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Invariant densities for piecewise linear maps of the unit interval

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Abstract. We find an explicit formula for the invariant density h of an arbitrary eventually expanding piecewise linear map τ of an interval $[0,1]$. We do not assume that the slopes of the branches are the same and we allow arbitrary number of shorter branches touching 0 or touching 1 or hanging in between. The construction involves matrix $\mathbf{S}$ which is defined in a way somewhat similar to the definition of the kneading matrix of a continuous piecewise monotonic map. Under some additional assumptions, we prove that if 1 is not an eigenvalue of $\mathbf{S}$, then dynamical system $(\tau, h\cdot m)$ is ergodic with full support.

1. *Introduction*

In this paper we continue the investigations of invariant densities (with respect to Lebesgue measure m) for piecewise linear maps of an interval. The first results about the classical β-maps were obtained by Rényi [22], Parry [19] and Gelfond [9]. Later, Parry generalized them further [20]. These maps have constant slope, all the branches are increasing and only the first or the last (or both) branches can be shorter.

The maps with both increasing and decreasing branches were investigated in [10]. Again, these maps have constant slope (in modulus) and shorter branches were allowed only as the first or the last one.

In this paper we consider arbitrary piecewise linear maps τ of $[0,1]$ onto itself. We do not assume that the slopes of the branches are the same and allow arbitrary number of shorter branches touching 0 or touching 1 or hanging in between. We assume that τ is onto and that it is eventually piecewise expanding, i.e., for some iterate $|(\tau^n)'| > 1$, wherever it exists.

In our main result, Theorem 2, we find an explicit formula for τ-invariant density h.

The construction of τ-invariant density h involves a matrix $\mathbf{S}$ defined in a way somewhat similar to the definition of the kneading matrix of a continuous piecewise monotonic map [1, 17]. In some simple cases, e.g., for greedy maps, we proved that if 1 is not an eigenvalue of $\mathbf{S}$, then dynamical system $(\tau, h \cdot m)$ is ergodic on $[0, 1]$. During the work on this paper we performed a great number of computer experiments, and we found that for piecewise increasing maps this always holds. Therefore, we state the following conjecture.

Conjecture 1: Let τ be piecewise linear, piecewise increasing and eventually piecewise expanding map. Then, 1 is not an eigenvalue of matrix $\mathbf{S} \Longrightarrow$ dynamical system $(\tau, h \cdot m)$ is ergodic on $[0, 1]$.

If τ is piecewise linear and eventually piecewise expanding map then we conjecture: 1 is not an eigenvalue of matrix $\mathbf{S} \Longrightarrow$ dynamical system $(\tau, h \cdot m)$ is ergodic.

There are matrix methods of detecting topological transitivity of piecewise monotone continuous interval maps [1, 17], which is implied by ergodicity for our class of maps. Perhaps, matrix $\mathbf{S}$ can be used for this purpose in a more general setting.

In Example 8 we construct not piecewise increasing map τ which is ergodic on a strict subset of $[0, 1]$ and whose matrix $\mathbf{S}$ does not have eigenvalue 1.

The converse of Conjecture 1 does not hold. It is shown in Example 5.

There are few papers dealing with absolutely continuous invariant measures of piecewise linear maps. The most general of them is Kopf's paper [13]. The author learned about Kopf's work after the previous version of this paper, containing results for piecewise linear, piecewise increasing maps was submitted for publication. Kopf's and our methods are related but different. The main differences are:

(a) Kopf makes a restrictive assumption $\tau(\{0, 1\}) \subset \{0, 1\}$. This is important for his method since it is based on comparing the behaviour of τ on both sides of its discontinuity points. For 0 or 1 there is no other side so these points have to behave like continuity points. As he points out, the general result can be obtained from this one by rescaling and adding extra branches to the map τ.

(b) Kopf uses all inner partition points for the construction of invariant density and of the system of equations defining its coefficients. We use only points whose images are different from 0 and 1, obtaining a more compact formula for the invariant density.

(c) To show solvability of the system defining the coefficients of the invariant density, Kopf uses a geometric property that straight lines which are the extensions of branches of τ do not intersect in a one point on the diagonal. To justify this he assumes $|\tau'| > 1$. If τ admits this property, then any iterate τ^n also has it. Thus, the eventual piecewise expanding implies it as well so this is not a restrictive assumption.

In contrast, we introduce τ-expansion of numbers and use it to reduce our system defining the coefficients of the invariant density and to prove its solvability.

(d) Kopf obtains all invariant densities, while our method usually gives only one version of invariant density for each ergodic component. This is not a real deficiency of our method as an arbitrary invariant density is a convex combination of these building blocks.

Other papers on the topic deal with more restrictive subclasses of piecewise linear maps. We mention papers which are related to our method. Absolutely continuous invariant measures for greedy maps with constant slope were investigated in [6, 7] by considering natural extensions of these maps. A two branches expanding-contracting (α, β)-maps were considered in [3]. These maps are eventually piecewise expanding so they are included in our model. This follows from Theorem (3.1) of [12] for $\beta \leq 2$ and from [8] for larger β.

In Section 2 we define all necessary notions and introduce τ-expansion of numbers in $[0, 1]$ related to our map τ. It is crucial in the considerations of this paper. Similar expansions were considered before under more restrictive assumptions. We followed mainly the ideas of Pedicini [21] who studied so called "greedy" expansions with deleted digits. More general expansions were studied in [5] which we recommend for further information and references.

In Section 3 we prove the main theorem giving the form of τ-invariant density.

In Section 4 we discuss the ergodic properties of piecewise linear maps.

In the next four sections we discuss special cases: piecewise increasing maps, greedy maps for which shorter branches touch 0, lazy maps with shorter branches touching 1 and the mixed type maps with shorter branches touching either 0 or 1 but not hanging in between. We prove a number of results which hold specifically for these classes. In particular, in Section 6 we discuss special cases of greedy maps with 2,3 or 4 branches.

In the last section we present an alternative method of finding invariant density for a general eventually piecewise expanding piecewise linear map τ. Using "Hofbauer's trick" we construct a piecewise increasing map τ_{inc} such that the original map τ is its 2-factor.

In this paper we are mainly interested in absolutely continuous τ-invariant measure. The general theory of such measures for piecewise expanding maps of an interval is well developed and we often refer to its results. The classical papers are [15] and [16] among many others. There is a number of books on the subject, see, e. g., [2] or [14].

While working on this project the author used extensively the computer program Maple 11. The programs with examples and illustrations, as well as their pdf printouts, are available at http://www.mathstat.concordia.ca/faculty/pgora/deleted .

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2. *Description of map τ and τ-expansion.*

In this section we introduce necessary notation and describe the maps we consider. We also introduce τ-expansion of numbers, crucial in the further considerations.

Throughout the paper δ(condition) will denote 1 when the condition is satisfied and 0 otherwise. We denote Lebesgue measure on $[0,1]$ by m.

Let τ be a piecewise linear map of interval $[0,1]$ onto itself. Let N denote the number of branches of τ and $K \leq N$ the number of shorter, not onto, branches. We allow $L \leq K$ shorter branches not to touch both 0 and 1. We will call them "hanging" branches.

The map τ can be described by three sequences of N numbers: the lengths of branches $\alpha_1, \alpha_2, \dots, \alpha_N$, with $0 < \alpha_j \leq 1$, $j = 1, \dots, N$; the heights of the lower endpoints of branches, i.e., the heights of left hand side endpoints for increasing and the heights of right hand side endpoints for decreasing branches, $\gamma_1, \gamma_2, \dots, \gamma_N$, with $0 \leq \gamma_j \leq 1 - \alpha_j$, $j = 1, \dots, N$; and the slopes of branches $\beta_1, \beta_2, \dots, \beta_N$. We assume $\beta_j \neq 0$, $j = 1, \dots, N$, and we have

$$\frac{\alpha_1}{|\beta_1|} + \frac{\alpha_2}{|\beta_2|} + \cdots + \frac{\alpha_N}{|\beta_N|} = 1 \ . \tag{1}$$

We do not assume that $1 < |\beta_i|$ but we assume that τ is eventually piecewise expanding, i.e., for some iterate τ^n we have $|(\tau^n)'| > 1$, whenever it is defined. This is necessary for the convergence of the series we consider below.

A shorter branch is called "greedy" if corresponding $\gamma_j = 0$, "lazy" if $\gamma_j + \alpha_j = 1$ and "hanging" if $0 < \gamma_j$ and $\gamma_j + \alpha_j < 1$. These names correspond to the names of piecewise increasing maps with such types of branches [5].

The endpoints of the domains of branches are $b_1 = 0$, $b_j = \frac{\alpha_1}{|\beta_1|} + \cdots + \frac{\alpha_{j-1}}{|\beta_{j-1}|}$, $j = 2, 3, \dots, N+1$. Note, $b_{N+1} = 1$.

We assume that the map τ is defined on the partition $\mathcal{P}_\tau = \{I_1, I_2, \dots, I_N\}$, where

$$\begin{aligned} I_1 &= [0, b_2) \ ; \\ I_j &= (b_j, b_{j+1}) \quad \text{for} \quad 2 \leq j \leq N-1 \ ; \\ I_N &= (b_N, 1] \ . \end{aligned} \tag{2}$$

This means that τ is not defined for a countable subset of $[0,1]$, the points b_j, $j = 2, \dots, N$ and their preimages. Since we will have to consider iterates of the points b_j we create two extensions τ_r (right) and τ_l (left) of τ. τ_r is the extension of τ by continuity to partition

$$\mathcal{P}_r = \{[0, b_2], (b_2, b_3], \dots, (b_{N-1}, b_N], (b_N, 1]\},$$

and τ_l is the extension of τ by continuity to partition

$$\mathcal{P}_l = \{[0, b_2), [b_2, b_3), \dots, [b_{N-1}, b_N), [b_N, 1]\}.$$

Now, we define the points c_i, $i = 1, 2, \dots, K+L$ which will play major role in the further study. They are the endpoints of the domains of shorter branches at which τ does not touch 0 or 1. Since a point can be the endpoint of two such domains we have to allow for duplication of them.

Each point c_i is actually a pair (c, j) where $c \in [0,1]$ and $1 \leq j \leq N$ and c is one of the endpoints of interval I_j. We define index function on points c_i: $j(c_i, k) = k$. We define $K + L$ points c_i. They are:

the right hand side endpoints of domains of shorter increasing branches touching 0 and the left hand side endpoints of domains of shorter decreasing branches touching 0 ("greedy" branches);

the left hand side endpoints of of domains of shorter increasing branches touching 1 and right hand side endpoints of of domains of shorter decreasing branches touching 1 ("lazy" branches);

both endpoints of domains of shorter "hanging" branches.

We number them in such a way that $c_1 < c_2 < \cdots < c_{K+L-1} < c_{K+L}$, where $(c,j) < (d,k)$ if either $c < d$ or $c = d$ and $j < k$. Note, the indices "i" of points c_i do not correspond directly to indices of intervals I_j.

We group c_i's into two disjoint sets: W_u containing "upper" c_i's associated with "greedy" branches, right hand side endpoints of domains of "hanging" increasing branches and left hand side endpoints of domains of "hanging" decreasing branches; W_l containing "lower" c_i's associated with "lazy" branches, left hand side endpoints of domains of "hanging" increasing branches and right hand side endpoints of domains of "hanging" decreasing branches.

We also group c_i's into "left" points U_l and "right" points U_r in obvious way. For piecewise increasing maps these two grouping coincide as then $W_u = U_r$ and $W_l = U_l$.

When we consider $\tau(c_i)$ we apply it to the first element of the pair. We always use τ_r to act on elements of U_r and τ_l to act on elements of U_l. Note,

$$\begin{aligned}
\tau(c_i) &= \tau_l(c_i) = \gamma_j && \text{for} \quad c_i \in U_l \cap W_l \ , \\
\tau(c_i) &= \tau_r(c_i) = \gamma_j && \text{for} \quad c_i \in U_r \cap W_l \ , \\
\tau(c_i) &= \tau_l(c_i) = \gamma_j + \alpha_j && \text{for} \quad c_i \in U_l \cap W_u \ , \\
\tau(c_i) &= \tau_r(c_i) = \gamma_j + \alpha_j && \text{for} \quad c_i \in U_r \cap W_u \ ,
\end{aligned}$$

where always $j = j(c_i)$.

Map τ can be conveniently represented using a set of "digits" $A = \{a_1, a_2, \ldots, a_N\}$, where

$$\begin{aligned}
&\text{if} \ \ \beta_j > 0, \ \ \text{then} \ \ a_j = \beta_j b_j - \gamma_j = \beta_j b_{j+1} - (\gamma_j + \alpha_j) \ , \\
&\text{if} \ \ \beta_j < 0, \ \ \text{then} \ \ a_j = \beta_j b_j - (\gamma_j + \alpha_j) = \beta_j b_{j+1} - \gamma_j \ , \qquad j = 1, \ldots, N \ .
\end{aligned}$$

Then, map τ is

$$\tau(x) = \beta_j \cdot x - a_j, \qquad \text{for} \qquad x \in I_j \ , \ j = 1, 2, \ldots, N \ .$$

Note that each a_j is between the minimal, "lazy" digit $a_j^l = \beta_j b_{j+1} - 1$ for $\beta_j > 0$ or $a_j^l = \beta_j b_j - 1$ for $\beta_j < 0$ and the maximal, "greedy" digit $a_j^u = \beta_j b_j$ for $\beta_j > 0$ or $a_j^u = \beta_j b_{j+1}$ for $\beta_j < 0$, $j = 1, 2, \ldots, N$. If the jth branch is onto, then $a_j = a_j^l = a_j^u$.

For any $x \in [0,1] \setminus \{b_2, \ldots, b_N\}$ we define its "index" $j(x)$ and its "digit" $a(x)$:

$$j(x) = j \qquad \text{for} \quad x \in I_j \ , \ j = 1, 2, \ldots, N \ ,$$

and

$$a(x) = a_{j(x)} .$$

We also define (for all $x \in [0,1]$) the indices $j_r(x)$, $j_l(x)$ and the digits $a_r(x)$, $a_l(x)$ using partitions $\mathcal{P}_r$ and $\mathcal{P}_l$, correspondingly.

We define the cumulative slopes for iterates of points as follows:

$$\beta(x,1) = \beta_{j(x)} \ ;$$
$$\beta(x,n) = \beta(x,n-1) \cdot \beta_{j(\tau^{n-1}(x))} \ \ , \ \ n \geq 2 \ .$$

The following proposition describes τ-expansion of numbers in $[0,1]$. It is similar to many known expansions, in particular to β-expansion [19] and "greedy" and "lazy" expansions with deleted digits [5].

PROPOSITION 1. *If τ is eventually expanding, then for any $x \in [0,1] \setminus \{b_2, \dots, b_N\}$ we have*

$$x = \sum_{n=1}^{\infty} \frac{a(\tau^{n-1}(x))}{\beta(x,n)} \ .$$

Moreover,

$$\tau^k(x) = \beta(x,k) \cdot \sum_{n=k+1}^{\infty} \frac{a(\tau^{n-1}(x))}{\beta(x,n)} \ ,$$

for any $k \geq 0$.

Proof: We have $\tau(x) = \beta_{j(x)} x - a(x)$ or

$$x = \frac{a(x)}{\beta(x,1)} + \frac{\tau(x)}{\beta(x,1)} \ .$$

Using this equality inductively n-times we obtain

$$x = \frac{a(x)}{\beta(x,1)} + \frac{a(\tau(x))}{\beta(x,2)} + \dots + \frac{a(\tau^{n-1}(x))}{\beta(x,n)} + \frac{\tau^n(x)}{\beta(x,n)} \ ,$$

which proves both statements. Since τ is eventually expanding $1/\beta(x,n) \to 0$ as $n \to +\infty$ and the series giving the expansion is convergent. □

We will call the representation defined in Proposition 1 the τ-expansion of x. In the same way we define "greedy" (or rather "right") and "lazy" (or "left") expansions using maps τ_r and τ_l. All three expansions are identical for almost all $x \in [0,1]$. Special attention has to be paid to represent points c_i or inner endpoints of the partition in general. Every time iteration of a c_i goes through a decreasing branch its side type changes. In a typical situation it does not matter but it is important when an image of a c_i hits one of the inner endpoints of the partition. If $c_i \in U_r$, then $\tau^n(c_i)$ is the right hand side endpoint of a small image interval if $\beta(c_i,n) > 0$ and $\tau^n(c_i)$ is the left hand side endpoint of a small image interval if $\beta(c_i,n) < 0$. Similarly, if $c_i \in U_l$ then, $\tau^n(c_i)$ is the left hand side endpoint of a small image interval if $\beta(c_i,n) > 0$ and $\tau^n(c_i)$ is the right hand side endpoint of a small image interval if $\beta(c_i,n) < 0$. To continue iteration, we should use appropriate version of τ.

3. *Invariant density of τ.*

An integrable nonnegative function h is a density of an m-absolutely continuous τ-invariant measure if and only if it satisfies Perron-Frobenius equation:

$$h(x) = \sum_{y:\tau(y)=x} h(y)/|\tau'(y)| = (P_\tau(h))(x),$$

for almost all $x \in [0,1]$. Operator P_τ is called Perron-Frobenius operator [2].

Let us define

$$S_{i,\,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i,n)|} \left[\delta(\beta(c_i,n)>0)\delta(\tau^n(c_i)>c_j) + \delta(\beta(c_i,n)<0)\delta(\tau^n(c_i)<c_j)\right] ,$$
$$\text{for} \quad c_i \in U_r \quad \text{and all} \quad c_j \ ,$$
$$S_{i,\,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i,n)|} \left[\delta(\beta(c_i,n)<0)\delta(\tau^n(c_i)>c_j) + \delta(\beta(c_i,n)>0)\delta(\tau^n(c_i)<c_j)\right] ,$$
$$\text{for} \quad c_i \in U_l \quad \text{and all} \quad c_j \ . \tag{3}$$

Let $\mathbf{S}$ be the matrix $(S_{i,j})_{1\le i,j\le K+L}$ and $\mathbf{Id}$ denote $(K+L)\times(K+L)$ identity matrix. Let $\mathbf{v} = [1,1,\ldots,1,1]$ be $(K+L)$-dimensional vector of 1's and let $D = [D_1,\ldots,D_{K+L}]$ denote the solution of the system

$$(-\mathbf{S}^T + \mathbf{Id})D^T = D_0\mathbf{v}^T \ , \tag{4}$$

where ^{T} denotes the transposition and parameter D_0 is either 1 or 0. We make here some comments about the parameter D_0 although their meaning may become clear only later. Since the non-normalized invariant density (6) is defined up to a multiplicative constant we consider only $D_0 = 1$ or $D_0 = 0$. In most cases we will use $D_0 = 1$. There may be a few reasons for the equation (4) to be unsolvable with $D_0 = 1$. First, τ can be ergodic but with support of invariant density I strictly smaller that $[0,1]$. In this case we consider τ restricted to I and rescaled back to $[0,1]$ rather than considering $D_0 = 0$. Secondly, τ may be either ergodic on $[0,1]$ or nonergodic with union of supports of invariant densities equal to $[0,1]$ but with matrix $\mathbf{S}$ having 1 as an eigenvalue. In these cases we consider $D_0 = 0$.

Let us define

$$\chi^s(\beta,x) = \begin{cases} \chi_{[0,x]} \ , & \text{for} \quad \beta > 0 \ , \\ \chi_{[x,1]} \ , & \text{for} \quad \beta < 0 \ . \end{cases} \tag{5}$$

THEOREM 2. *Let τ will be the map defined in the previous section, i.e., any piecewise linear map which is eventually piecewise expanding. System (4) always has a non-vanishing solution. If 1 is not an eigenvalue of $\mathbf{S}$, then with $D_0 = 1$. If 1 is an eigenvalue of $\mathbf{S}$, then at least with $D_0 = 0$. Let*

$$h = D_0 + \sum_{c_i\in U_r} D_i \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_i,n),\tau^n(c_i))}{|\beta(c_i,n)|} + \sum_{c_i\in U_l} D_i \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_i,n),\tau^n(c_i))}{|\beta(c_i,n)|} \ , \tag{6}$$

where constants D_i, $i=1,\dots,K$, satisfy the system (4). Then h is τ-invariant.

If all values $\tau(c_i)$, $i=1,\dots,K+L$, are different, then the inverse statement also holds: If h is τ-invariant, then the constants $D_0, D_1, \dots, D_{K+L}$ satisfy the system (4).

In particular, system (4) is uniquely solvable (i.e., 1 is not an eigenvalue of $\mathbf{S}$) if $\min_{1\le j\le N}|\beta_j| > K+L+1$.

Proof: Let $x\in[0,1]$ and $x(j)$, $j=1,2,\dots,N$ be the jth τ-preimage of x, if it exists. We need to show that

$$h(x)=\sum_{j=1}^{N}\frac{h(x(j))}{|\beta_j|}\ ,$$

for almost all $x\in[0,1]$.

We have

$$\sum_{j=1}^{N}\frac{1(x(j))}{|\beta_j|}=\sum_{j=1}^{N}\frac{1}{|\beta_j|}-\sum_{c_k\in W_u}\frac{\delta(x>\tau(c_k))}{|\beta_{j(c_k)}|}-\sum_{c_k\in W_l}\frac{\delta(x<\tau(c_k))}{|\beta_{j(c_k)}|}\ . \tag{7}$$

For $c_k\in U_r\cup U_l$ we have

$$\begin{aligned}\sum_{j=1}^{N}\frac{\chi_{[0,\tau^n(c_k)]}(x(j))}{|\beta_j|} &= \sum_{j=1}^{j(\tau^n(c_k))-1}\frac{1}{|\beta_j|}+\frac{\delta(\beta_{j(\tau^n(c_k))}>0)\delta(x\le\tau^{n+1}(c_k))}{|\beta_{j(\tau^n(c_k))}|}\\ &\quad+\frac{\delta(\beta_{j(\tau^n(c_k))}<0)\delta(x\ge\tau^{n+1}(c_k))}{|\beta_{j(\tau^n(c_k))}|}\\ -\sum_{c_i\in W_u}\frac{\delta(x>\tau(c_i))\cdot\delta(\tau^n(c_k)>c_i)}{|\beta_{j(c_i)}|}&-\sum_{c_i\in W_l}\frac{\delta(x<\tau(c_i))\cdot\delta(\tau^n(c_k)>c_i)}{|\beta_{j(c_i)}|}\ ,\end{aligned} \tag{8}$$

and

$$\begin{aligned}\sum_{j=1}^{N}\frac{\chi_{[\tau^n(c_k),1]}(x(j))}{|\beta_j|} &= \sum_{j=j(\tau^n(c_k))+1}^{N}\frac{1}{|\beta_j|}+\frac{\delta(\beta_{j(\tau^n(c_k))}>0)\delta(x\ge\tau^{n+1}(c_k))}{|\beta_{j(\tau^n(c_k))}|}\\ &\quad+\frac{\delta(\beta_{j(\tau^n(c_k))}<0)\delta(x\le\tau^{n+1}(c_k))}{|\beta_{j(\tau^n(c_k))}|}\\ -\sum_{c_i\in W_u}\frac{\delta(x>\tau(c_i))\cdot\delta(\tau^n(c_k)<c_i)}{|\beta_{j(c_i)}|}&-\sum_{c_i\in W_l}\frac{\delta(x<\tau(c_i))\cdot\delta(\tau^n(c_k)<c_i)}{|\beta_{j(c_i)}|}\ .\end{aligned} \tag{9}$$

Let us define

$$
\begin{aligned}
S_k &= \sum_{n=1}^{\infty} \frac{1}{|\beta(c_k,n)|}\Big[\delta(\beta(c_k,n)>0)\Big(\sum_{j=1}^{j(\tau^n(c_k))-1} \frac{1}{|\beta_j|}\Big) \\
&\qquad + \delta(\beta(c_k,n)<0)\Big(\sum_{j=j(\tau^n(c_k))+1}^{N} \frac{1}{|\beta_j|}\Big)\Big] , \quad \text{for } \ c_k \in U_r \ , \\
S_k &= \sum_{n=1}^{\infty} \frac{1}{|\beta(c_k,n)|}\Big[\delta(\beta(c_k,n)<0)\Big(\sum_{j=1}^{j(\tau^n(c_k))-1} \frac{1}{|\beta_j|}\Big) \\
&\qquad + \delta(\beta(c_k,n)>0)\Big(\sum_{j=j(\tau^n(c_k))+1}^{N} \frac{1}{|\beta_j|}\Big)\Big] , \quad \text{for } \ c_k \in U_l \ .
\end{aligned} \tag{10}
$$

Using previous equalities and $\beta_{j(\tau^n(c_k))}\cdot\beta(c_k,n)=\beta(c_k,n+1)$, we write

$$
\begin{aligned}
\sum_{j=1}^{N} \frac{h(x(j))}{|\beta_j|} &= D_0\Big[\sum_{j=1}^{N}\frac{1}{|\beta_j|} - \sum_{c_k\in W_u}\frac{\delta(x>\tau(c_k))}{|\beta_{j(c_k)}|} - \sum_{c_k\in W_l}\frac{\delta(x<\tau(c_k))}{|\beta_{j(c_k)}|}\Big] \\
&+ \sum_{c_k\in U_r} D_k\Big[S_k + \sum_{n=2}^{\infty}\frac{\chi^s(\beta(c_k,n),\tau^n(c_k))}{|\beta(c_k,n)|} \\
&\qquad - \sum_{c_i\in W_u}\delta(x>\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|} - \sum_{c_i\in W_l}\delta(x<\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|}\Big] \\
&+ \sum_{c_k\in U_l} D_k\Big[S_k + \sum_{n=2}^{\infty}\frac{\chi^s(-\beta(c_k,n),\tau^n(c_k))}{|\beta(c_k,n)|} \\
&\qquad - \sum_{c_i\in W_u}\delta(x>\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|} - \sum_{c_i\in W_l}\delta(x<\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|}\Big] \ .
\end{aligned} \tag{11}
$$

To eliminate $h(x)$ from the right hand side of (11) we need to add to it and to subtract from it D_0, $\sum_{c_k\in U_r} D_k \frac{\chi^s(\beta(c_k,1),\tau(c_k))}{|\beta(c_k,1)|}$ and $\sum_{c_k\in U_l} D_k \frac{\chi^s(-\beta(c_k,1),\tau(c_k))}{|\beta(c_k,1)|}$. Note,

$$
\begin{aligned}
&\text{for} \quad c_k\in U_r\cap W_u \quad \text{we have} \quad \chi^s(\beta(c_k,1),\tau(c_k)) = \delta(x\le\tau(c_k)) \ , \\
&\text{for} \quad c_k\in U_l\cap W_u \quad \text{we have} \quad \chi^s(-\beta(c_k,1),\tau(c_k)) = \delta(x\le\tau(c_k)) \ , \\
&\text{for} \quad c_k\in U_r\cap W_l \quad \text{we have} \quad \chi^s(\beta(c_k,1),\tau(c_k)) = \delta(x\ge\tau(c_k)) \ , \\
&\text{for} \quad c_k\in U_l\cap W_l \quad \text{we have} \quad \chi^s(-\beta(c_k,1),\tau(c_k)) = \delta(x\ge\tau(c_k)) \ .
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\sum_{c_k\in U_r} D_k\frac{\chi^s(\beta(c_k,1),\tau(c_k))}{|\beta(c_k,1)|} + \sum_{c_k\in U_l} D_k\frac{\chi^s(-\beta(c_k,1),\tau(c_k))}{|\beta(c_k,1)|} &= \\
\sum_{c_k\in W_u} D_k\frac{\delta(x\le\tau(c_k))}{|\beta(c_k,1)|} + \sum_{c_k\in W_l} D_k\frac{\delta(x\ge\tau(c_k))}{|\beta(c_k,1)|} \ .
\end{aligned}
$$

We eliminate $h(x)$ from the right hand side of (11) and we see that we are looking for constants D_i, $i=1,\dots,K+L$, such that the following equality (12) is satisfied for all $x\in[0,1]$ except possibly the images of points c_i.

$$\begin{aligned}
&\sum_{c_k\in W_u} D_k\Big[S_k-\frac{\delta(x\le\tau(c_k))}{|\beta(c_k,1)|}\\
&\qquad\qquad -\sum_{c_i\in W_u}\delta(x>\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|}-\sum_{c_i\in W_l}\delta(x<\tau(c_i))\frac{S_{k,i}}{\beta_{j(c_i)}}\Big]\\
&+\sum_{c_k\in W_l} D_k\Big[S_k-\frac{\delta(x\ge\tau(c_k))}{|\beta(c_k,1)|}\\
&\qquad\qquad -\sum_{c_i\in W_u}\delta(x>\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|}-\sum_{c_i\in W_l}\delta(x<\tau(c_i))\frac{S_{k,i}}{|\beta_{j(c_i)}|}\Big]\\
&=D_0\Big[1-\sum_{j=1}^{N}\frac{1}{|\beta_j|}+\sum_{c_k\in W_u}\frac{\delta(x>\tau(c_k))}{|\beta_{j(c_k)}|}+\sum_{c_k\in W_l}\frac{\delta(x<\tau(c_k))}{|\beta_{j(c_k)}|}\Big]\ .
\end{aligned}\tag{12}$$

Let us assume tentatively that all values $\tau(c_i)$, $i=1,\dots,K+L$, are different. Then, they divide interval $(0,1)$ into $K+L+1$ disjoint open subintervals. Let us chose one point x from each of the subintervals and number them in the increasing order $x_0<x_1<x_2<\dots<x_{K+L}$. If equality (12) holds for these points, then it holds for almost every $x\in[0,1]$. Substituting points x_i into (12) we obtain equations which we denote by E_i, $i=0,\dots,K+L$. Together, we obtain system of $K+L+1$ equations which we denote by ES. Rather than write it down we create from it a simplified equivalent system denoted by EQS. We proceed as follows: consider two consecutive points $x_i<\tau(c_k)<x_{i+1}$. If $c_k\in W_u$, then the difference $EQ_k=E_{i+1}-E_i$ is

$$-\sum_{\substack{j=1\\ j\ne k}}^{K+L} D_j\frac{S_{k,j}}{|\beta_{j(c_k)}|}-D_k\Big[\frac{S_{k,k}}{|\beta_{j(c_k)}|}-\frac{1}{|\beta_{j(c_k)}|}\Big]=\frac{D_0}{|\beta_{j(c_k)}|}\ .\tag{13}$$

If $c_k\in W_l$, then the difference $EQ_k=E_i-E_{i+1}$ is of the above form. The equations $\{EQ_1,EQ_2,\dots,EQ_{K+L}\}$ form the system EQS which is obviously equivalent to the system $\{|\beta_{j(c_1)}|EQ_1,|\beta_{j(c_2)}|EQ_2,\dots,|\beta_{j(c_{K+L})}|EQ_{K+L}\}$, which is the system (4). In ES we have one more equation which can be reduced to EQ_{K+L+1} of the form

$$\sum_{k=1}^{K+L} D_k\Big[S_k-\frac{1}{|\beta_{j(c_k)}|}\Big]=D_0\Big[1-\sum_{j=1}^{N}\frac{1}{|\beta_j|}\Big]\ .\tag{14}$$

If some level x_i intersects all branches of τ, then equation E_i is of form (14). If not, then we take x_i which level intersects most branches of τ and reduce if to form (14) subtracting appropriate equations EQ_k.

The systems ES and $EQS\cup\{EQ_{K+L+1}\}$ are equivalent since we can recover equations of ES from equations $EQ_1,\dots,EQ_{K+L},EQ_{K+L+1}$. To prove the equivalence of systems ES and EQS it is enough to show that EQ_{K+L+1} is a linear combination of equations EQ_i, $i = 1, \dots, K+L$. We will do it as follows: If $c_k \in W_u$ we set $\eta_k = 1 - \gamma_{j(c_k)} - \alpha_{j(c_k)}$. If $c_k \in W_l$ we set $\eta_k = \gamma_{j(c_k)}$. Note that if c_k is associated with a greedy or a lazy branch, then $\eta_k = 1 - \alpha_{j(c_k)}$. Then, we have both left and right hand sides of

$$EQ_{K+L+1} + \sum_{k=1}^{K+L} \eta_k \cdot EQ_k \ ,$$

sum to zero.

First, let us consider the right hand side of the summed up equations. We have

$$\begin{aligned} 1 - \sum_{j=1}^{N} \frac{1}{|\beta_j|} + \sum_{k=1}^{K+L} \eta_k \frac{1}{|\beta_{j(c_k)}|} &= 1 - \sum_{j=1}^{N} \frac{1}{|\beta_j|} + \sum_{\substack{1 \le k \le N \\ k\text{-th branch is shorter}}} \frac{1-\alpha_k}{|\beta_k|} \\ &= 1 - \sum_{j=1}^{N} \frac{\alpha_j}{|\beta_j|} = 0 \ . \end{aligned} \tag{15}$$

Now, let us consider the summed up coefficients of D_k (summed up k-th column of the system). We have to show

$$S_k - \frac{1}{|\beta_{j(c_k)}|} - \sum_{j=1}^{K+L} \eta_j \frac{S_{k,j}}{|\beta_{j(c_j)}|} + \eta_k \frac{1}{|\beta_{j(c_k)}|} = 0 \ . \tag{16}$$

First, we consider $c_k \in U_r$. Then, we have

$$\begin{aligned} &S_k - \sum_{j=1}^{K+L} \eta_j \frac{S_{k,j}}{|\beta_{j(c_j)}|} \\ &= \sum_{n=1}^{\infty} \frac{1}{|\beta(c_k,n)|} \Bigg(\delta(\beta(c_k,n)>0) \Big[\sum_{j=1}^{j(\tau^n(c_k))-1} \frac{1}{|\beta_j|} - \sum_{j=1}^{K+L} \eta_j \frac{\delta(\tau^n(c_k) > c_j)}{|\beta_{j(c_j)}|} \Big] \\ &\qquad + \delta(\beta(c_k,n)<0) \Big[\sum_{j=j(\tau^n(c_k))+1}^{N} \frac{1}{|\beta_j|} - \sum_{j=1}^{K+L} \eta_j \frac{\delta(\tau^n(c_k) < c_j)}{|\beta_{j(c_j)}|} \Big] \Bigg) \ . \end{aligned} \tag{17}$$

Let us fix n for a moment and consider the expressions in the square brackets above. Let $j_0 = j(\tau^n(c_k))$. If $\beta_{j_0} > 0$, then the expression in the first square bracket is equal to

$$\sum_{j=1}^{j_0-1} \frac{1}{|\beta_j|} - \sum_{\substack{j<j_0 \\ j\text{-th branch is shorter}}} \frac{1-\alpha_{j(c_j)}}{|\beta_{j(c_j)}|} - \frac{\gamma_{j_0}}{|\beta_{j_0}|} = b_{j_0} - \frac{\gamma_{j_0}}{|\beta_{j_0}|} = \frac{a(\tau^n(c_k))}{\beta_{j(\tau^n(c_k))}} \ ,$$

and the expression in the second square bracket is equal to

$$\begin{aligned} &\sum_{j=j_0+1}^{N} \frac{1}{|\beta_j|} - \sum_{\substack{j>j_0 \\ j\text{-th branch is shorter}}} \frac{1-\alpha_{j(c_j)}}{|\beta_{j(c_j)}|} - \frac{1-\gamma_{j_0}-\alpha_{j_0}}{|\beta_{j_0}|} \\ &= 1 - b_{j_0+1} - \frac{1-\gamma_{j_0}-\alpha_{j_0}}{|\beta_{j_0}|} = 1 - \frac{1}{\beta_{j(\tau^n(c_k))}} - \frac{a(\tau^n(c_k))}{\beta_{j(\tau^n(c_k))}} \ . \end{aligned}$$

If $\beta_{j_0} < 0$, then these sums are correspondingly equal to

$$b_{j_0} - (1-\gamma_{j_0} - \alpha_{j_0})\frac{1}{|\beta_{j_0}|} = \frac{1}{\beta_{j(\tau^n(c_k))}} + \frac{a_{j(\tau^n(c_k))}}{\beta_{j(\tau^n(c_k))}} ,$$

and

$$1 - b_{j_0+1} - \gamma_{j_0}\frac{1}{|\beta_{j_0}|} = 1 - \frac{a_{j(\tau^n(c_k))}}{\beta_{j(\tau^n(c_k))}} .$$

Thus, using the notation $j_0^{(n)} = j(\tau^n(c_k))$, the sum on the right hand side of (17) is equal to

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{|\beta(c_k,n)|}\Bigg(\delta(\beta(c_k,n)>0)\Big[\ \delta(\beta_{j_0^{(n)}}>0)\frac{a_{j_0^{(n)}}}{\beta_{j_0^{(n)}}} + \delta(\beta_{j_0^{(n)}}<0)\frac{a_{j_0^{(n)}}}{\beta_{j_0^{(n)}}}\Big] \\
&\qquad\qquad + \delta(\beta(c_k,n)<0)\Big[\ \delta(\beta_{j_0^{(n)}}>0)\frac{-a_{j_0^{(n)}}}{\beta_{j_0^{(n)}}} + \delta(\beta_{j_0^{(n)}}<0)\frac{-a_{j_0^{(n)}}}{\beta_{j_0^{(n)}}}\Big]\Bigg) \\
&+\sum_{n=1}^{\infty}\frac{1}{|\beta(c_k,n)|}\Bigg(\delta(\beta(c_k,n)>0)\Big[\ \delta(\beta_{j_0^{(n)}}>0)\cdot 0 + \delta(\beta_{j_0^{(n)}}<0)\frac{1}{\beta_{j_0^{(n)}}}\Big] \\
&\qquad\qquad + \delta(\beta(c_k,n)<0)\Big[\ \delta(\beta_{j_0^{(n)}}>0)\Big(1-\frac{1}{\beta_{j_0^{(n)}}}\Big) + \delta(\beta_{j_0^{(n)}}<0)\cdot 1\Big]\Bigg) .
\end{aligned} \tag{18}$$

The first infinite sum equals to

$$\sum_{n=1}^{\infty}\frac{1}{\beta(c_k,n)}\frac{a(\tau^n(c_k))}{\beta_{j(\tau^n(c_k))}} = \sum_{n=1}^{\infty}\frac{a(\tau^n(c_k))}{\beta(c_k,n+1)} = \frac{\tau(c_k)}{\beta_{j(c_k)}} . \tag{19}$$

The second infinite sum is a telescopic sum equal to 0 if $\beta_{j(c_k)} > 0$ and to $\frac{-1}{\beta_{j(c_k)}}$ if $\beta_{j(c_k)} < 0$. We can prove it as follows. First, let us assume that $\beta(c_k,1) = \beta_{j(c_k)} > 0$. Then, the sum starts with the summand of the first kind, i.e., $\delta(\beta_{j_0^{(n)}} > 0)\cdot 0 + \delta(\beta_{j_0^{(n)}} < 0)\frac{1}{\beta_{j_0^{(n)}}}$. Let us assume, more generally, that at some point during the summation, the partial sum $S = 0$ and $\beta(c_k,n) > 0$. If $\beta_{j(\tau^n(c_k))} > 0$, then the next step starts in the same situation. If $\beta_{j(\tau^n(c_k))} < 0$, then we have

$$S = \frac{1}{\beta(c_k,n)\beta_{j(\tau^n(c_k))}} = \frac{1}{\beta(c_k,n+1)} ,$$

and $\beta(c_k,n+1) < 0$. Thus, the next summand is of the second type. If $\beta_{j(\tau^{n+1}(c_k))} < 0$, then we have

$$S = \frac{1}{\beta(c_k,n+1)} + \frac{1}{|\beta(c_k,n+1)|} = 0 ,$$

and $\beta(c_k,n+2) > 0$. We are back in the situation we started with.

If $\beta_{j(\tau^{n+1}(c_k))} > 0$, then

$$S = \frac{1}{\beta(c_k,n+1)} + \frac{1}{|\beta(c_k,n+1)|} + \frac{1}{\beta(c_k,n+2)} = \frac{1}{\beta(c_k,n+2)} ,$$

and $\beta(c_k, n+2) < 0$. The situation repeats until for some i we have $\beta_{j(\tau^{n+i}(c_k))} < 0$ and S reduces to 0. If this never happens, then the sum is 0 since $\lim_{i\to\infty} \frac{1}{\beta(c_k,n+i)} = 0$.

Now, let us assume that $\beta(c_k, 1) = \beta_{j(c_k)} < 0$. The second infinite sum in (18) starts with a summand of the second type. If $\beta_{j(\tau(c_k))} < 0$, then we have

$$S = \frac{1}{|\beta_{j(c_k)}|} \ ,$$

and $\beta(c_k, 2) > 0$. Reasoning exactly as in the previous case, we show that the complete sum is $\frac{1}{|\beta_{j(c_k)}|} = \frac{-1}{\beta_{j(c_k)}}$.

If $\beta_{j(\tau(c_k))} > 0$, then

$$S = \frac{1}{|\beta_{j(c_k)}|} + \frac{1}{\beta(c_k, 2)},$$

and $\beta(c_k, 2) < 0$. Again, this is one of the situations considered in the previous case. Again, we obtain that the complete sum equals $\frac{-1}{\beta_{j(c_k)}}$.

To complete the proof of (16) we consider sum $\frac{-1}{|\beta_{j(c_k)}|} + \eta_k \frac{1}{|\beta_{j(c_k)}|}$. If $\beta_{j(c_k)} > 0$, then

$$\frac{-1}{|\beta_{j(c_k)}|} + \eta_k \frac{1}{|\beta_{j(c_k)}|} = \frac{-(\gamma_{j(c_k)} + \alpha_{j(c_k)})}{\beta_{j(c_k)}} = \frac{-\tau(c_k)}{\beta_{j(c_k)}} \ . \tag{20}$$

If $\beta_{j(c_k)} < 0$, then

$$\frac{-1}{|\beta_{j(c_k)}|} + \eta_k \frac{1}{|\beta_{j(c_k)}|} = \frac{-1 + \gamma_{j(c_k)}}{|\beta_{j(c_k)}|} = \frac{1 - \tau(c_k)}{\beta_{j(c_k)}} \ . \tag{21}$$

Equalities (19), (20), (21) and the results about the second infinite sum in (18) complete the proof of (16) for $c_k \in U_r$. The proof for $c_k \in U_l$ is very similar.

We have proved the equivalence of the systems ES and EQS (or (4)) when all values $\tau(c_i)$, $i = 1, \ldots, K + L$, are different.

Now, we briefly describe the situation when some of the values $\tau(c_i)$, $i = 1, \ldots, K+L$, coincide. The systems ES and (4) may not be equivalent but solutions of (4) always satisfy ES as well.

If $\tau(c_{i_1}) = \tau(c_{i_2})$ and the points c_{i_1}, c_{i_2} are of different type, i.e., $c_{i_1} \in W_u$ and $c_{i_2} \in W_l$ or vice versa, then substituting point $x = \tau(c_{i_1}) = \tau(c_{i_2})$ into (12) gives us an equation which "separates" c_{i_1} and c_{i_2}. Everything proceeds as in the case of different values $\tau(c_i)$.

If $\tau(c_{i_1}) = \tau(c_{i_2})$ and the points c_{i_1}, c_{i_2} are of the same type (or there are more points with this property), then we cannot produce sufficient number of test points x_i and the number of equations in ES is smaller than $K+L+1$. Similarly as before we can obtain equations EQ_i for c_i with distinct values and an equation EQ_{i_1,i_2} corresponding to points c_{i_1}, c_{i_2}. If more groups of of c_i's of the same type with equal values occurs, then there will be more such common equations. The equation EQ_{i_1,i_2} is the sum of two equations of the form (13) corresponding to indices $k = i_1$ and $k = i_2$. Any common equation is a sum of the corresponding equations of the form (13). Thus, any solution of the system (4) satisfies the system ES. The linear dependence of the extra equation (14) is proved exactly as above. (Note that if $\tau(c_{i_1}) = \tau(c_{i_2})$ and they are of the same type, i.e., both in W_u or both in W_l, then $\eta_{i_1} = \eta_{i_2}$.) This completes the proof of the first part of the theorem.

In the proof of the second part we will use the following fragment of Perron-Frobenius theorem for non-negative matrices [18].

THEOREM 3. *If* $\mathbf{S} = (S_{i,j})_{1\le i,j\le M}$ *is a matrix with non-negative entries, then all eigenvalues* λ *of* $\mathbf{S}$ *satisfy*

$$|\lambda| \le \max_{1\le i\le M} \sum_{j=1}^{M} S_{i,j} \ . \tag{22}$$

Note that the assumptions of the second part imply that $\beta = \min_{1\le j\le N} |\beta_j| > 1$. For each $S_{i,j}$ we have

$$S_{i,j} \le \sum_{n=1}^{\infty} \frac{1}{\beta^n} = \frac{1}{\beta - 1} \ .$$

Thus, if $\beta > K + L + 1$ we have $\frac{K+L}{\beta-1} < 1$ which by Perron-Frobenius estimate implies that 1 is not an eigenvalue of $\mathbf{S}$ and the system (4) is uniquely solvable. □

In the three examples below we illustrate the proof of Theorem 2.

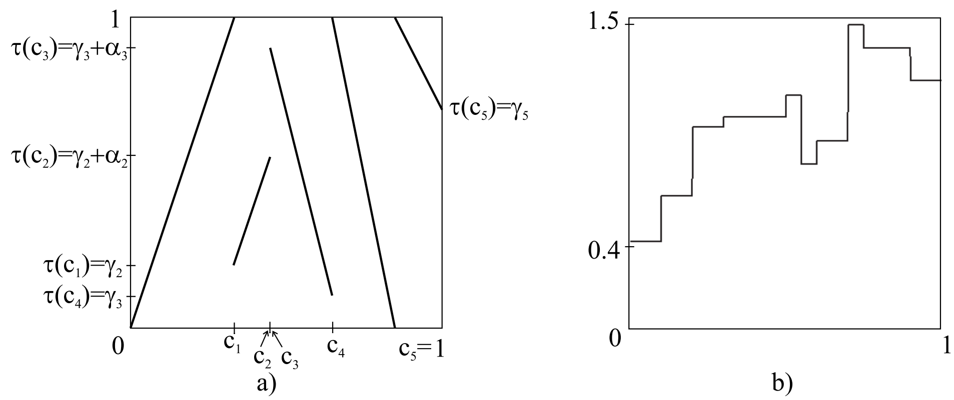

FIGURE 1. Map τ of Example 1 and its invarianr density.

Example 1: In this example all values $\tau(c_i)$ are different. Let $N = 5$ and let τ be defined by the vectors

$$\alpha = [1, 0.35, 0.8, 1, 0.3] \ \ , \ \ \beta = [3, 3, -4, -5, -2] \ \ , \ \ \gamma = [0, 0.2, 0.1, 0, 0.7] \ .$$

We have $K = 3$ and $L = 2$. The graph of τ is shown in Figure 1 a). The digits are $\{0, 0.8, -2.7, -4.25, -2.7\}$. The first and the fourth branches of τ is onto, the second and the third are hanging and the last one is lazy. The points c_i are $c_1 = 1/3$, $c_2 = 0.45 = (0.45, 2)$, $c_3 = 0.45 = (0.45, 3)$, $c_4 = 0.65$, $c_5 = 1$. $c_2, c_3 \in W_u$ and $c_1, c_4, c_5 \in W_l$. $c_1, c_3 \in U_l$ and $c_2, c_4, c_5 \in U_r$. We have $0 < \tau(c_4) < \tau(c_1) < \tau(c_2) < \tau(c_5) < \tau(c_3) < 1$ and taking the points

$x_0 < x_1 < x_2 < x_3 < x_4 < x_5$ between them we obtain the system ES: we show only the coefficients, the first three columns and the next three separately.

Columns 1–3:

$$\begin{array}{lll}
S_1 - \frac{S_{1,1}}{|\beta_2|} - \frac{S_{1,4}}{|\beta_3|} - \frac{S_{1,5}}{|\beta_5|} & S_2 - \frac{S_{2,1}}{|\beta_2|} - \frac{S_{2,4}}{|\beta_3|} - \frac{S_{2,5}}{|\beta_5|} - \frac{1}{|\beta_2|} & S_3 - \frac{S_{3,1}}{|\beta_2|} - \frac{S_{3,4}}{|\beta_3|} - \frac{S_{3,5}}{|\beta_5|} - \frac{1}{|\beta_3|} \\
S_1 - \frac{S_{1,1}}{|\beta_2|} - \frac{S_{1,5}}{|\beta_5|} & S_2 - \frac{S_{2,1}}{|\beta_2|} - \frac{S_{2,5}}{|\beta_5|} - \frac{1}{|\beta_2|} & S_3 - \frac{S_{3,1}}{|\beta_2|} - \frac{S_{3,5}}{|\beta_5|} - \frac{1}{|\beta_3|} \\
S_1 - \frac{S_{1,5}}{|\beta_5|} - \frac{1}{|\beta_2|} & S_2 - \frac{S_{2,5}}{|\beta_5|} - \frac{1}{|\beta_2|} & S_3 - \frac{S_{3,5}}{|\beta_5|} - \frac{1}{|\beta_3|} \\
S_1 - \frac{S_{1,2}}{|\beta_2|} - \frac{S_{1,5}}{|\beta_5|} - \frac{1}{|\beta_2|} & S_2 - \frac{S_{2,2}}{|\beta_2|} - \frac{S_{2,5}}{|\beta_5|} & S_3 - \frac{S_{3,2}}{|\beta_2|} - \frac{S_{3,5}}{|\beta_5|} - \frac{1}{|\beta_3|} \\
S_1 - \frac{S_{1,2}}{|\beta_2|} - \frac{1}{|\beta_2|} & S_2 - \frac{S_{2,2}}{|\beta_2|} & S_3 - \frac{S_{3,2}}{|\beta_2|} - \frac{1}{|\beta_3|} \\
S_1 - \frac{S_{1,2}}{|\beta_2|} - \frac{S_{1,3}}{|\beta_3|} - \frac{1}{|\beta_2|} & S_2 - \frac{S_{2,2}}{|\beta_2|} - \frac{S_{2,3}}{|\beta_3|} & S_3 - \frac{S_{3,2}}{|\beta_2|} - \frac{S_{3,3}}{|\beta_3|}
\end{array}$$

Columns 4–6:

$$\begin{array}{lll}
S_4 - \frac{S_{4,1}}{|\beta_2|} - \frac{S_{4,4}}{|\beta_3|} - \frac{S_{4,5}}{|\beta_5|} & S_5 - \frac{S_{5,1}}{|\beta_2|} - \frac{S_{5,4}}{|\beta_3|} - \frac{S_{5,5}}{|\beta_5|} & 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_4|} \\
S_4 - \frac{S_{4,1}}{|\beta_2|} - \frac{S_{4,5}}{|\beta_5|} - \frac{1}{|\beta_3|} & S_5 - \frac{S_{5,1}}{|\beta_2|} - \frac{S_{5,5}}{|\beta_5|} & 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_3|} - \frac{1}{|\beta_4|} \\
S_4 - \frac{S_{4,5}}{|\beta_5|} - \frac{1}{|\beta_3|} & S_5 - \frac{S_{5,5}}{|\beta_5|} & 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_2|} - \frac{1}{|\beta_3|} - \frac{1}{|\beta_4|} \\
S_4 - \frac{S_{4,2}}{|\beta_2|} - \frac{S_{4,5}}{|\beta_5|} - \frac{1}{|\beta_3|} & S_5 - \frac{S_{5,2}}{|\beta_2|} - \frac{S_{5,5}}{|\beta_5|} & 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_3|} - \frac{1}{|\beta_4|} \\
S_4 - \frac{S_{4,2}}{|\beta_2|} - \frac{1}{|\beta_3|} & S_5 - \frac{S_{5,2}}{|\beta_2|} - \frac{1}{|\beta_5|} & 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_3|} - \frac{1}{|\beta_4|} - \frac{1}{|\beta_5|} \\
S_4 - \frac{S_{4,2}}{|\beta_2|} - \frac{S_{4,3}}{|\beta_3|} - \frac{1}{|\beta_3|} & S_5 - \frac{S_{5,2}}{|\beta_2|} - \frac{S_{5,3}}{|\beta_3|} - \frac{1}{|\beta_5|} & 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_4|} - \frac{1}{|\beta_5|}
\end{array}$$

System ES is simplified to equivalent system $EQS \cup \{EQ_{K+L+1}\}$: $EQ_1 = E_1 - E_2$, $EQ_2 = E_3 - E_2$, $EQ_3 = E_5 - E_4$, $EQ_4 = E_0 - E_1$ and $EQ_5 = E_3 - E_4$. The sixth equation can be obtained as $EQ_6 = E_4 - EQ_2$.

$$\begin{array}{llllll}
-\frac{S_{1,1}}{|\beta_2|} + \frac{1}{|\beta_2|} & -\frac{S_{2,1}}{|\beta_2|} & -\frac{S_{3,1}}{|\beta_2|} & -\frac{S_{4,1}}{|\beta_2|} & -\frac{S_{5,1}}{|\beta_2|} & \frac{1}{|\beta_2|} \\
-\frac{S_{1,2}}{|\beta_2|} & -\frac{S_{2,2}}{|\beta_2|} + \frac{1}{|\beta_2|} & -\frac{S_{3,2}}{|\beta_2|} & -\frac{S_{4,2}}{|\beta_2|} & -\frac{S_{5,2}}{|\beta_2|} & \frac{1}{|\beta_2|} \\
-\frac{S_{1,3}}{|\beta_3|} & -\frac{S_{2,3}}{|\beta_3|} & -\frac{S_{3,3}}{|\beta_3|} + \frac{1}{|\beta_3|} & -\frac{S_{4,3}}{|\beta_3|} & -\frac{S_{5,3}}{|\beta_3|} & \frac{1}{|\beta_3|} \\
-\frac{S_{1,4}}{|\beta_3|} & -\frac{S_{2,4}}{|\beta_3|} & -\frac{S_{3,4}}{|\beta_3|} & -\frac{S_{4,4}}{|\beta_3|} + \frac{1}{|\beta_3|} & -\frac{S_{5,4}}{|\beta_3|} & \frac{1}{|\beta_3|} \\
-\frac{S_{1,5}}{|\beta_5|} & -\frac{S_{2,5}}{|\beta_5|} & -\frac{S_{3,5}}{|\beta_5|} & -\frac{S_{4,5}}{|\beta_5|} & -\frac{S_{5,5}}{|\beta_5|} + \frac{1}{|\beta_5|} & \frac{1}{|\beta_5|} \\
S_1 - \frac{1}{|\beta_2|} & S_2 - \frac{1}{|\beta_2|} & S_3 - \frac{1}{|\beta_3|} & S_4 - \frac{1}{|\beta_3|} & S_5 - \frac{1}{|\beta_5|} & A
\end{array} ,$$

where $A = 1 - \frac{1}{|\beta_1|} - \frac{1}{|\beta_2|} - \frac{1}{|\beta_3|} - \frac{1}{|\beta_4|} - \frac{1}{|\beta_5|}$. For $D_0 = 1$ the solution of system (4) is $D \simeq [-2.133, -2.133, -2.133, -1.885, -2.510]$. The normalizing constant is $\simeq -2.069$. The normalized τ-invariant density is shown in Figure 1 b).

Example 2: In this example all values $\tau(c_i)$ are different. All branches are increasing. Let $N = 4$ and let τ be defined by the vectors

$$\alpha = [0.7, 0.2, 1, 0.45] \ , \quad \beta = [2, 3, 4, 1.35] \ , \quad \gamma = [0, 0.2, 0, 0.55] \ .$$

We have $K = 3$ and $L = 1$. The graph of τ is shown in Figure 2 a). The digits are $\{0, 0.85, 1.66\dots, 0.35\}$. The first branch of τ is greedy, the second hanging, the third onto and the last one is lazy. The points c_i are $c_1 = 0.35 = (0.35, 1)$, $c_2 = 0.35 = (0.35, 2)$, $c_3 = 0.4166\dots$, $c_4 = 0.66\dots$. $c_1, c_3 \in W_u$ and $c_2, c_4 \in W_l$. We have $0 < \tau(c_2) < \tau(c_3) < \tau(c_4) < \tau(c_1) < 1$ and taking the points $x_0 < x_1 < x_2 < x_3 < x_4$ between them we obtain the system ES: (we show only the coefficients)

$$\begin{array}{lllll}
S_1 - \frac{S_{1,2}}{\beta_2} - \frac{S_{1,4}}{\beta_4} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,2}}{\beta_2} - \frac{S_{2,4}}{\beta_4} & S_3 - \frac{S_{3,2}}{\beta_2} - \frac{S_{3,4}}{\beta_4} - \frac{1}{\beta_2} & S_4 - \frac{S_{4,2}}{\beta_2} - \frac{S_{4,4}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_3} \\
S_1 - \frac{S_{1,4}}{\beta_4} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,4}}{\beta_4} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,4}}{\beta_4} - \frac{1}{\beta_2} & S_4 - \frac{S_{4,4}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} \\
S_1 - \frac{S_{1,3}}{\beta_2} - \frac{S_{1,4}}{\beta_4} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,3}}{\beta_2} - \frac{S_{2,4}}{\beta_4} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,3}}{\beta_2} - \frac{S_{3,4}}{\beta_4} & S_4 - \frac{S_{4,3}}{\beta_2} - \frac{S_{4,4}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_3} \\
S_1 - \frac{S_{1,3}}{\beta_2} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,3}}{\beta_2} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,3}}{\beta_2} & S_4 - \frac{S_{4,3}}{\beta_2} - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_3} - \frac{1}{\beta_4} \\
S_1 - \frac{S_{1,1}}{\beta_1} - \frac{S_{1,3}}{\beta_2} & S_2 - \frac{S_{2,1}}{\beta_1} - \frac{S_{2,3}}{\beta_2} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,1}}{\beta_1} - \frac{S_{3,3}}{\beta_2} & S_4 - \frac{S_{4,1}}{\beta_1} - \frac{S_{4,3}}{\beta_2} - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_3} - \frac{1}{\beta_4}
\end{array}$$

System ES is simplified to equivalent system $EQS \cup \{EQ_{K+L+1}\}$: $EQ_1 = E_4 - E_3$, $EQ_2 = E_0 - E_1$, $EQ_3 = E_2 - E_1$ and $EQ_4 = E_3 - E_2$. The fifth equation can be obtained as $EQ_5 = E_3 - EQ_3$.

$$\begin{array}{lllll}
-\frac{S_{1,1}}{\beta_1} + \frac{1}{\beta_1} & -\frac{S_{2,1}}{\beta_1} & -\frac{S_{3,1}}{\beta_1} & -\frac{S_{4,1}}{\beta_1} & \frac{1}{\beta_1} \\
-\frac{S_{1,2}}{\beta_2} & -\frac{S_{2,2}}{\beta_2} + \frac{1}{\beta_2} & -\frac{S_{3,2}}{\beta_2} & -\frac{S_{4,2}}{\beta_2} & \frac{1}{\beta_2} \\
-\frac{S_{1,3}}{\beta_2} & -\frac{S_{2,3}}{\beta_2} & -\frac{S_{3,3}}{\beta_2} + \frac{1}{\beta_2} & -\frac{S_{4,3}}{\beta_2} & \frac{1}{\beta_2} \\
-\frac{S_{1,4}}{\beta_4} & -\frac{S_{2,4}}{\beta_4} & -\frac{S_{3,4}}{\beta_4} & -\frac{S_{4,4}}{\beta_4} + \frac{1}{\beta_4} & \frac{1}{\beta_4} \\
S_1 - \frac{1}{\beta_1} & S_2 - \frac{1}{\beta_2} & S_3 - \frac{1}{\beta_2} & S_4 - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} - \frac{1}{\beta_4}
\end{array}$$

For $D_0 = 1$ the solution of system (4) is $D \simeq [-0.876, -0.876, -0.883, -16.539]$. The normalizing constant is $\simeq -7.812$. The normalized τ-invariant density is shown in Figure 3 a).

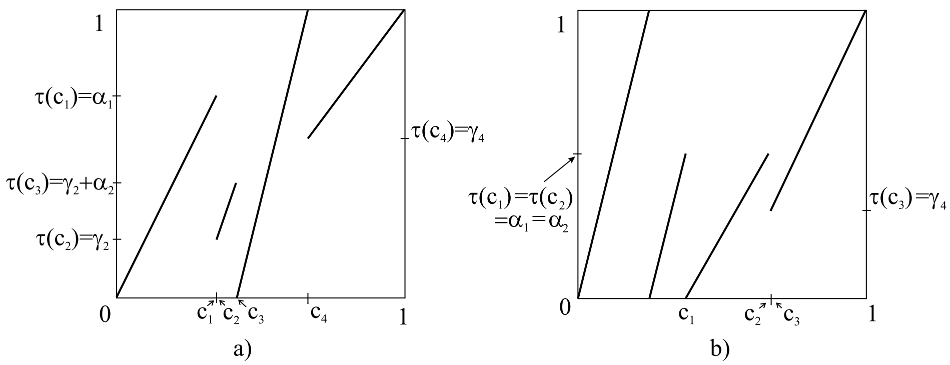

FIGURE 2. Maps τ of a) Example 2 and b) Example 3.

Example 3: Here, we have $\tau(c_1) = \tau(c_2)$. Let $N = 4$ and let τ be defined by the vectors

$$\alpha = [1, 0.5, 0.5, 0.7] \quad , \quad \beta = [4, 3, 2, 2.1] \quad , \quad \gamma = [0, 0, 0, 0.3] \, .$$

We have $K = 3$, $L = 0$. The graph of τ is shown in Figure 2 b). The digits are $\{0, 0.75, 0.833\ldots, 1.1\}$. The first branch of τ is onto, the second and third are greedy and the last one is lazy. The points c_i are $c_1 = 0.4166\ldots$, $c_2 = 0.66\cdots = (0.66\ldots, 3)$, $c_3 = 0.66\cdots = (0.66\ldots, 4)$. $c_1, c_2 \in W_u$ and $c_3 \in W_l$. We have $0 < \tau(c_3) < \tau(c_1) = \tau(c_2) < 1$ and taking the points $x_0 < x_1 < x_2$ between them we obtain the system ES: (again, we show only the coefficients)

$$\begin{array}{llll}
S_1 - \frac{S_{1,3}}{\beta_4} - \frac{1}{\beta_2} & S_2 - \frac{S_{2,3}}{\beta_4} - \frac{1}{\beta_3} & S_3 - \frac{S_{3,3}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} \\
S_1 - \frac{1}{\beta_2} & S_2 - \frac{1}{\beta_3} & S_3 - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} - \frac{1}{\beta_4} \\
S_1 - \frac{S_{1,1}}{\beta_2} - \frac{S_{1,2}}{\beta_3} & S_2 - \frac{S_{2,1}}{\beta_2} - \frac{S_{2,2}}{\beta_3} & S_3 - \frac{S_{3,1}}{\beta_2} - \frac{S_{3,2}}{\beta_3} - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_4}
\end{array}$$

Again, system ES is simplified to equivalent system $EQS \cup \{EQ_{K+L+1}\}$: $EQ_1 = EQ_2 = E_2 - E_1$, $EQ_3 = E_0 - E_1$. The third (or formally the fourth) equation can be obtained as $EQ_4 = E_1$.

$$\begin{array}{llll}
-\frac{S_{1,1}}{\beta_2} - \frac{S_{1,2}}{\beta_3} + \frac{1}{\beta_2} & -\frac{S_{2,1}}{\beta_2} - \frac{S_{2,2}}{\beta_3} + \frac{1}{\beta_3} & -\frac{S_{3,1}}{\beta_2} - \frac{S_{3,2}}{\beta_3} & \frac{1}{\beta_2} + \frac{1}{\beta_3} \\
-\frac{S_{1,3}}{\beta_4} & S_2 - \frac{S_{2,3}}{\beta_4} & -\frac{S_{3,3}}{\beta_4} + \frac{1}{\beta_4} & \frac{1}{\beta_4} \\
S_1 - \frac{1}{\beta_2} & S_2 - \frac{1}{\beta_3} & S_3 - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} - \frac{1}{\beta_4}
\end{array} \tag{23}$$

The solution of system (4), for $D_0 = 1$, is $D \simeq [8.794, 3.382, 3.382]$. System (23) is not equivalent to (4), but solution of (4) satisfies also (23). System (23) has infinitely many solutions $D^{(t)} \simeq [t, 9.2447 - 0.6667t, 3.382]$. We have $D = D^{(t)}$ for $t = D_1$. The functions

$$h_1 = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]} \frac{1}{\beta(c_1, n)} \quad \text{and} \quad h_2 = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_2)]} \frac{1}{\beta(c_2, n)}$$

are proportional, $\beta_2 h_1 = \beta_3 h_2$, and the invariant density h stays the same whether we use constants D_1, D_2, D_3 or $D_1^{(t)}, D_2^{(t)}, D_3^{(t)}$ for arbitrary t. The normalizing constant is $\simeq 5.989$. The normalized τ-invariant density is shown in Figure 3 b).

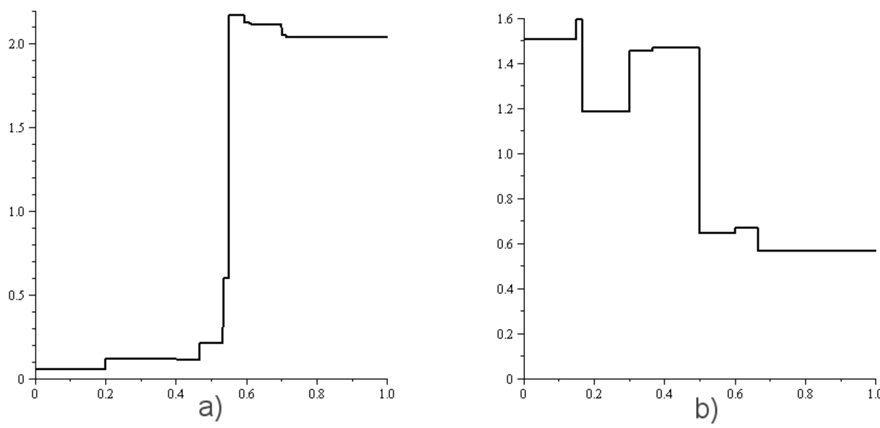

FIGURE 3. Invariant densities for maps of a) Example 2 and b) Example 3.

In the next example we show a map τ which is not ergodic. Matrix $\mathbf{S}$ has an eigenvalue 1. The system (4) with $D_0 = 1$ is solvable (non-uniquely). Both methods, i.e., using $D_0 = 1$ or $D_0 = 0$, of finding τ-invariant density agree.

Example 4: Let $N = 8$ and τ be defined by the constant slope $\beta = 3$ and the vectors

$$\alpha = [0.5, 0.25, 0.25, 0.5, 0.5, 0.25, 0.25, 0.5] \quad , \quad \gamma = [0, 0, 0.1, 0, 0.5, 0.65, 0.75, 0.5] \; .$$

The graph of τ is shown in Figure 4 a). The matrix

$$\mathbf{S} \simeq \begin{bmatrix} 0.5 & 0.5 & 0.34654 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.34654 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.34654 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.487037 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.45 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.45 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.116667 & 0.5 & 0.5 \end{bmatrix} .$$

For $D_0 = 1$ system (4) has solutions

$$D(t) \simeq \Big[t,\ \frac{2t}{3},\ \frac{2t}{3},\ 0.769231t,\ 0,$$
$$0,\ -0.691358t - 2.074074,\ -\frac{2t}{3} - 2,\ -\frac{2t}{3} - 2,\ -0.77778t - \frac{7}{3}\Big]\ .$$

The eigenvector of $\mathbf{S}$ corresponding to the eigenvalue 1 is

$$D_v \simeq [-0.943423,\ -0.628949,\ -0.628949,\ -0.725710,\ 0,$$
$$0,\ 0.652243,\ 0.628949,\ 0.628949,\ 0.733774]\ .$$

The τ-invariant densities are shown in Figure 4 b). Density for $D_0 = 1$ and constants $D(-0.5)$ is shown in black, density for $D_0 = 1$ and constants $D(-1.9)$ is shown in gray, and density for $D_0 = 0$ and constants D_v is shown in gray dash line. The last one happens to be a combination of negative density for one ergodic component and a positive density for the other one.

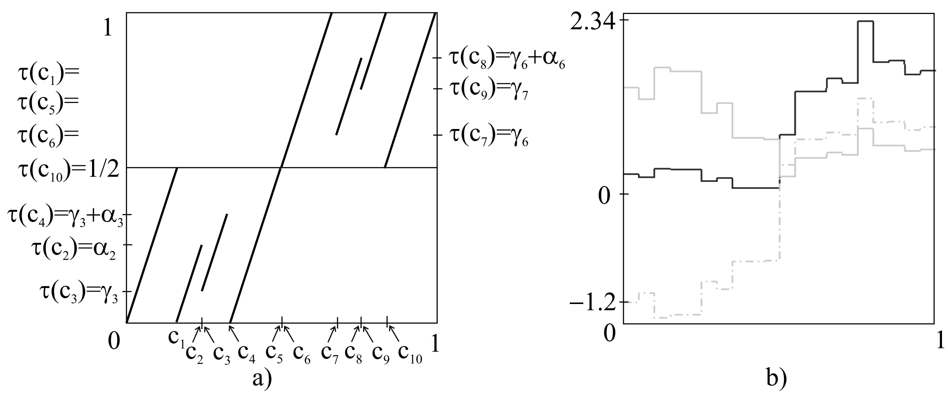

FIGURE 4. Map τ of Example 4 and three versions of its invariant density.

4. *Ergodic properties of piecewise linear maps*

In this section we discuss the ergodic implication of having invariant density with full support. In particular, this applies to any τ satisfying the assumptions of Proposition 5, or any greedy map for which 1 is not an eigenvalue of $\mathbf{S}$.

THEOREM 4. *Let τ be a piecewise linear and eventually piecewise expanding map which admits an invariant density supported on* $[0, 1]$*. Then, if at least one branch of τ is onto then τ has at most two ergodic components. If at least two branches are onto, then τ is exact.*

Remark: It follows from the general theory (e.g., Theorem 8.4.1 of [2]) that an exact piecewise expanding map τ is weakly Bernoulli.

Proof: It follows from the general theory (for example [2, Chapter 8]) that τ has finite number of ergodic components and the support of each ergodic component consists of a finite number of intervals. To prove exactness of an ergodic component it is enough to show that the images of arbitrarily small interval in the component grow to cover the whole domain of the component.

If τ has an onto branch, then let x_0 be a fixed point in the domain of this branch. There are two possibilities:

a) Some neighborhood J of x_0 is contained in one ergodic component of τ. Then, the images $\tau^n(J)$ grow to cover the whole $[0,1]$ and τ has one exact component.

b) τ has at least two ergodic components and some intervals J_1 of one component and J_2 of the second component touch x_0. Let $J_1 \subset [0,x_0)$. Then, the images $\tau^n(J_1)$ grow to cover $[0,x_0)$ and the images $\tau^n(J_2)$ grow to cover $(x_0,1]$. τ has two ergodic components.

If τ has at least two onto branches, then the fixed points in these branches, x_0 and x_1 are different. Each of the intervals $[0,x_0]$, $[0,x_1]$, $[x_0,1]$, $[x_1,1]$, is completely contained in a support of an ergodic component. Thus, we have at most one ergodic component. Since arbitrary neighborhood of any of these fixed points grows under iteration to cover the whole $[0,1]$ the system is exact. □

Proposition 5. *If spectral radius ρ of matrix $\mathbf{S}$ satisfies $\rho < 1$, then the system (4) is solvable for $D_0 = 1$ and the solution vector $D > 0$. This implies that the invariant density h is strictly positive.*

Proof: Let again $\mathbf{v} = [1,1,\dots,1,1]$ be $(K+L)$-dimensional vector of 1's. System (4) can be rewritten as

$$D = \mathbf{S}^T D^T + v \ . \tag{24}$$

Let us define map $F : \mathbb{R}^{K+L} \to \mathbb{R}^{K+L}$ as

$$F(x) = \mathbf{S}^T x^T + v \ ,$$

and consider the sequence of vectors v, $F(v) = \mathbf{S}^T v^T + v$, $\dots$, $F^n(v) = (S^T)^n v^T + \dots + \mathbf{S}^T v^T + v$, $\dots$ Map F preserves the cone C^+ of nonnegative vectors in $\mathbb{R}^{K+L}$ and all $F^n(v) \in C^+$, $n = 0,1,2,\dots$ Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^{K+L}$. Since $\rho < 1$ we have

$$\begin{aligned} \|F^{n+m}(v) - F^n(v)\| &= \|(S^T)^{n+1}v^T + (S^T)^{n+2}v^T + \dots + +(S^T)^{n+m}v^T\| \\ &\le \text{const} \sum_{k=n+1}^{m} \rho^k \|v\| \to 0 \ , \end{aligned} \tag{25}$$

as $n,m \to +\infty$. Thus, the sequence $\{F^n(v)\}_{n\ge 0}$ converges to a vector $\bar{D} \in C^+$, which is a solution of (24). Since the solution is unique we have $\bar{D} = D$ and $D \in C^+$. Then, $\mathbf{S}^T D^T \in C^+$ and $D = \mathbf{S}^T D^T + v > 0$. □

Remark: If the system (4) is solvable for $D_0 = 1$ and the solution vector $D \ge 0$, then the following statements hold:

(a) $D > 0$;
(b) The spectral radius of $\mathbf{S}$ satisfies $\rho < 1$;
(c) $S_{j,j} < 1$ for $j = 1, \dots, K + L$.
(a) is proved as in the proof of Proposition 5. (b) follows from the theory of M-matrices and (c) follows by Collatz–Wielandt formula (e.g., [18]).
In Example 5 we show that the converse of Conjecture 1 is not always true.

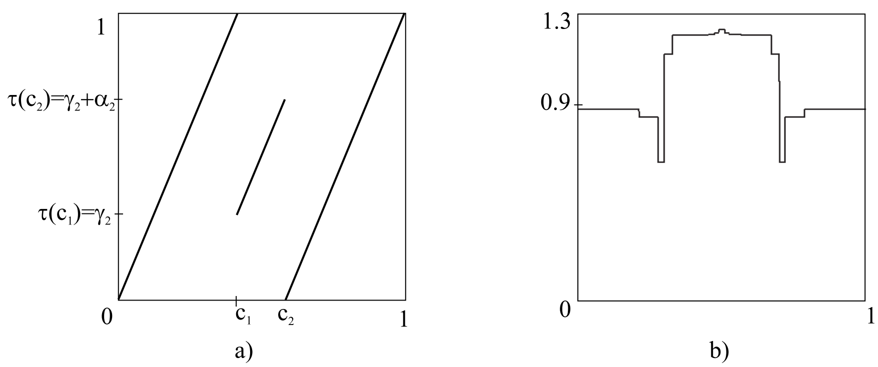

FIGURE 5. Map of Example 5 and its invariant density.

Example 5: Let τ be as in Figure 5 a). The slope β is constant, the first and the third branches are onto, the second is hanging. Let $\alpha = \alpha_2 < 1$ and $\gamma = \gamma_2 = \frac{1-\alpha}{2}$. Then, $\beta = 2 + \alpha$. The digits are $\{0, (1 + \alpha)/2, 1 + \alpha\} = \{0 \cdot d, 1 \cdot d, 2 \cdot d\}$, where $d = (1 + \alpha)/2$. Using the symmetry of map τ and definition (10) in this special case we obtain

$$\begin{aligned} S_1 &= \sum_{n=1}^{\infty} \frac{N - j(\tau^n(c_1))}{\beta^{n+1}} = \sum_{n=1}^{\infty} \frac{j(\tau^n(c_2)) - 1}{\beta^{n+1}} \\ &= \frac{1}{d\beta} \sum_{n=1}^{\infty} \frac{(j(\tau^n(c_2)) - 1) \cdot d}{\beta^n} = \frac{\tau(c_2)}{d\beta} = \frac{1}{\beta} \ . \end{aligned} \tag{26}$$

By the symmetry of τ we have $S_{1,1} = S_{2,2}$ and $S_{1,2} = S_{2,1}$. We will show that $S_{1,1} + S_{2,1} = 1$ (and also $S_{1,2} + S_{2,2} = 1$). In the proof of Theorem 2 we showed that

$$\frac{-S_{1,1} + 1}{\beta}\gamma_2 + \frac{-S_{1,2}}{\beta}(1 - \gamma_2 - \alpha_2) = S_1 - \frac{1}{\beta} \ .$$

In our case we have $\gamma_2 = 1 - \gamma_2 - \alpha_2 = (1 - \alpha)/2$ so equality (26) implies

$$-S_{1,1} + 1 - S_{1,2} = 0 \ ,$$

which in turn gives $S_{1,1}+S_{2,1} = 1$ and $S_{1,2}+S_{2,2} = 1$. This shows that the matrix $\mathbf{S}$ has eigenvalue 1. At the same time τ is exact and has unique absolutely continuous invariant measure supported on $[0, 1]$. For $D_0 = 1$ the system (4) is contradictory and does not have any solutions. For $D_0 = 0$ it is solvable and $D_1 = D_2 = 1$ is one of the solutions. Thus, τ-invariant density is

$$h = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_2)]} \frac{1}{\beta^n} + \sum_{n=1}^{\infty} \chi_{[\tau^n(c_1),1]} \frac{1}{\beta^n} \ .$$

For $\alpha = 0.4$, it is shown in Figure 5 b).

Let us note that the smallest change from the symmetry of this example results in a solvable system (4) with $D_0 = 1$ and the invariant density for τ can be obtained as a limit of densities for perturbed maps with perturbations converging to zero.

Another example with the same properties is given by τ^2. It preserves the same density h. Also, if we make the middle branch decreasing with the same slope in modulus, the resulting map τ is ergodic and $\mathbf{S}$ has 1 as an eigenvalue. This τ preserves different density h. □

In the following example we show that τ with one ergodic component is not necessarily exact.

Example 6: Let $N = 4$ and let τ be defined by the vectors

$$\alpha = [0.5, 0.5, 0.5, 0.5] \ \ , \ \ \beta = [2, 2, 2, 2] \ \ , \ \ \gamma = [0.5, 0.5, 0, 0] \ .$$

We have $K = 4$ and $L = 0$. τ is obviously ergodic and τ^2 has two exact components. System (4) with $D_0 = 1$ is solvable, $D_1 = D_4 = -0.5$, $D_2 = D_3 = -1$ and normalizing factor is -1. $h \equiv 1$.

Example 7 shows a non-ergodic map τ. Matrix $\mathbf{S}$ has 1 as an eigenvalue, although $h \equiv 1$ is a τ-invariant density.

Example 7: Let $N = 3$, and let τ be defined by the vectors

$$\alpha = [0.5, 1, 0.5] \ \ , \ \ \beta = [2, 2, 2] \ \ , \ \ \gamma = [0, 0, 0.5] \ .$$

We have $K = 2$ and $L = 0$. τ obviously has two exact components and $h \equiv 1$ is a τ-invariant density. Matrix $\mathbf{S}$ has an eigenvalue 1 and system (4) is not solvable for $D_0 = 1$. For $D_0 = 0$, any pair D_1, D_2 satisfies system (4) which agrees with the fact that

$$h_1 = D_1 \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]} \frac{1}{2^n} \quad \text{and} \quad h_2 = D_2 \sum_{n=1}^{\infty} \chi_{[\tau^n(c_2),1]} \frac{1}{2^n} \ ,$$

are invariant densities for the ergodic components of τ.

The next example shows that Conjecture 1 fails for not piecewise increasing maps τ.

Example 8: Let $N = 2$, and let τ be defined by the vectors

$$\alpha = [1, 0.8] \ \ , \ \ \beta = [1.8, -1.8] \ \ , \ \ \gamma = [0, 0.2] \ .$$

We have $K = 1$ and $L = 0$. τ is ergodic but on a smaller interval $[0.2, 1]$. Matrix $\mathbf{S} = [S_{1,1}] = [1.125]$ has an eigenvalue 1.125 and system (4) is solvable for $D_0 = 1$. We have $D_1 = -0.8$.

For the corresponding piecewise increasing map, i.e., if we keep the same α's and γ's and change β to $\beta = [1.8, 1.8]$, matrix $\mathbf{S} = [S_{1,1}] = [1]$ has an eigenvalue 1.

5. *Special case: piecewise increasing maps.*

In this section we briefly show simplifications occurring when we consider only piecewise linear, piecewise increasing maps.

We assume here that all slopes $\beta_j > 0$, $j = 1, \dots, N$. Then, we have $W_u = U_r$ and $W_l = U_l$. The formulas (3) simplify to

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_i,n)} \delta(\tau_u^n(c_i) > c_j) \ , \qquad \text{for} \quad c_i \in W_u \ \text{ and all } \ c_j \ , \tag{27}$$

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_i,n)} \delta(\tau_l^n(c_i) < c_j) \ , \qquad \text{for} \quad c_i \in W_l \ \text{ and all } \ c_j \ .$$

The formula for τ-invariant density h simplifies to

$$h(x) = D_0 + \sum_{i \in W_u} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta(c_i,n)} + \sum_{i \in W_l} D_i \sum_{n=1}^{\infty} \chi_{[\tau^n(c_i),1]} \frac{1}{\beta(c_i,n)} \ , \tag{28}$$

where constants D_i, $i = 1, \dots, K$, satisfy the system (4).

We have the following condition for the solvability of the system (4) with $D_0 = 1$.

Proposition 6. *Let τ be piecewise linear, piecewise increasing and eventually expanding map. If $\{0,1\} \subset W_u \cup W_l$, then condition $\min_{1 \le j \le N} |\beta_j| > K + L$ is sufficient for the solvability of the system (4) with $D_0 = 1$.*

Proof: If $c_1 = 0$ and $c_{K+L} = 1$, then for any $c_i \in W_l$ we have $S_{i,1} = 0$ and for any $c_i \in W_u$ we have $S_{i,K+L} = 0$. Thus, there is at least one 0 in each row of $\mathbf{S}$ and Perron-Frobenius estimate implies that 1 is not an eigenvalue of $\mathbf{S}$ for $\beta > K + L$. □

6. *Special case: Greedy maps*

In this section we discuss maps related to the greedy expansion with deleted digits [5, 21], i.e., piecewise linear, piecewise increasing maps for which all shorter branches touch 0. They are called greedy since the digits are the largest possible for given α's and β's.

Absolutely continuous invariant measures for such maps with constant slope were investigated in [6, 7] by other methods.

Our definition of a greedy map is a little more general than the one usually used. We give the standard definition for reference. It is assumed that the last branch is onto and the slope is constant $\beta > 1$. Under these conditions the digits define the map τ. Let the digits be $\{a_1, a_2, \dots, a_N\}$. We want to define τ on $[0,1]$ so we will make some unrestrictive assumptions: $a_1 = 0$ and

$$M_a = \max_{1 \le j \le N-1} (a_{j+1} - a_j) = 1 \ . \tag{29}$$

Any set of digits can be shifted and scaled to satisfy these assumptions. The maps for both sets are linearly conjugated. Now, we set $\beta = a_N + 1$ and define $b_i = a_i/\beta$,

$i = 1, \ldots, N$, $b_{N+1} = 1$. We have $\alpha_i = \frac{b_{i+1} - b_i}{\beta}$, for $i = 1, \ldots, N$. All γ's are 0 by assumption.

We return to our, slightly more general, setting. For greedy maps we have $\gamma_i = 0$ for all $i = 1, \ldots, N$. We assume that at least one branch is onto as otherwise τ should be considered on a different interval. Since the set W_l is in this case empty, we have

$$h = D_0 + \sum_{i=1}^{K} D_i \cdot \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta(c_i, n)} \ . \tag{30}$$

We will prove a number of results specific to the greedy maps.

Theorem 7. *Let us assume that τ is a greedy map. If the system (4) is solvable, then h is a non-normalized τ-invariant density. If the system (4) is solvable for $D_0 = 1$, then the system $(\tau, h \cdot m)$ is exact on $[0, 1]$.*

In particular, system (4) is uniquely solvable (1 is not an eigenvalue of $\mathbf{S}$) if $\min_{1 \le j \le N} \beta_j > K + 1$. If the last branch is shorter, then the condition $\min_{1 \le j \le N} \beta_j > K$ is sufficient and the coefficient $D_K = 1$.

Proof: Most of the claims of Theorem 7 follow by Theorem 2. We will prove exactness. From general theory (for example [**2**, Chapter 8]), we know that the support of each ergodic component contains a neighborhood J of some inner partition point. Then, the image $\tau(J)$ touches 0. This proves there is only one ergodic component C. We will show that if the system (4) is solvable for $D_0 = 1$, then $C = [0, 1]$. Assume that $C \subsetneq [0, 1]$. We will show that then $C = [0, a]$ with some $a < 1$. Let $K_b \le K$ be the number of shorter branches to the left of the first onto branch. We consider $K_b > 0$ as otherwise $C = [0, 1]$. Let x_j denote the fixed point on the jth branch, if it exists. In particular $x_1 = 0$ and x_{K_b+1} is the fixed point on the first onto branch. We say that branches with indices $1 = j_1 < j_2 < \cdots < j_{s-1} < j_s \le K_b + 1$ form an increasing sequence if for each j_k, $k = 1, \ldots, s-1$, the branch with index j_{k+1} is the first branch to the right of the branch j_k such that

$$\tau(c_{j_{k+1}}) > \tau(c_{j_k}) > x_{j_{k+1}} \ .$$

It is easy to see that if an increasing sequence of branches with $j_s = K_b + 1$ exists, then $C = [0, 1]$. Since we assumed $C \subsetneq [0, 1]$ such sequence ends before reaching the first onto branch. Let $1 \le j_l < K_b + 1$ be the index of the last branch in the increasing sequence. Then, $C = [0, \tau(c_{j_l})]$ is the support of the unique τ-invariant density.

Now, we will show that if there are any c_j with $j > K_b$, i.e., to the right of the first onto branch, then the corresponding D_i, $i > K_b$, in the formula (30) are equal to 0. The invariant density h is zero (a.e.) in the interval $[\tau(c_{j_l}), 1]$. In particular it is 0 around all points c_j with $j > K_b$.

Note that, if

$$c_j \notin \bigcup_{1 \le i \le K, \ n \ge 1} \{\tau^n c_i\} \ ,$$

then for any $i = 1, \dots, K$, comparing with (27) we obtain

$$\sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]}(c_j) \frac{1}{\beta(c_i, n)} = S_{i,j} \ ,$$

and thus, using (30) we have

$$h(c_j) = 1 + \sum_{i=1}^{K} D_i S_{i,j} \ ,$$

assuming that the value of h at c_j is given exactly by the formula (30).

In general we proceed as follows. Let fix a c_j, $j > K_b$. Since h is a.e. 0 around c_j and for any $m \ge 1$ the set

$$\bigcup_{1 \le i \le K, \ 1 \le n \le m} \{\tau^n c_i\} \ ,$$

is finite we can find a point x_m (close to c_j) such that $h(x_m) = 0$ and

$$\sum_{n=1}^{m} \chi_{[0,\tau^n(c_i)]}(x_m) \frac{1}{\beta(c_i, n)} = \sum_{n=1}^{m} \delta(\tau^n(c_i) > c_j) \frac{1}{\beta(c_i, n)} \ .$$

The left hand side is close to $h_i(x_m) = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]}(x_m) \frac{1}{\beta(c_i,n)}$ and the right hand side is close to $S_{i,j}$. Summing over $i = 1 \dots, K$, we obtain

$$\left| 1 + \sum_{i=1}^{K} D_i S_{i,j} \right| \le 2 \cdot \sum_{1 \le i \le K} |D_i| \sum_{n=m+1}^{\infty} \frac{1}{\beta(c_i, n)} \ ,$$

for every $m \ge 1$, which implies $1 + \sum_{i=1}^{K} D_i S_{i,j} = 0$. By equation (4), $D_j = 1 + \sum_{i=1}^{K} D_i S_{i,j} = 0$.

We have proved that

$$h = 1 + \sum_{i=1}^{K_b} D_i \cdot \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta(c_i, n)} \ .$$

Since all points $\{\tau^n(c_i) : 1 \le i \le K_b, n \ge 1\}$ are contained in the interval $[0, \tau(c_{j_l})]$ we have $h(x) = 1$ for almost all points x in $[\tau(c_{j_l}), 1]$. This contradicts what we have proved before. We proved that solvability of (4) with $D_0 = 1$ implies that h is supported on $[0, 1]$.

To show exactness, note that for arbitrarily small neighborhood J_1 of the fixed point on the onto branch its images $\tau^n(J_1)$ grow to cover the whole $[0, 1]$.

If the last branch is shorter, then $c_K = 1$. We have $S_{i,K} = 0$ for all $i = 1, \dots, K$ and Perron-Frobenius estimate on the modulus of eigenvalues of $\mathbf{S}$ is $\frac{K-1}{\beta-1}$. Thus, $\beta > K$ is sufficient in this case. The last equation in system (4) is then $D_K \cdot 1 = 1$ and $D_K = 1$. □

In a very special case of greedy map with only one shorter branch, $K = 1$, and constant slope β we have the following

PROPOSITION 8. *Let τ be a greedy map with $K = 1$ and constant slope $\beta = \beta_i$, $i = 1, \dots, N$. If $\beta > 2$ or the first branch is onto, then the non-normalized τ-invariant density h is given by the formula*

$$h = 1 + D_1 \cdot \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]} \frac{1}{\beta^n} \ , \tag{31}$$

where $D_1 = \frac{1}{1-S_{1,1}}$, and the system $(\tau, h \cdot m)$ is exact.

If $\beta < 2$ and the first branch is not onto, then the support of τ-invariant absolutely continuous measure is the interval $[0, \alpha_1]$. Moreover, $S_{1,1} = 1$. The restricted map $\tau_{|[0,\alpha_1]}$ is again a greedy map with one shorter branch.

Proof: If $\beta > 2$, then $S_{1,1} \leq \frac{1}{\beta-1} < 1$, the density h of (31) is well defined and supported on $[0, 1]$. The system is exact.

If $\beta < 2$, then τ has two branches (as $K = 1$) and we consider two cases:

a) The second branch is shorter: Then, $c_1 = 1$ and $\delta(\tau^n(c_1) > c_1) = 0$ for all $n \geq 1$. Thus, $S_{1,1} = 0$ and $D_1 = 1$. We obtained classical Parry's formula [19].

b) The first branch is shorter and the second one is onto. The τ-invariant absolutely continuous measure is supported on $[0, \alpha_1]$. τ restricted to this interval is a map from case a). We will prove that $S_{1,1} = 1$. If $S_{1,1} \neq 1$ then the invariant density h of (31) is well defined. In particular, for any $x > \tau(c_1)$ we have $h(x) = 1$ which is impossible. □

We have proved

PROPOSITION 9. *If τ is a greedy map with $K = 1$ and constant slope β, then τ is ergodic on $[0, 1]$ if and only if $S_{1,1} \neq 1$.*

Example 9: Let τ be a greedy map with $K = 1$ and the first branch shorter. If $\beta_2 = \min_{2 \leq j \leq N} \beta_j$ and $\alpha_1 = \tau(c_1) > x_0$, where x_0 is the fixed point on the second branch, then $S_{1,1} \neq 1$ and the claims of Proposition 8 hold.

The second branch of τ is $\tau(x) = \beta_2 x - \beta_2 \frac{\alpha_1}{\beta_1}$. Thus, $x_0 = \frac{\alpha_1 \beta_2}{\beta_1(\beta_2 - 1)}$ and $\tau(c_1) > x_0$ gives $\frac{\beta_2}{\beta_1(\beta_2-1)} < 1$. At the same time, we have

$$S_{1,1} \leq \sum_{n=1}^{\infty} \frac{1}{\beta_1 \beta_2^{n-1}} = \frac{\beta_2}{\beta_1(\beta_2 - 1)} < 1 \ .$$

□

Now, we consider greedy maps τ with constant slope $\beta > 1$ with $K = 2$ shorter branches satisfying $\beta \leq 3$, or $\beta \leq 2$ if the last branch is shorter.

We will first consider cases when τ has two shorter branches, $\beta \leq 2$ and the last branch is shorter. This means that τ has 3 branches.

(A) The first branch is onto: Then, τ is exact, which can be proved as in Theorem 7. Since $c_2 = 1$ we have $S_{1,2} = S_{2,2} = 0$ and $D_2 = 1$. D_1 has to satisfy $D_1(-S_{1,1} + 1) = 1 + S_{2,1}$. We will show that

$$S_{1,1} < 1 \ . \tag{32}$$

Let us assume that $\tau(c_1) = \alpha_2 \geq \alpha_3 = \tau(c_2)$. We have $\beta = 1 + \alpha_1 + \alpha_2 \leq 2$ so $\alpha_2 < \beta - 1$. The fixed point on the second branch would be x_0 such that $\beta x_0 - 1 = x_0$ which gives $x_0 = \frac{1}{\beta - 1} \geq 1$. Thus, the second branch is always below the diagonal. In particular, $\alpha_2 < c_1$. Also, whenever $\tau^n(c_1) > c_1$ then $\tau^{n+1}(c_1) \leq \alpha_3 < c_1$. Thus, $S_{1,1} < \frac{1}{\beta^2 - 1}$ and (32) is shown at least for $\beta > \beta^{(1)} = \sqrt{2}$ such that $(\beta^{(1)})^2 - 1 = 1$.

Assume that $\beta \leq \beta^{(1)}$. Then, $(\beta + 1)(\beta - 1) \leq 1$ or $\beta - 1 < \frac{1}{\beta + 1}$. Since $\alpha_2 < \beta - 1$ this means that $\alpha_2 < \frac{1}{\beta}$ and $\tau^2(c_1) = \beta\alpha_2 \leq \frac{\beta}{\beta+1}\frac{\beta}{\beta} \leq \frac{2}{\beta+1}\frac{1}{\beta} < \frac{1}{\beta} < c_1$. Thus, $\tau(c_1) < c_1$ and $\tau^2(c_1) < c_1$. Moreover, whenever $\tau^n(c_1) > c_1$ then the next two iterates are smaller then $\frac{1}{\beta}$. Thus, $S_{1,1} < \frac{1}{\beta^3 - 1}$ and (32) is shown at least for $\beta > \beta^{(2)} = \sqrt[3]{2}$ such that $(\beta^{(2)})^3 - 1 = 1$.

Assume again that $\beta \leq \beta^{(2)}$. Then, $(\beta^2 + \beta + 1)(\beta - 1) \leq 1$ or $\alpha_2 < \frac{1}{\beta^2 + \beta + 1}$ which means that $\tau^k(c_1) < c_1$ for $k = 1, 2, 3, 4$. Moreover, whenever $\tau^n(c_1) > c_1$ then the next four iterates are smaller then $\frac{1}{\beta}$. Thus, $S_{1,1} < \frac{1}{\beta^5 - 1}$ and (32) is shown at least for $\beta > \beta^{(3)} = \sqrt[5]{2}$ such that $(\beta^{(3)})^5 - 1 = 1$.

Since the roots $\sqrt[n]{2}$ converge to 1 as n converges to infinity, repeating the above reasoning inductively, we can prove (32) for all $\beta > 1$.

Now, let us assume that $\tau(c_1) = \alpha_2 < \alpha_3 = \tau(c_2)$. The proof is similar. Again, $\tau(c_1) \leq c_1$ which gives $S_{1,1} \leq \frac{1}{\beta(\beta - 1)}$. Thus, (32) is shown at least for $\beta > \beta^{(0)} = (1 + \sqrt{5})/2 \simeq 1.618$ such that $\beta^{(0)}(\beta^{(0)} - 1) = 1$.

Assume that $\beta \leq \beta^{(0)}$. Then, $\beta(\beta - 1) \leq 1$ or $\beta - 1 < \frac{1}{\beta}$. Since $\alpha_2 < \alpha_3 < \beta - 1$, we have $\tau(c_1) \leq c_1$ and whenever $\tau^n(c_1) > c_1$ then $\tau^{n+1}(c_1) \leq \alpha_3 < \frac{1}{\beta}$. This gives $S_{1,1} \leq \frac{1}{(\beta^2 - 1)}$. Thus, (32) is shown at least for $\beta > \beta^{(1)} = \sqrt{2}$ such that $(\beta^{(1)})^2 - 1 = 1$. Then, the proof proceeds as in the previous case.

Example 10: τ considered in case (A) gives an example of maps for which invariant density h exists although β can be arbitrarily close to 1.

(B) The first branch is shorter. Then, the fixed point in the middle onto branch is $x_0 = \alpha_1/(\beta - 1)$ and $x_0 \geq \alpha_1$. The support of absolutely continuous invariant measure is the interval $[0, \alpha_1]$ and τ restricted to this interval is classical β-map.

Now, we consider situation where the last branch is onto and $\beta \leq 3$. This means that τ has 3 or 4 branches.

3 branches case: Since the last branch of τ is onto, the first and the second branch are shorter.

(C) $\alpha_1 \leq \alpha_2$: There are two possibilities:

(Ca) α_1 is below the fixed point on the second branch (or this fix point does not exist). Then, map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_1]$. τ restricted to this interval is a classical β-map and the invariant density can be found by Parry's formula (or our formula after rescaling).

(Cb) The image of the first branch covers the fixed point on the second branch. Then, map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first and the last branches shorter. This situation is considered in (B).

(D) $\alpha_1 > \alpha_2$: Map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_1]$. τ restricted to this interval has the first branch onto. This situation is considered in (A).

4 branches case: The last branch of τ is onto.

(E) The first branch is onto. $2 < \beta \le 3$. τ is exact. We will prove that 1 is not an eigenvalue of $\mathbf{S}$.

First, we will show that it is not possible for both α_2, α_3 to be above the point $c_1 = \frac{1+\alpha_2}{\beta}$.

Assume $\alpha_2 \le \alpha_3$. Since $\beta = 2+\alpha_2+\alpha_3 \le 3$ we have $\alpha_2 \le \frac{1}{2}$. Then, if $\alpha_2 > \frac{1+\alpha_2}{\beta}$, we would have $\beta > \frac{1+\alpha_2}{\alpha_2} \ge 3$, a contradiction.

Assume $\alpha_2 > \alpha_3$. Now, we have $\alpha_3 \le \frac{1}{2}$. If $\alpha_3 > \frac{1+\alpha_2}{\beta} > \frac{1+\alpha_3}{\beta}$, we would have $\beta > \frac{1+\alpha_3}{\alpha_3} \ge 3$, again a contradiction.

Thus, at least one of the images $\tau(c_i)$, $i = 1, 2$ is below both points c_1, c_2. This makes Perron-Frobenius estimate on eigenvalues of $\mathbf{S}$ (or $\mathbf{S}^T$) equal to $\frac{1}{\beta-1} + \frac{1}{\beta(\beta-1)}$. Let $\beta^{(1)}$ be the positive solution of

$$\frac{1}{\beta - 1} + \frac{1}{\beta(\beta - 1)} = 1 \ .$$

We proved that 1 is not an eigenvalue of $\mathbf{S}$ for $\beta > \beta^{(1)} = \sqrt{2} + 1$.

Now, we assume that $\beta \le \beta^{(1)}$. We have $\alpha_2 + \alpha_3 \le \beta^{(1)} - 2$. We will show that both α_2, α_3 are below the point $c_1 > \frac{1}{\beta}$. The worst case scenario is when the smaller of α's is almost 0 and and the other one is almost $\beta^{(1)} - 2$. Since $\frac{1}{\beta^{(1)}} = \beta^{(1)} - 2$, inequality $\beta^{(1)} - 2 \le \frac{1}{\beta}$ is satisfied for all $2 < \beta \le \beta^{(1)}$. We proved that both images $\tau(c_i)$, $i = 1, 2$ are below both points c_1, c_2. Now, Perron-Frobenius estimate becomes $\frac{2}{\beta(\beta-1)}$. Since

$$\frac{2}{\beta(\beta - 1)} < 1 \ ,$$

for all $\beta > 2$ we completed the proof.

(F) The two first branches are shorter. $2 < \beta \le 3$.

Assume first $\alpha_1 \le \alpha_2$: Since the fixed point in the second branch is $x_0 = \frac{\alpha_1}{\beta-1} < \alpha_1$ the image of the first branch covers it. There are two cases:

(Fa) If α_2 is above the fixed point in the third, onto branch, then τ is exact. The third branch is $\tau(x) = \beta x - (\alpha_1 + \alpha_2)$ so this fixed point is $x_0 = \frac{\alpha_1+\alpha_2}{\beta-1}$. Conditions $\alpha_2 > x_0$ and $\alpha_1 + \alpha_2 < 1$ lead to inequality

$$\alpha_1 < \min\{1 - \alpha_2, \frac{\alpha_2^2}{1 - \alpha_2}\} \ .$$

(Fb) If α_2 is below the fixed point in the third, onto branch, then map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first and the last branches shorter. This situation is considered in (B).

Now, assume $\alpha_1 > \alpha_2$: Again, there are two cases:

(Fc) If α_1 is above the fixed point in the third, onto branch, then τ is exact. This fixed point is again $x_0 = \frac{\alpha_1+\alpha_2}{\beta-1}$. Conditions $\alpha_1 > x_0$ and $\alpha_1 + \alpha_2 < 1$ lead to inequality

$$\alpha_2 < \min\{1-\alpha_1, \frac{\alpha_1^2}{1-\alpha_1}\} \; .$$

(Fd) If α_1 is below the fixed point in the third, onto branch, then map τ has unique absolutely continuous invariant measure supported on $[0,\alpha_1]$. τ restricted to this interval has the second and the third (the last) branches shorter. This situation is considered in (A).

(G) The first and the third branches are shorter. $2<\beta\le 3$. Since again the image of the first branch covers the fixed point in the second onto branch, map τ is exact.

We have $c_2 = 1-\frac{1}{\beta}$. We will find when both α_1 and α_3 are below the point c_2. Let $\alpha=\max\{\alpha_1,\alpha_3\}$. We need $\alpha\le c_2$. Since $\alpha<\beta-2$ it is enough to have $\beta-2\le 1-\frac{1}{\beta}$. Let $\beta^{(2)}=(3+\sqrt{5})/2\simeq 2.618$ be the larger solution of equation $\beta-2=1-\frac{1}{\beta}$. For $\beta\le\beta^{(2)}$ Perron-Frobenius estimate on eigenvalues of $\mathbf{S}$ is $\frac{1}{\beta-1}+\frac{1}{\beta(\beta-1)}$. For $\beta>\beta^{(1)}\simeq 2.414$ of case (E), this implies that 1 is not an eigenvalue of $\mathbf{S}$. Thus, this holds in our case for $\beta^{(1)}<\beta\le\beta^{(2)}$ or $2.414<\beta\le 2.618$.

We have proved the following

PROPOSITION 10. *If τ is a greedy map with $K=2$ and constant slope β and τ satisfies assumptions of case (A), (E) or (G) with $2.414<\beta\le 2.618$ then τ is ergodic on $[0,1]$ if and only if 1 is not an eigenvalue of $\mathbf{S}$. For cases (B), (C), (D), (Fb) and (Fd) analogous statement is true for τ restricted to a smaller interval. Cases (Fa), (Fc) and (G) outside the mentioned interval of β's are open to further investigation.*

In all computer experiments we performed during the work on this paper, matrices $\mathbf{S}$ for greedy maps had spectral radius $\rho\le 1$, and if the maps were exact on $[0,1]$, then $\mathbf{S}$ never had an eigenvalue 1.

Therefore we state the following conjecture.

Conjecture 2: Let τ be a greedy map, i.e., a piecewise linear, piecewise increasing map with shorter branches touching 0. Then,

a) 1 is not an eigenvalue of matrix $\mathbf{S}\iff$ dynamical system $(\tau, h\cdot m)$ is exact on $[0,1]$;

b) the spectral radius ρ of $\mathbf{S}$ satisfies $\rho\le 1$.

7. *Special case: Lazy maps.*

In this section we consider piecewise linear maps of an interval $[0,1]$ with all branches increasing and such that the images of shorter branches touch 1. This means that $\alpha_i+\gamma_i=1$ for all $i=1,\dots,N$. Such maps are related to so called "lazy expansions with deleted digits" [5]. They are called lazy since the digits are the smallest possible for the given α's and β's.

We will show that any lazy map is conjugated by a linear map to a corresponding greedy map so all results proven in the previous section hold, after necessary changes, for lazy maps as well.

Let $\tilde{\tau}$ be a lazy map. Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma} = 1 - \tilde{\alpha}$ denote vectors of α's, β's and γ's defining $\tilde{\tau}$. The partition points are defined, as in the general case, by

$$\tilde{b}_1 = 0 \quad , \quad \tilde{b}_j = \sum_{i=1}^{j-1} \frac{\tilde{\alpha}_i}{\tilde{\beta}_i} \ , \quad j = 2 \dots, N+1 \ .$$

Note, that $\tilde{b}_{N+1} = 1$. Let $\tilde{I}_j = (\tilde{b}_j, \tilde{b}_{j+1})$, $j = 1 \dots, N$. The digits $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N\}$, are as before defined by

$$\tilde{a}_j = \tilde{\beta}_j \tilde{b}_j - \tilde{\gamma}_j = \tilde{\beta}_j \tilde{b}_{j+1} - 1 \ , \quad j = 1, \dots, N \ .$$

We will now show that lazy map $\tilde{\tau}$ is conjugated to some greedy map τ by diffeomorphism $f(x) = 1 - x$ on $[0, 1]$. First we define "conjugated" vectors α, β and γ by

$$\begin{aligned} \alpha_j &= \tilde{\alpha}_{N-j+1} \quad , \\ \beta_j &= \tilde{\beta}_{N-j+1} \quad , \qquad j = 1, 2, \dots, N \ , \\ \gamma_j &= 0 \quad . \end{aligned}$$

This defines the "conjugated" partition points

$$b_1 = 0 \ , \quad b_j = \sum_{i=1}^{j-1} \frac{\alpha_i}{\beta_i} = \sum_{i=1}^{j-1} \frac{\tilde{\alpha}_{N-i+1}}{\tilde{\beta}_{N-i+1}} = 1 - \tilde{b}_{N-j+2} \ , \quad j = 2 \dots, N+1 \ .$$

This defines also the conjugated set of digits $A = \{a_1, a_2, \dots, a_N\}$ with

$$a_j = \beta_j b_j = \tilde{\beta}_{N-j+1}(1 - \tilde{b}_{N-j+2}) = \tilde{\beta}_{N-j+1} - 1 - \tilde{a}_{N-j+1} \ , \quad j = 1, 2, \dots, N \ .$$

In particular, $a_1 = 0$. For standard greedy and lazy maps this reduces to $a_j = \tilde{a}_N - \tilde{a}_{N-j+1}$, $j = 1, 2, \dots, N$. The lengths of intervals $\tilde{I}_j$ and I_{N-j+1} are equal since $b_{N-j+2} - b_{N-j+1} = (1 - \tilde{b}_j) - (1 - \tilde{b}_{j+1}) = \tilde{b}_{j+1} - \tilde{b}_j$, $j = 1, 2, \dots, N$.

THEOREM 11. *The maps $\tilde{\tau}$ and τ are conjugated by the diffeomorphism $f(x) = 1 - x$. If h is a τ-invariant density, then the density $\tilde{h}(x) = h(1-x)$ is $\tilde{\tau}$-invariant. We have*

$$\tilde{h}(x) = D_0 + \sum_{i=1}^{K} \tilde{D}_i \sum_{n=1}^{\infty} \chi_{[\tilde{\tau}^n(\tilde{c}_i),1]} \frac{1}{\tilde{\beta}(\tilde{c}_i, n)} \ ,$$

where constants $\tilde{D}_i = D_{K-i+1}$, $i = 1, \dots, K$, satisfy the system (4) (for $\tilde{\tau}$), and points $\tilde{c}_i = 1 - c_i$, $i = 1, \dots, K$ are the special points for $\tilde{\tau}$.

Proof: Both τ and $f \circ \tilde{\tau} \circ f^{-1}$ are piecewise linear, piecewise increasing maps and the images of shorter intervals touch 0. The equality of the lengths of the intervals I_j and $\tilde{I}_{N-j+1}$ and of the slopes $\beta_j = \tilde{\beta}_{N-j+1}$, $j = 1, 2, \dots, N$, proves that they are identical. Then, $\tilde{h}(x) = h(1-x)$ since $|f'| = 1$. The formula for $\tilde{h}$ follows by the general Theorem 2. □

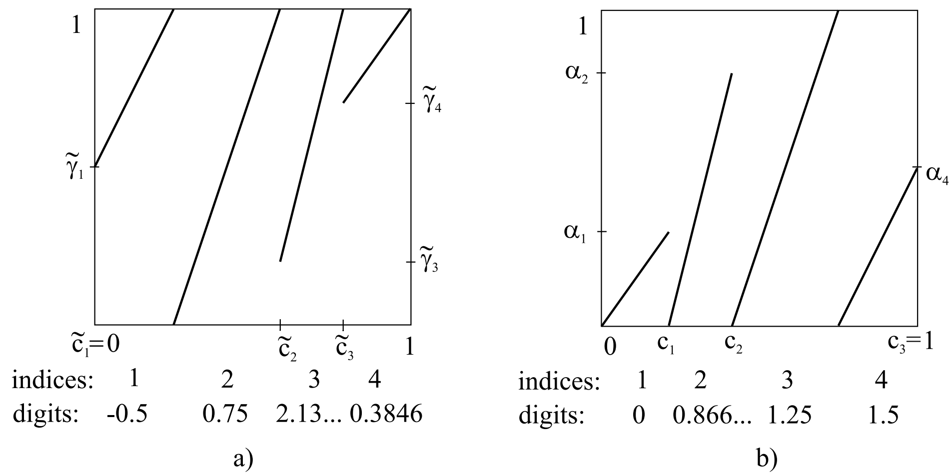

FIGURE 6. Graphs of a) lazy map and b) greedy map of Example 11.

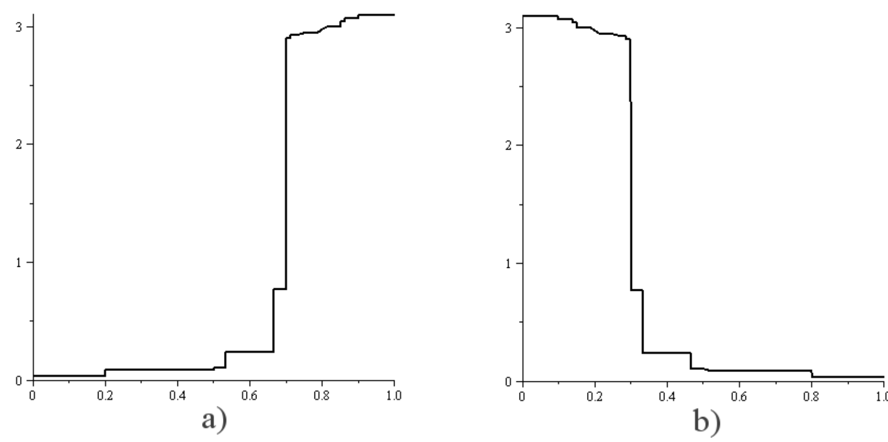

FIGURE 7. Invariant densities of a) lazy map and b) greedy map of Example 11.

Example 11: Let the lazy map $\tilde{\tau}$ be defined by $N = 4$, $K = 3$ and

$$\tilde{\alpha} = [0.5, 1, 0.8, 0.3] \quad , \quad \tilde{\beta} = [2, 3, 4, 1.3846] \quad , \quad \tilde{\gamma} = [0.5, 1, 0.2, 0.7] \; .$$

The digits are $\tilde{A} = \{-0.5, 0.75, 2.13\ldots, 0.3846\}$. The graph of $\tilde{\tau}$ is shown in Figure 6 a). The conjugated greedy map τ is defined by

$$\alpha = [0.3, 0.8, 1, 0.5] \quad , \quad \beta = [1.3846, 4, 3, 2] \quad , \quad \gamma = [0, 0, 0, 0] \; .$$

The digits are $A = \{0, 0.866\ldots, 1.25, 1.5\}$. The graph of map τ is shown in Figure 6 b). Using Maple 11 we calculated, for $D_0 = 1$, $\tilde{D}_1 = 1$, $\tilde{D}_2 \simeq 7.9992$, $\tilde{D}_3 \simeq 99.671$. We have $D_i = \tilde{D}_{K-i+1}$, $i = 1, \ldots, K$. The normalizing constant of the density is $\simeq 33.7996$. The graph of $\tilde{\tau}$-invariant density is shown in Figure 7 a)and the graph of τ-invariant density is shown in Figure 7 b) .

8. *Special case: mixed greedy-lazy maps.*

In this section we consider maps with some shorter branches touching 0 and others touching 1. We do not assume that there is at least one onto branch.

We prove some results which are specific for mixed type maps.

THEOREM 12. *Let τ be an eventually piecewise expanding map of mixed type. Let h be the τ-invariant density. Then the dynamical system $\{\tau, h \cdot m\}$ can have at most two ergodic components. If the invariant density h has full support and τ has at least two onto branches, then $\{\tau, h \cdot m\}$ is exact.*

Proof: It follows from the general theory that the support of each ergodic component contains neighborhood of some inner endpoint of the partition. Since the image of each branch touches either 0 or 1, there can be at most two ergodic components. The second statement was proved in general in Theorem 4. □

Example 7 shows that mixed type map can actually have two ergodic components. In this specific case system (4) is not solvable for $D_0 = 1$.

We will describe the situation in the case of two ergodic components in more detail.

Let τ be a mixed type map with an invariant density h with support equal to $[0, 1]$. Let us assume there are two ergodic components. Since 0 belongs to one component and 1 belongs to the other component we will denote the supports of the components by C_0 and C_1 respectively. There are two possibilities:

(C1): there exists $x_0 \in [0, 1]$ such that $C_0 = [0, x_0]$ and $C_1 = [x_0, 1]$. Let $\tau_0 = \tau_{|C_0}$ and $\tau_1 = \tau_{|C_1}$. For example, this happens if τ has at least one onto branch.

We have $\tau^n(c_k) \le c_j$ for all $n \ge 1$ and all $c_k \in C_0$, $c_j \in C_1$ and $\tau^n(c_k) \ge c_j$ for all $n \ge 1$ and all $c_k \in J_1$, $c_j \in J_0$. Thus, matrix $\mathbf{S}$ is a block matrix

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_0 = (S_{i,j})_{1 \le i, j \le M} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 = (S_{i,j})_{M+1 \le i, j \le K+L} \end{pmatrix},$$

where $c_1, \dots, c_M \in C_0$ and $c_{M+1}, \dots, c_{K+L} \in C_1$.

The image of at least one $c_{i_0} \in C_0$ and at least one $c_{i_1} \in C_1$ is equal to x_0 as otherwise there would be a hole in the support of h. Even if x_0 is a fixed point in a common onto branch od τ, there must exist such points.

Since h has full support, each of the systems $(\tau_0, h \cdot m_{|C_0})$, $(\tau_1, h \cdot m_{|C_1})$ is exact by Theorem 7. Each can be considered separately and the invariant densities can be combined.

In this case we can prove that matrix $\mathbf{S}$ has an eigenvalue 1.

PROPOSITION 13. *Let τ be mixed type map described in (C1). Then, 1 is an eigenvalue of $\mathbf{S}$.*

Proof: All branches of τ with domains in C_0 are greedy and all branches of τ with domains in C_1 are lazy. Matrix $\mathbf{S}_0$ is identical with the matrix $\mathbf{S}$ of the greedy map τ_g constructed as follows: on $[0, 1/2]$ define τ_g as $\tau_{|C_0}$ scaled to transform $[0, 1/2]$

onto $[0, 1/2]$ and on $(1/2, 1]$ put $\tau_g = 2(x - 1/2)$. We proved in Theorem 7 that such matrix $\mathbf{S}$ has 1 as an eigenvalue. □

Proposition 13 can be generalized to the case when $C_0 = [0, x_0]$ and $C_1 = [x_1, 1]$ with $x_0 < x_1$.

(C2): Each component C_0 and C_1 consists of some number of disjoint subintervals separated by the subintervals of the other component. A map τ with each C_i consisting of 2 subintervals is given in Example 12 and a map where each C_i has 3 subintervals is given in Example 13. Examples with more subintervals in each C_i can be constructed in analogous way.

Example 12: Let $N = 4$ and τ be defined by vectors

$$\alpha = \left[\frac{2}{4}, \frac{1}{4}, \frac{2}{4}, \frac{1}{4}\right] \quad , \quad \beta = [1, 2, 2, 2] \quad , \quad \gamma = \left[\frac{2}{4}, 0, 0, \frac{3}{4}\right] \ .$$

τ is eventually expanding and $C_0 = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$, $C_1 = [\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1]$.

Example 13: Let $N = 4$ and τ be defined by vectors

$$\alpha = \left[\frac{4}{6}, \frac{1}{6}, \frac{2}{6}, \frac{1}{6}\right] \quad , \quad \beta = [1, 2, 2, 2] \quad , \quad \gamma = \left[\frac{2}{6}, 0, 0, \frac{5}{6}\right] \ .$$

τ is eventually expanding and $C_0 = [0, \frac{1}{6}] \cup [\frac{2}{6}, \frac{3}{6}] \cup [\frac{4}{6}, \frac{5}{6}]$, $C_1 = [\frac{1}{6}, \frac{2}{6}] \cup [\frac{3}{6}, \frac{4}{6}] \cup [\frac{5}{6}, 1]$.

Example 14: In [10] we considered "generalized" β-maps τ_β. In the current notation they can be described as maps with the slopes of constant modulus $\beta > 1$, with $N = \mathrm{Int}(\beta) + 1$, $\alpha_j = 1$ for $j = 1, 2, \ldots, N-1$ and $\alpha_N = \beta - \mathrm{Int}(\beta)$, $\gamma_j = 0$ for $j = 1, 2, \ldots, N-1$ and $\gamma_N = 0$ if $\beta_N > 0$ and $\gamma_N = 1 - \alpha_N$ otherwise. We found a formula for the invariant density of τ_β:

$$h = 1 + \sum_{n=1}^{\infty} \chi[0, \tau_\beta^n(1)] \frac{1}{\beta(1, n)} \ .$$

This representation of h is different from the one obtained in this paper

$$h = 1 + D_1 \cdot \sum_{n=1}^{\infty} \chi^s[\beta(1, n), \tau_\beta^n(1)] \frac{1}{|\beta(1, n)|} \ ,$$

but both define the same function. Usually, $D_1 \neq 1$.

9. *Reduction of the general case to the case of piecewise increasing map.*

In this section we describe an alternative method to obtain invariant density for a general piecewise linear map by reduction to a case of a piecewise increasing map. It is based on so called "Hofbauer's trick" [11].

Let τ be an eventually expanding piecewise linear map of an interval $[0, 1]$ into itself. We will construct a piecewise increasing map τ_{inc} of $[0, 2]$ into itself such that τ is a 2-factor of τ_{inc}. This allows us to obtain τ-invariant density from the τ_{inc}-invariant density (see Figure 9).

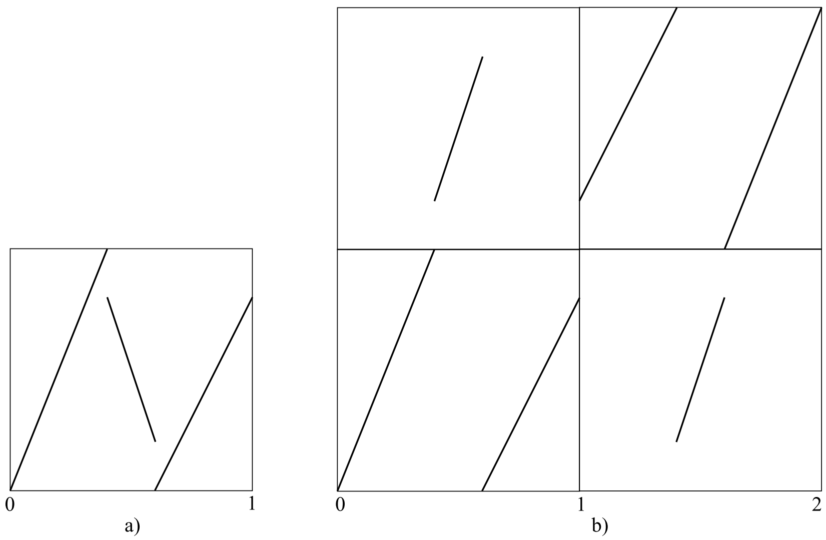

FIGURE 8. "Hofbauer's trick": a) map τ and b) map $\tau_{\rm inc}$

We construct $\tau_{\rm inc}$ as follows (see Figure 8). Recall that $I_1, I_2, \ldots, I_N$ are the domains of the branches of τ. First we define a map $\tau_t : [0,1] \to [0,2]$:

$$\tau_t(x) = \begin{cases} \tau(x) & , \text{ for } \quad x \in [0,1], \ \ x \in I_j, \ \tau \text{ is increasing on } I_j \ ; \\ 2 - \tau(x) & , \text{ for } \quad x \in [0,1], \ \ x \in I_j, \ \tau \text{ is decreasing on } I_j \ ; \end{cases}$$

and then the map $\tau_{\rm inc} : [0,2] \to [0,2]$:

$$\tau_{\rm inc}(x) = \begin{cases} \tau_t(x) & , \text{ for } \quad x \in [0,1] \ ; \\ 2 - \tau_t(2-x) & , \text{ for } \quad x \in (1,2] \ . \end{cases}$$

Then, $\tau_{\rm inc}$ is piecewise increasing and $(\tau, [0,1])$ is the 2-factor of $(\tau_{\rm inc}, [0,2])$ via piecewise diffeomorphism

$$\phi(x) = \begin{cases} x & , \text{ for } \quad x \in [0,1] \ ; \\ 2 - x & , \text{ for } \quad x \in (1,2] \ , \end{cases}$$

i.e., $\tau \circ \phi = \phi \circ \tau_{\rm inc}$. Let $h_{\rm inc}$ be the $\tau_{\rm inc}$-invariant density (we can easily rescale $\tau_{\rm inc}$ to $[0,1]$, use formula (28) and then rescale back). Since the slopes of ϕ are 1 in modulus the τ-invariant density is

$$h(x) = h_{\rm inc}(x) + h_{\rm inc}(2-x) = 2h_{\rm inc}(x) \ , \quad x \in [0,1] \ .$$

The Figures 8 and 9 are prepared for the map τ defined using $\alpha = [1, 0.6, 0.8]$, $\beta = [2.5, -3, 2]$ and $\gamma = [0, 0.2, 0]$, i.e.,

$$\tau(x) = \begin{cases} 2.5x & , \text{ for } \quad x \in [0, 0.4] \ ; \\ -3x + 2 & , \text{ for } \quad x \in (0.4, 0.6] \ ; \\ \ \ 2x - 1.2 & , \text{ for } \quad x \in (0.6, 1] \ . \end{cases}$$

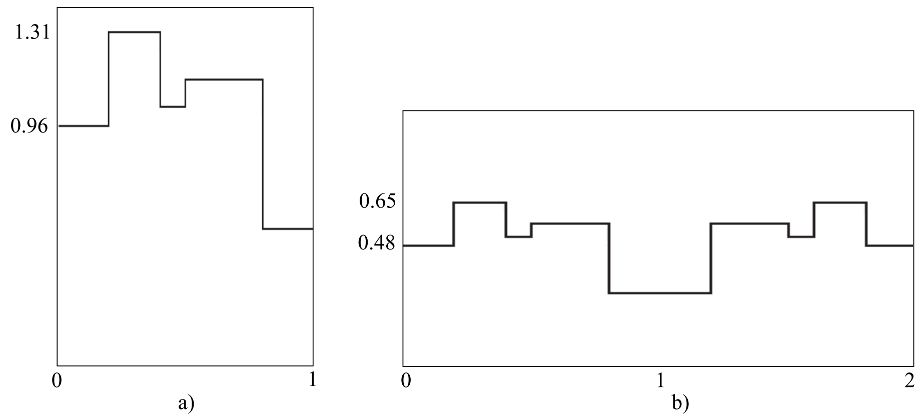

FIGURE 9. Invariant densities of a) map τ and b) map τ_{inc}

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