpha_s && \text{for} \quad c_i \in W_l \ \text{ if } \ k_s = j(c_i),\ U_{j(c_i)} = 1 \ , \\
\tau(c_i) &= \gamma_z && \text{for} \quad c_i \in W_l \ \text{ if } \ l_z = j(c_i),\ UU_{j(c_i)} = 1 \ .
\end{aligned}$$

When we consider $\tau(c_i)$ we apply it to the first element of the pair. Since we always use τ_g to act on elements of W_g and τ_l to act on elements of W_l there is no problem with recognizing the image.

We define the set of "digits" $A = \{a_1, a_2, \dots, a_N\}$, where

$$a_j = j - 1 - \sum_{i=1}^{K} \left[\delta(j > k_i - U_j)(1 - \alpha_{k_i}) - \delta(j = k_i)\gamma_s\right] \ ,$$

$$\text{where} \quad l_s = k_i = j \ , \quad j = 1, \dots, N \ .$$

a_j can be also calculated as

$$\begin{aligned}
a_j &= \beta b_j, && \text{if } \ U_j = UU_j = 0 \ ; \\
a_j &= \beta b_j - \gamma_s \ , && \text{if } \ l_s = j,\ U_j = 0,\ UU_j = 1 \ ; \\
a_j &= \beta b_{j+1} - 1, && \text{if } \ U_j = 1 \ .
\end{aligned}$$

Note that each a_j is between the minimal, "lazy" digit $a_j^l = \beta b_{j+1} - 1$ and maximal, "greedy" digit $a_j^g = \beta b_j$, $j = 1, 2, \dots, N$. If the jth branch is onto then $a_j = a_j^l = a_j^g$.

Now, we can write map τ as

$$\tau(x) = \beta \cdot x - a_j, \qquad \text{for} \qquad x \in I_j \ , \ j = 1, 2, \dots, N \ .$$

For any $x \in [0, 1] \setminus \{b_2, \dots, b_N\}$ we define its "index" $j(x)$ and its "digit" $a(x)$:

$$j(x) = j \qquad \text{for} \quad x \in I_j \ , \ j = 1, 2, \dots, N \ ,$$

and

$$a(x) = a_{j(x)}.$$

As in the previous section, we also define (for all $x \in [0, 1]$) the indices $j_g(x)$, $j_l(x)$ and digits $a_g(x)$, $a_l(x)$. The expansions analogous to described in Proposition 1 are then defined and analogues of of this proposition hold (for almost all x in case of τ-expansion). These expansions are identical for almost all $x \in [0,1]$. To represent points c_i we will use "greedy" expansion if $c_i \in W_g$ and "lazy" expansion if $c_i \in W_l$.

We define the $(K+L) \times (K+L)$ matrix $\mathbf{S}$, the $K+L$-dimensional vector of constants $D = [D_1, \dots, D_{K+L}]$ and the density h in the same way as in the previous section by equations (15)-(17).

The theorem below describes invariant density for all piecewise linear, piecewise increasing maps τ of constant slope $\beta > 1$, at least for β large enough .

THEOREM 10. *Let $\tau = \tau_A$ will be the map defined in this section, i.e, any piecewise linear, piecewise increasing map of constant slope $\beta > 1$. If the system (16) is solvable, then h is τ-invariant density and dynamical system $\{\tau, h \cdot m\}$ is ergodic.*

In particular, system (16) is uniquely solvable if $\beta > K+L+1$. If the last branch is "greedy " or hanging or if the first branch is "lazy" or hanging, then condition $\beta > K+L$ is sufficient. When both possibilities are realized it is enough to have $\beta > K+L-1$. Whenever one of these cases happens the corresponding constant $D_i = 1$.

The dynamical system $\{\tau, h{\cdot}m\}$ can have any finite number of ergodic subsystems. If this number is larger than 1, then $\frac{1}{\beta}$ is an eigenvalue of matrix $\mathbf{S}$ and system (16) is not solvable.

If τ has one onto branch, then it has at most two ergodic components. If τ has at least two onto branches, then τ is exact.

Remark 3: If the system (16) is solvable, then it is uniquely solvable. This is proved exactly as Remark 2.

Again, the proof of Theorem 10 closely follows the proofs from the previous sections.

We again proved the following

COROLLARY 11. *Let τ be a piecewise linear, piecewise increasing map of constant slope $\beta > 1$. If $1/\beta$ is not an eigenvalue of matrix $\mathbf{S}$, then system $\tau, h \cdot m$ is ergodic.*

Conjecture 3: Let τ be a piecewise linear, piecewise increasing map of constant slope $\beta > 1$. $1/\beta$ is not an eigenvalue of matrix $\mathbf{S} \iff$ dynamical system τ, μ is ergodic, where μ is absolutely continuous τ-invariant measure supported on $[0,1]$.

Example 9: Let $N = 6$, $K = 5$, $L = 2$, $k_1 = 1$, $\alpha_1 = 0.4$, $k_2 = 3$, $\alpha_2 = 0.5$, $k_3 = 4$, $\alpha_3 = 0.3$, $k_4 = 5$, $\alpha_4 = 0.6$, $k_5 = 6$, $\alpha_5 = 0.7$, $U = [1,0,0,0,1,0]$. Also, $\gamma_1 = 0.3$, $\gamma_2 = 0.2$ and $UU = [0,0,1,0,0,1]$. Then, greedy digits are $\{0, 0.4, 1.4, 1.9, 2.2, 2.8\}$, lazy digits are $\{-0.6, 0.4, 0.9, 1.2, 1.8, 2.5\}$ and the digits are $A = \{-0.6, 0.4, 1.1, 1.9, 1.8, 2.6\}$. We have $\beta = 3.5$ and $c_1 = 0$, $c_2 = 0.4$, $c_3 \simeq 0.54$, $c_4 = (c_4, 4) \simeq 0.63$, $c_5 = (c_5, 5) \simeq 0.63$, $c_6 = 0.8$, $c_7 = 1$. $c_3, c_4, c_7 \in W_g$ and the four others are in W_l. Using Maple 11 we calculated $D_1 \simeq -0.613$, $D_2 \simeq -1.076$, $D_3 \simeq -1.554$, $D_4 = D_5 \simeq -1.652$, $D_6 \simeq -1.226$, $D_7 \simeq -0.826$. The normalizing constant is $\simeq -0.161$. Map τ is shown in Figure 7 a) and the normalized density h in Figure 8 a).

Example 10: Let $N = 9$, $K = 9$, $L = 3$, $k_i = i$ for $i = 1, \dots, 9$, $\alpha_1 = \alpha_2 = \alpha_3 = 0.3$, $\alpha_4 = \alpha_5 = \alpha_6 = 0.2$, $\alpha_7 = \alpha_8 = \alpha_9 = 0.5$, $U = [0,0,0,0,0,0,1,1,1]$. Also, $\gamma_1 = \gamma_2 = \gamma_3 = 0.3$ and $UU = [0,0,0,1,1,1,0,0,0]$.

Then, greedy digits are $\{0, 0.3, 0.6, 0.9, 1.1, 1.3, 1.5, 2.0, 2.5\}$, lazy digits are $\{-0.7, -0.4, -0.1, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0\}$ and the digits are $A = \{0, 0.3, 0.6, 0.6, 0.8, 1.0, 1.0, 1.5, 2.0\}$. We have $\beta = 3$ and $c_1 = 0.1$, $c_2 = 0.2$, $c_3 = (c_3, 3) = 0.3$, $c_4 = (c_4, 4) = 0.3$, $c_5 = (c_5, 4) \simeq 0.367$, $c_6 = (c_6, 5) \simeq 0.367$, $c_7 = (c_7, 5) \simeq 0.433$, $c_8 = (c_8, 6) \simeq 0.433$, $c_9 = (c_9, 6) = 0.5$, $c_{10} = (c_{10}, 7) = 0.5$, $c_{11} =\simeq 0.667$, $c_{12} \simeq 0.833$. We have $c_1, c_2, c_3, c_5, c_7, c_9 \in W_g$ and the six others are in W_l. τ has 3 ergodic components and $1/3$ is an eigenvalue of matrix $\mathbf{S}$. Lebesgue measure is obviously invariant. Map τ is shown in Figure 7 b).

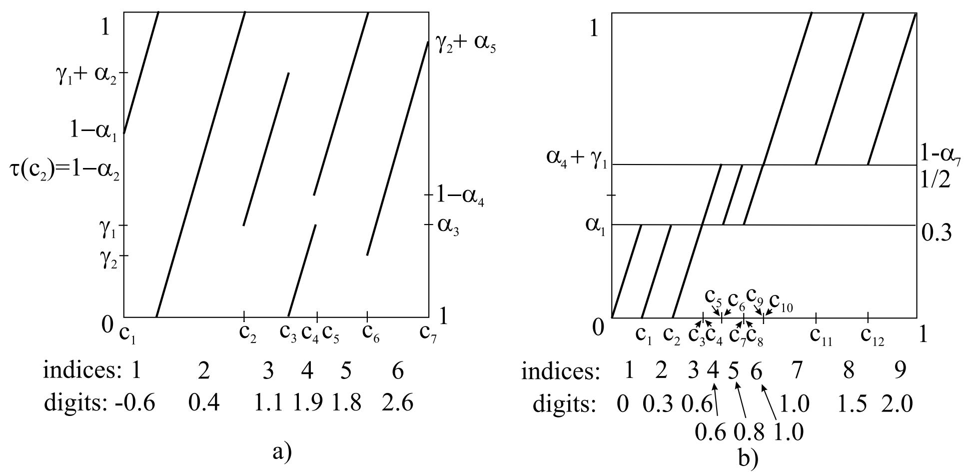

FIGURE 7. Maps of a) Example 9 and b) Example 10.

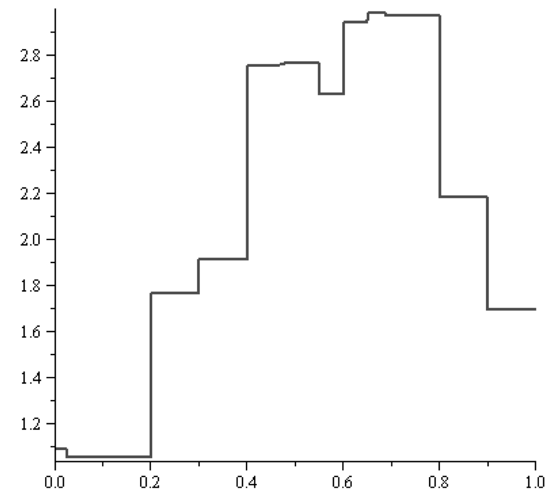

FIGURE 8. Invariant density for τ of Example 9.

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