Ergod. Th. & Dynam. Sys. (2008), **XX**, 1–10
Printed in the United Kingdom

Invariant densities for piecewise linear, piecewise increasing maps

† Paweł Góra

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada (e-mail: pgora@mathstat.concordia.ca) (*Received* ?? *March* 2008)

Abstract. We find an explicit formula for the invariant density h of a piecewise linear, piecewise increasing map τ of an interval $[0,1]$. We do not assume that the slopes of the branches are the same and we allow arbitrary number of shorter branches touching 0 or touching 1 or hanging in between. The construction involves matrix $\mathbf{S}$ which is defined in a way somewhat similar to the definition of the kneading matrix of a continuous piecewise monotonic map. Under some additional assumptions, we prove that if 1 is not an eigenvalue of $\mathbf{S}$, then dynamical system $(\tau, h\cdot m)$ is ergodic with full support.

1. *Introduction*

In this paper we continue the investigations of invariant densities (with respect to Lebesgue measure m) for piecewise linear, piecewise increasing maps. The first results about the classical β-maps were obtained by Rényi [19], Parry [16] and Gelfond [8]. Later, Parry generalized [17] them further. These maps have constant slope, all the branches increasing and only the first or the last (or both) branches can be shorter.

The maps with both increasing and decreasing branches were investigated in [9]. Again, these maps have constant slope (in modulus) and shorter branches were allowed only as the first or the last one.

In this paper we consider piecewise linear maps τ of $[0,1]$ onto itself with increasing branches. We do not assume that the slopes of the branches are the same and allow arbitrary number of shorter branches touching 0 or touching 1 or hanging in between. We assume that τ is onto and that it is eventually piecewise expanding, i.e., for some iterate $|(\tau^n)'| > 1$, wherever it exists.

In our main result, Theorem 2, we find an explicit formula for τ-invariant density h.

The construction of τ-invariant density h involves a matrix $\mathbf{S}$ defined in a way somewhat similar to defining of the kneading matrix of a continuous piecewise monotonic map [1, 14]. In some simple cases we proved that if 1 is not an eigenvalue of $\mathbf{S}$, then dynamical system $(\tau, h \cdot m)$ is ergodic on $[0,1]$. During the work on this paper we performed great number of computer experiments and always found that this holds. Therefore, we state the following conjecture.

Conjecture 1: Let τ be piecewise linear, piecewise increasing and eventually piecewise expanding map . Then, 1 is not an eigenvalue of matrix $\mathbf{S} \Longrightarrow$ dynamical system $(\tau, h \cdot m)$ is ergodic on $[0,1]$.

There are matrix methods of detecting topological transitivity of piecewise monotone continuous interval maps [1, 14], which is implied by ergodicity for our class of maps. Perhaps, matrix $\mathbf{S}$ can be used for this purpose in a more general setting.

The inverse of Conjecture 1 does not hold. It is shown in Example 4.

There are few papers dealing with our type of piecewise linear maps. Absolutely continuous invariant measures for greedy maps with constant slope were investigated in [6, 7] by other methods. A two branches expanding-contracting (α, β)-maps were considered in [3]. Since Theorem (3.1) of [10] implies that (α, β)-maps considered there are eventually piecewise expanding, they are included in our model.

In Section 2 we define all necessary notions and prove the main theorem. In Proposition 1 we introduce τ-expansion of numbers in $[0,1]$ related to our map τ. It is crucial in the considerations of this paper. Similar expansions were considered before under more restrictive assumptions. We followed mainly the ideas of Pedicini [18] who studied so called "greedy" expansions with deleted digits. More general expansions were studied in [5] which we recommend for further information and references.

In Section 3 we discuss the ergodic properties of maps we consider.

In the next three sections we discuss special cases: greedy maps for which shorter branches touch 0, lazy maps with shorter branches touching 1 and the mixed type maps with shorter branches touching either 0 or 1 but not hanging in between. We prove a number of results which hold specifically for these classes. In particular, in Section 4 we discuss special cases of greedy maps with 2,3 or 4 branches.

In this paper we are mainly interested in absolutely continuous τ-invariant measure. The general theory of such measures for piecewise expanding maps of an interval is well developed and we often refer to its results. The classical papers are [12] and [13] among many others. There is a number of books on the subject, see, e. g., [2] or [11].

While working on this project the author used extensively the computer program Maple 11. The programs with examples and illustrations, as well as their pdf printouts, are available at http://www.mathstat.concordia.ca/faculty/pgora/deleted .

2. *Description of the map and the main result.*

In this section we introduce necessary notation and describe the maps we consider. then, we prove the main theorem.

Throughout the paper δ(condition) will denote 1 when the condition is satisfied and 0 otherwise. We denote Lebesgue measure on $[0,1]$ by m.

Let τ be a piecewise linear, piecewise increasing map of interval $[0,1]$ onto itself. Let N denote the number of branches of τ and $K \le N$ the number of shorter, not onto, branches. We allow $L \le K$ shorter branches not to touch 0 or 1. We will call them "hanging" branches.

The map τ can be described by three sequences of N numbers: the lengths of branches $\alpha_1, \alpha_2, \dots, \alpha_N$, with $0 < \alpha_j \le 1$, $j = 1, \dots, N$; the heights of the left hand side endpoints of branches $\gamma_1, \gamma_2, \dots, \gamma_N$, with $0 \le \gamma_j \le 1 - \alpha_j$, $j = 1, \dots, N$; and the slopes of branches $\beta_1, \beta_2, \dots, \beta_N$. We assume $0 < \beta_j$, $j = 1, \dots, N$ and we have

$$\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \cdots + \frac{\alpha_N}{\beta_N} = 1 \ . \tag{1}$$

We do not assume that $1 < \beta_i$ but we will assume that τ is eventually piecewise expanding, i.e., for some iterate τ^n we have $(\tau^n)' > 1$, whenever it is defined. This is necessary for the convergence of the series we consider below.

A shorter branch is called "greedy" if corresponding $\gamma_j = 0$, "lazy" if $\gamma_j + \alpha_j = 1$ and "hanging" if $0 < \gamma_j$ and $\gamma_j + \alpha_j < 1$.

The endpoints of the domains of branches are $b_1 = 0$, $b_j = \frac{\alpha_1}{\beta_1} + \cdots + \frac{\alpha_{j-1}}{\beta_{j-1}}$, $j = 2, 3, \dots, N+1$. Note, $b_{N+1} = 1$.

We assume that map τ is defined on the partition $\mathcal{P}_\tau = \{I_1, I_2, \dots, I_N\}$, where

$$\begin{aligned} I_1 &= [0, b_2) \ ; \\ I_j &= (b_j, b_{j+1}) \quad \text{for} \quad 2 \le j \le N-1 \ ; \\ I_N &= (b_N, 1] \ . \end{aligned} \tag{2}$$

This means that τ is not defined for a countable subset of $[0,1]$, the points b_j, $j = 2, \dots, N$ and their preimages. Since we will have to consider iterates of the points b_j we create two extensions τ_u (upper) and τ_l (lower) of τ. τ_u is the extension of τ by continuity to partition

$$\mathcal{P}_u = \{[0, b_2], (b_2, b_3], \dots, (b_{N-1}, b_N], (b_N, 1]\},$$

and τ_l is the extension of τ by continuity to partition

$$\mathcal{P}_l = \{[0, b_2), [b_2, b_3), \dots, [b_{N-1}, b_N), [b_N, 1]\}.$$

Now, we define the points c_i, $i = 1, 2, \dots, K+L$ which will play major role in the further study. They are the endpoints of the domains of shorter branches at which τ does not touch 0 or 1. Since a point can be the endpoint of two such domains we have to allow for duplication of them.

Each point c_i is actually a pair (c, j) where $c \in [0,1]$ and $1 \le j \le N$ and c is one of the endpoints of interval I_j. We define index function on points c_i: $j(c_i, k) = k$. We define $K + L$ points c_i. They are:

the right hand side endpoints of domains of shorter branches touching 0 ("greedy" branches);

the left hand side endpoints of of domains of shorter branches touching 1 ("lazy" branches);

both endpoints of domains of shorter "hanging" branches.

We number them in such a way that $c_1 < c_2 < \cdots < c_{K+L-1} < c_{K+L}$, where $(c, j) < (d, k)$ if either $c < d$ or $c = d$ and $j < k$. Note, the indices "i" of points c_i do not correspond directly to indices of intervals I_j. We group c_i's into two disjoint sets: W_u containing c_i's associated with "greedy" branches and right hand side endpoints of domains of "hanging" branches ; W_l containing c_i's associated with "lazy" branches and left hand side endpoints of domains of "hanging" branches.

When we consider $\tau(c_i)$ we apply it to the first element of the pair. We always use τ_u to act on elements of W_u and τ_l to act on elements of W_l. Note,

$$\begin{aligned} \tau(c_i) &= \tau_u(c_i) = \alpha_j + \gamma_j \qquad &\text{for} \quad c_i \in W_u \ , \quad \text{where} \quad j = j(c_i) \ , \\ \tau(c_i) &= \tau_l(c_i) = \gamma_j \qquad &\text{for} \quad c_i \in W_l \ , \quad \text{where} \quad j = j(c_i) \ . \end{aligned}$$

Map τ can be conveniently represented using a set of "digits" $A = \{a_1, a_2, \dots, a_N\}$, where

$$a_j = \beta_j b_j - \gamma_j = \beta_j b_{j+1} - (\gamma_j + \alpha_j) \ , \quad j = 1, \dots, N \ .$$

Then, map τ is

$$\tau(x) = \beta \cdot x - a_j, \quad \text{for} \quad x \in I_j \ , \ j = 1, 2, \dots, N \ .$$

Note that each a_j is between the minimal, "lazy" digit $a_j^l = \beta b_{j+1} - 1$ and maximal, "greedy" digit $a_j^u = \beta b_j$, $j = 1, 2, \dots, N$. If the jth branch is onto, then $a_j = a_j^l = a_j^u$.

For any $x \in [0, 1] \setminus \{b_2, \dots, b_N\}$ we define its "index" $j(x)$ and its "digit" $a(x)$:

$$j(x) = j \qquad \text{for} \quad x \in I_j \ , \ j = 1, 2, \dots, N \ ,$$

and

$$a(x) = a_{j(x)}.$$

We can also define (for all $x \in [0, 1]$) the indices $j_u(x)$, $j_l(x)$ and the digits $a_u(x)$, $a_l(x)$ using partitions $\mathcal{P}_u$ and $\mathcal{P}_l$, correspondingly.

We define the cumulative slopes for iterates of points as follows:

$$\begin{aligned} &\beta(x, 1) = \beta_{j(x)} \ ; \\ &\beta(x, n) = \beta(x, n-1) \cdot \beta_{j(\tau^{n-1}(x))} \ , \quad n \geq 2 \ . \end{aligned}$$

The following proposition describes τ-expansion of numbers in $[0, 1]$. It is similar to many known expansions, in particular to β-expansion [16] and "greedy" and "lazy" expansions with deleted digits [5].

PROPOSITION 1. *If τ is eventually expanding, then for any $x \in [0, 1] \setminus \{b_2, \dots, b_N\}$ we have*

$$x = \sum_{n=1}^{\infty} \frac{a(\tau^{n-1}(x))}{\beta(x, n)} \ .$$

Moreover,

$$\tau^k(x) = \beta(x,k) \cdot \sum_{n=k+1}^{\infty} \frac{a(\tau^{n-1}(x))}{\beta(x,n)} \ ,$$

for any $k \geq 0$*.*

Proof: We have $\tau(x) = \beta_{j(x)} x - a(x)$ or

$$x = \frac{a(x)}{\beta(x,1)} + \frac{\tau(x)}{\beta(x,1)} \ .$$

Using this equality inductively n-times we obtain

$$x = \frac{a(x)}{\beta(x,1)} + \frac{a(\tau(x))}{\beta(x,2)} + \cdots + \frac{a(\tau^{n-1}(x))}{\beta(x,n)} + \frac{\tau^n(x)}{\beta(x,n)} \ ,$$

which proves both statements. Since τ is eventually expanding $\beta(x,n) \to 0$ as $n \to +\infty$ and the series giving the expansion is convergent. □

We will call the representation defined in Proposition 1 the τ-expansion of x. In the same way we define "greedy" and "lazy" expansions using maps τ_u and τ_l. All three expansions are identical for almost all $x \in [0,1]$. To represent points c_i we will use "greedy" expansion if $c_i \in W_u$ and "lazy" expansion if $c_i \in W_l$.

An integrable nonnegative function h is a density of an m-absolutely continuous τ-invariant measure if and only if it satisfies Perron-Frobenius equation:

$$h(x) = \sum_{y:\tau(y)=x} h(y)/|\tau'(y)| = (P_\tau(h))(x),$$

for almost all $x \in [0,1]$. Operator P_τ is called Perron-Frobenius operator [2].

Let us define

$$\begin{aligned} S_{i,j} &= \sum_{n=1}^{\infty} \frac{1}{\beta(c_i,n)} \delta(\tau_u^n(c_i) > c_j) \ , \qquad \text{for} \quad c_i \in W_u \ \text{ and all } \ c_j \ , \\ S_{i,j} &= \sum_{n=1}^{\infty} \frac{1}{\beta(c_i,n)} \delta(\tau_l^n(c_i) < c_j) \ , \qquad \text{for} \quad c_i \in W_l \ \text{ and all } \ c_j \ . \end{aligned} \tag{3}$$

Let $\mathbf{S}$ be the matrix $(S_{i,j})_{1 \leq i,j \leq K+L}$ and $\mathbf{Id}$ denote $(K+L) \times (K+L)$ identity matrix. Let $\mathbf{v} = [1,1,\ldots,1,1]$ be $(K+L)$-dimensional vector of 1's and let $D = [D_1, \ldots, D_{K+L}]$ denote the solution of the system

$$(-\mathbf{S}^T + \mathbf{Id})D = D_0 \mathbf{v} \ , \tag{4}$$

where A^T denotes the transpose of A and parameter D_0 is either 1 or 0. We make here some comments about the parameter D_0 although their meaning may become clear only later. Since the non-normalized invariant density (5) is defined up to a multiplicative constant we consider only $D_0 = 1$ or $D_0 = 0$. In most cases we will use $D_0 = 1$. There may be a few reasons for the equation (4) to be unsolvable with $D_0 = 1$. First, τ can be ergodic but with support of invariant density I strictly smaller that $[0,1]$. In this case we consider τ restricted to I and rescaled back to $[0,1]$ rather than considering $D_0 = 0$. Secondly, τ may be either ergodic on $[0,1]$ or nonergodic with union of supports of invariant densities equal to $[0,1]$ but with matrix $\mathbf{S}$ having 1 as an eigenvalue. In these cases we consider $D_0 = 0$.

Theorem 2. *Let τ will be the map defined in this section, i.e., any piecewise linear, piecewise increasing map which is eventually piecewise expanding. System (4) always has a non-vanishing solution. If 1 is not an eigenvalue of* $\mathbf{S}$*, then with* $D_0 = 1$*. If 1 is an eigenvalue of* $\mathbf{S}$*, then at least with* $D_0 = 0$*. Let*

$$h(x) = D_0 + \sum_{i\in W_u} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau_u^n(c_i)]} \frac{1}{\beta(c_i,n)} + \sum_{i\in W_l} D_i \sum_{n=1}^{\infty} \chi_{[\tau_l^n(c_i),1]} \frac{1}{\beta(c_i,n)} \ , \qquad (5)$$

where constants D_i, $i = 1,\dots,K$, satisfy the system (4). Then h is τ-invariant.

If all values $\tau(c_i)$, $i = 1,\dots,K+L$, are different, then the inverse statement also holds: If h is τ-invariant, then the constants $D_0, D_1,\dots,D_{K+L}$ satisfy the system (4).

In particular, system (4) is uniquely solvable (i.e., 1 is not an eigenvalue of $\mathbf{S}$*) if $\min_{1\le j\le N}\beta_j > K+L+1$. If the last branch is "greedy " or "hanging" and the first branch is "lazy" or "hanging", then condition $\min_{1\le j\le N}\beta_j > K+L$ is sufficient.*

Proof: Let $x \in [0,1]$ and $x(j)$, $j = 1,2,\dots,N$ be the jth τ-preimage of x, if it exists. We need to show that

$$h(x) = \sum_{j=1}^{N} \frac{h(x(j))}{\beta_j} \ ,$$

for almost all $x \in [0,1]$.

We have

$$\sum_{j=1}^{N} \frac{1(x(j))}{\beta_j} = \sum_{j=1}^{N} \frac{1}{\beta_j} - \sum_{c_k\in W_u} \frac{\delta(x > \tau(c_k))}{\beta_{j(c_k)}} - \sum_{c_k\in W_l} \frac{\delta(x < \tau(c_k))}{\beta_{j(c_k)}} \ . \qquad (6)$$

For $c_k \in W_u$ we have

$$\begin{aligned}\sum_{j=1}^{N} \frac{\chi_{[0,\tau^n(c_k)]}(x(j))}{\beta_j} &= \sum_{j=1}^{j(\tau^n(c_k))-1} \frac{1}{\beta_j} + \frac{\delta(x \le \tau^{n+1}(c_k))}{\beta_{j(\tau^n(c_k))}} \\ &- \sum_{c_i\in W_u} \frac{\delta(x > \tau(c_i))\cdot\delta(\tau^n(c_k) > c_i)}{\beta_{j(c_i)}} \\ &- \sum_{c_i\in W_l} \frac{\delta(x < \tau(c_i))\cdot\delta(\tau^n(c_k) > c_i)}{\beta_{j(c_i)}} \ .\end{aligned} \qquad (7)$$

For $c_k \in W_l$ we have

$$\begin{aligned}\sum_{j=1}^{N} \frac{\chi_{[\tau^n(c_k),1]}(x(j))}{\beta_j} &= \sum_{j=j(\tau^n(c_k))+1}^{N} \frac{1}{\beta_j} + \frac{\delta(x \ge \tau^{n+1}(c_k))}{\beta_{j(\tau^n(c_k))}} \\ &- \sum_{c_i\in W_u} \frac{\delta(x > \tau(c_i))\cdot\delta(\tau^n(c_k) < c_i)}{\beta_{j(c_i)}} \\ &- \sum_{c_i\in W_l} \frac{\delta(x < \tau(c_i))\cdot\delta(\tau^n(c_k) < c_i)}{\beta_{j(c_i)}} \ .\end{aligned} \qquad (8)$$

Let us define

$$
\begin{aligned}
S_k &= \sum_{n=1}^{\infty} \sum_{j=1}^{j(\tau^n(c_k))-1} \frac{1}{\beta_j \cdot \beta(c_k,n)} \ , \quad \text{for} \quad c_k \in W_u \ , \\
S_k &= \sum_{n=1}^{\infty} \sum_{j=j(\tau^n(c_k))+1}^{N} \frac{1}{\beta_j \cdot \beta(c_k,n)} \ , \quad \text{for} \quad c_k \in W_l \ .
\end{aligned} \tag{9}
$$

Using previous equalities and $\beta_{j(\tau^n(c_k))} \cdot \beta(c_k,n) = \beta(c_k,n+1)$, we write

$$
\begin{aligned}
\sum_{j=1}^{N} \frac{h(x(j))}{\beta_j} = D_0 \Big[\sum_{j=1}^{N} \frac{1}{\beta_j} &- \sum_{c_k \in W_u} \frac{\delta(x > \tau(c_k))}{\beta_{j(c_k)}} - \sum_{c_k \in W_l} \frac{\delta(x < \tau(c_k))}{\beta_{j(c_k)}} \Big] \\
&+ \sum_{c_k \in W_u} D_k \Big[S_k + \sum_{n=1}^{\infty} \frac{\delta(x \le \tau^{n+1}(c_k))}{\beta(c_k,n+1)} \\
&\qquad - \sum_{c_i \in W_u} \delta(x > \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} - \sum_{c_i \in W_l} \delta(x < \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} \Big] \\
&+ \sum_{c_k \in W_l} D_k \Big[S_k + \sum_{n=1}^{\infty} \frac{\delta(x \ge \tau^{n+1}(c_k))}{\beta(c_k,n+1)} \\
&\qquad - \sum_{c_i \in W_u} \delta(x > \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} - \sum_{c_i \in W_l} \delta(x < \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} \Big] \ .
\end{aligned} \tag{10}
$$

Adding and subtracting D_0, $\sum_{c_k \in W_u} D_k \frac{\delta(x \le \tau(c_k))}{\beta(c_k,1)}$ and $\sum_{c_k \in W_l} D_k \frac{\delta(x \ge \tau(c_k))}{\beta(c_k,1)}$ we eliminate $h(x)$ from the right hand side of (10) and we see that we are looking for constants D_i, $i = 1, \ldots, K+L$, such that the following equality (11) is satisfied for all $x \in [0,1]$ except possibly the images of points c_i.

$$
\begin{aligned}
&\sum_{c_k \in W_u} D_k \Big[S_k - \frac{\delta(x \le \tau(c_k))}{\beta(c_k,1)} \\
&\qquad - \sum_{c_i \in W_u} \delta(x > \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} - \sum_{c_i \in W_l} \delta(x < \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} \Big] \\
&+ \sum_{c_k \in W_l} D_k \Big[S_k - \frac{\delta(x \ge \tau(c_k))}{\beta(c_k,1)} \\
&\qquad - \sum_{c_i \in W_u} \delta(x > \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} - \sum_{c_i \in W_l} \delta(x < \tau(c_i)) \frac{S_{k,i}}{\beta_{j(c_i)}} \Big] \\
&= D_0 \Big[1 - \sum_{j=1}^{N} \frac{1}{\beta_j} + \sum_{c_k \in W_u} \frac{\delta(x > \tau(c_k))}{\beta_{j(c_k)}} + \sum_{c_k \in W_l} \frac{\delta(x < \tau(c_k))}{\beta_{j(c_k)}} \Big] \ .
\end{aligned} \tag{11}
$$

Let us assume tentatively that all values $\tau(c_i)$, $i = 1, \ldots, K+L$, are different. Then, they divide interval $(0,1)$ into $K+L+1$ disjoint open subinterval. Let us chose one point x from each of the subintervals and number them in the increasing order $x_0 < x_1 < x_2 < \cdots < x_{K+L}$. If equality (11) holds for these points, then it holds for almost every $x \in [0,1]$. Substituting points x_i into (11) we obtain equations which we denote by E_i, $i = 0, \dots, K+L$. Together, we obtain system of $K+L+1$ equations which we denote by ES. Rather than write it down we create from it a simplified equivalent system denoted by EQS. We proceed as follows: consider two consecutive points $x_i < \tau(c_k) < x_{i+1}$. If $c_k \in W_u$, then the difference $EQ_k = E_{i+1} - E_i$ is

$$-\sum_{\substack{j=1 \\ j\neq k}}^{K+L} D_j \frac{S_{k,j}}{\beta_{j(c_k)}} - D_k\Big[\frac{S_{k,k}}{\beta_{j(c_k)}} - \frac{1}{\beta_{j(c_k)}}\Big] = \frac{D_0}{\beta_{j(c_k)}} \ . \tag{12}$$

If $c_k \in W_l$, then the difference $EQ_k = E_i - E_{i+1}$ is of the above form. The equations $\{EQ_1, EQ_2, \dots, EQ_{K+L}\}$ form the system EQS which is obviously equivalent to the system $\{\beta_{j(c_1)}EQ_1, \beta_{j(c_2)}EQ_2, \dots, \beta_{j(c_{K+L})}EQ_{K+L}\}$, which is the system (4). In ES we have one more equation which can be reduced to EQ_{K+L+1} of the form

$$\sum_{k=1}^{K+L} D_k\Big[S_k - \frac{1}{\beta_{j(c_k)}}\Big] = D_0\Big[1 - \sum_{j=1}^{N}\frac{1}{\beta_j}\Big] \ . \tag{13}$$

If some level x_i intersects all branches of τ, then equation E_i is of form (13). If not, then we take x_i which level intersects most branches of τ and reduce if to form (13) subtracting appropriate equations EQ_k.

The systems ES and $EQS \cup \{EQ_{K+L+1}\}$ are equivalent since we can recover equations of ES from equations $EQ_1, \dots, EQ_{K+L}, EQ_{K+L+1}$. To prove the equivalence of systems ES and EQS it is enough to show that EQ_{K+L+1} is a linear combination of equations EQ_i, $i = 1, \dots, K+L$. We will do it as follows: If $c_k \in W_u$ we set $\eta_k = 1 - \gamma_{j(c_k)} - \alpha_{j(c_k)}$. If $c_k \in W_l$ we set $\eta_k = \gamma_{j(c_k)}$. Note that if c_k is the right hand side endpoint of the domain of greedy branch, then $\eta_k = 1 - \alpha_{j(c_k)}$ and if c_k is the left hand side endpoint of the domain of lazy branch, then also $\eta_k = 1 - \alpha_{j(c_k)}$. Then, we have

$$EQ_{K+L+1} + \sum_{k=1}^{K+L} \eta_k \cdot EQ_k \quad \Longleftrightarrow \quad 0 = 0 \ .$$

First, let us consider the right hand side of the summed up equations. We have

$$\begin{aligned} 1 - \sum_{j=1}^{N}\frac{1}{\beta_j} + \sum_{k=1}^{K+L}\eta_k\frac{1}{\beta_{j(c_k)}} &= 1 - \sum_{j=1}^{N}\frac{1}{\beta_j} + \sum_{\substack{1\le k\le N \\ k\text{-th branch is shorter}}} \frac{1-\alpha_k}{\beta_k} \\ &= 1 - \sum_{j=1}^{N}\frac{\alpha_j}{\beta_j} = 0 \ . \end{aligned} \tag{14}$$

Now, let us consider the summed up coefficients of D_k (summed up k-th column of the system). We have to show

$$S_k - \frac{1}{\beta_{j(c_k)}} - \sum_{j=1}^{K+L}\eta_j\frac{S_{k,j}}{\beta_{j(c_j)}} + \eta_k\frac{1}{\beta_{j(c_k)}} = 0 \ . \tag{15}$$

First, we consider $c_k \in W_u$. Then, we have

$$\frac{-1}{\beta_{j(c_k)}} + \eta_k \frac{1}{\beta_{j(c_k)}} = \frac{-(\gamma_{j(c_k)} + \alpha_{j(c_k)})}{\beta_{j(c_k)}} = \frac{-\tau(c_k)}{\beta_{j(c_k)}} \ . \tag{16}$$

On the other hand

$$S_k - \sum_{j=1}^{K+L} \eta_j \frac{S_{k,j}}{\beta_{j(c_j)}} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_k, n)} \Big[\sum_{j=1}^{j(\tau^n(c_k))-1} \frac{1}{\beta_j} - \sum_{j=1}^{K+L} \eta_j \frac{\delta(\tau^n(c_k) > c_j)}{\beta_{j(c_j)}} \Big] \ . \tag{17}$$

Let us fix n for a moment and consider the expression in the brackets above. Let $j_0 = j(\tau^n(c_k))$. The expression in the brackets is equal to

$$\sum_{j=1}^{j_0-1} \frac{1}{\beta_j} - \sum_{\substack{j<j_0 \\ j\text{-th branch is shorter}}} \frac{1-\alpha_{j(c_j)}}{\beta_{j(c_j)}} - \frac{\gamma_{j_0}}{\beta_{j_0}} = b_{j_0} - \frac{\gamma_{j_0}}{\beta_{j_0}} = \frac{a(\tau^n(c_k))}{\beta_{j(\tau^n(c_k))}} \ .$$

Thus, the sum on the right hand side of (17) is

$$\sum_{n=1}^{\infty} \frac{1}{\beta(c_k, n)} \frac{a(\tau^n(c_k))}{\beta_{j(\tau^n(c_k))}} = \sum_{n=1}^{\infty} \frac{a(\tau^n(c_k))}{\beta(c_k, n+1)} = \frac{\tau(c_k)}{\beta_{j(c_k)}} \ .$$

With (16) this proves (15) for $c_k \in W_u$.

Now, let us consider $c_k \in W_l$. We have

$$\frac{-1}{\beta_{j(c_k)}} + \eta_k \frac{1}{\beta_{j(c_k)}} = \frac{-1+\gamma_{j(c_k)}}{\beta_{j(c_k)}} = \frac{-(1-\tau(c_k))}{\beta_{j(c_k)}} \ , \tag{18}$$

and

$$S_k - \sum_{j=1}^{K+L} \eta_j \frac{S_{k,j}}{\beta_{j(c_j)}} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_k, n)} \Big[\sum_{j=j(\tau^n(c_k))+1}^{N} \frac{1}{\beta_j} - \sum_{j=1}^{K+L} \eta_j \frac{\delta(\tau^n(c_k) < c_j)}{\beta_{j(c_j)}} \Big] \ . \tag{19}$$

Let us fix n for a moment and consider the expression in the brackets above. Let $j_0 = j(\tau^n(c_k))$. This expression is equal to

$$\begin{aligned} &\sum_{j=j_0+1}^{N} \frac{1}{\beta_j} - \sum_{\substack{j>j_0 \\ j\text{-th branch is shorter}}} \frac{1-\alpha_{j(c_j)}}{\beta_{j(c_j)}} - \frac{1-\gamma_{j_0}-\alpha_{j_0}}{\beta_{j_0}} \\ &= 1 - b_{j_0+1} - \frac{1-\gamma_{j_0}-\alpha_{j_0}}{\beta_{j_0}} = 1 - \frac{1}{\beta_{j(\tau^n(c_k))}} - \frac{a(\tau^n(c_k))}{\beta_{j(\tau^n(c_k))}} \ . \end{aligned}$$

Thus, the series on the right hand side of (19) equals

$$\sum_{n=1}^{\infty} \frac{1}{\beta(c_k, n)} - \sum_{n=1}^{\infty} \frac{1}{\beta(c_k, n+1)} - \sum_{n=1}^{\infty} \frac{a(\tau^n(c_k))}{\beta(c_k, n=1)} = \frac{1}{\beta_{j(c_k)}} - \frac{\tau(c_k))}{\beta_{j(c_k)}} \ .$$

With (18) this proves (15) for $c_k \in W_l$. We have proved the equivalence of the systems ES and EQS (or (4)) when all values $\tau(c_i)$, $i = 1, \ldots, K+L$, are different.

Now, we briefly describe the situation when some of the values $\tau(c_i)$, $i = 1,\dots,K+L$, coincide. The systems ES and (4) may not be equivalent but solutions of (4) always satisfy ES as well.

If $\tau(c_{i_1}) = \tau(c_{i_2})$ and the points c_{i_1}, c_{i_2} are of different type, i.e., $c_{i_1} \in W_u$ and $c_{i_2} \in W_l$ or vice versa, then substituting point $x = \tau(c_{i_1}) = \tau(c_{i_2})$ into (11) gives us an equation which "separates" c_{i_1} and c_{i_2}. Everything proceeds as in the case of different values $\tau(c_i)$.

If $\tau(c_{i_1}) = \tau(c_{i_2})$ and the points c_{i_1}, c_{i_2} are of the same type, then we cannot produce sufficient number of test points x_i and the number of equations in ES is smaller than $K + L + 1$. Similarly as before we can obtain equations EQ_i for c_i with distinct values and an equation EQ_{i_1,i_2} corresponding to points c_{i_1}, c_{i_2}. If more groups of of c_i's of the same type with equal values occurs, then there will be more such common equations. The equation EQ_{i_1,i_2} is the sum of two equations of the form (12) corresponding to indices $k = i_1$ and $k = i_2$. Any common equation is a sum of the corresponding equations of the form (12). Thus, any solution of the system (4) satisfies the system ES. The linear dependence of the extra equation (13) is proved exactly as above. (Note that if $\tau(c_{i_1}) = \tau(c_{i_2})$ and they are of the same type, then $\eta_{i_1} = \eta_{i_2}$.) This completes the proof of the first part of the theorem.

In the proof of the second part we will use the following fragment of Perron-Frobenius theorem for non-negative matrices [15].

THEOREM 3. *If* $\mathbf{S} = (S_{i,j})_{1\le i,j\le M}$ *is a matrix with non-negative entries, then all eigenvalues* λ *of* $\mathbf{S}$ *satisfy*

$$|\lambda| \le \max_{1\le i\le M} \sum_{j=1}^{M} S_{i,j} \ . \tag{20}$$

Note that the assumptions of the second part imply that $\beta = \min_{1\le j\le N} \beta_j > 1$. For each $S_{i,j}$ we have

$$S_{i,j} \le \sum_{n=1}^{\infty} \frac{1}{\beta^n} = \frac{1}{\beta - 1} \ .$$

Thus, if $\beta > K + L + 1$ we have $\frac{K+L}{\beta-1} < 1$ which by Perron-Frobenius estimate implies that 1 is not an eigenvalue of $\mathbf{S}$ and the system (4) is uniquely solvable.

If the last branch of τ is greedy or hanging then $c_{K+L} = 1$. Then, for any $c_i \in W_u$ we have $S_{i,K+L} = 0$. Similarly, if the first branch is lazy or hanging, then $c_1 = 0$ and for any $c_i \in W_l$ we have $S_{i,1} = 0$. Thus, if both conditions occur at the same time there is at least one 0 in each row of $\mathbf{S}$ and Perron-Frobenius estimate implies that 1 is not an eigenvalue of $\mathbf{S}$ for $\beta > K + L$. □

In the two examples below we illustrate the proof of Theorem 2.

Example 1: In this example all values $\tau(c_i)$ are different. Let $N = 4$ and let τ be defined by the vectors

$$\alpha = [0.7, 0.2, 1, 0.45] \ \ , \ \ \beta = [2, 3, 4, 1.35] \ \ , \ \ \gamma = [0, 0.2, 0, 0.55] \ .$$

We have $K = 3$ and $L = 1$. The graph of τ is shown in Figure 1 a). The digits are $\{0, 0.85, 1.66\dots, 0.35\}$. The first branch of τ is greedy, the second hanging, the third onto and the last one is lazy. The points c_i are $c_1 = 0.35 = (0.35, 1)$, $c_2 = 0.35 = (0.35, 2)$, $c_3 = 0.4166\ldots$, $c_4 = 0.66\ldots$. $c_1, c_3 \in W_u$ and $c_2, c_4 \in W_l$. We have $0 < \tau(c_2) < \tau(c_3) < \tau(c_4) < \tau(c_1) < 1$ and taking the points $x_0 < x_1 < x_2 < x_3 < x_4$ between them we obtain the system ES: (we show only the coefficients)

$$
\begin{array}{lllll}
S_1 - \frac{S_{1,2}}{\beta_2} - \frac{S_{1,4}}{\beta_4} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,2}}{\beta_2} - \frac{S_{2,4}}{\beta_4} & S_3 - \frac{S_{3,2}}{\beta_2} - \frac{S_{3,4}}{\beta_4} - \frac{1}{\beta_2} & S_4 - \frac{S_{4,2}}{\beta_2} - \frac{S_{4,4}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_3} \\
S_1 - \frac{S_{1,4}}{\beta_4} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,4}}{\beta_4} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,4}}{\beta_4} - \frac{1}{\beta_2} & S_4 - \frac{S_{4,4}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} \\
S_1 - \frac{S_{1,3}}{\beta_2} - \frac{S_{1,4}}{\beta_4} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,3}}{\beta_2} - \frac{S_{2,4}}{\beta_4} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,3}}{\beta_2} - \frac{S_{3,4}}{\beta_4} & S_4 - \frac{S_{4,3}}{\beta_2} - \frac{S_{4,4}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_3} \\
S_1 - \frac{S_{1,3}}{\beta_2} - \frac{1}{\beta_1} & S_2 - \frac{S_{2,3}}{\beta_2} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,3}}{\beta_2} & S_4 - \frac{S_{4,3}}{\beta_2} - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_3} - \frac{1}{\beta_4} \\
S_1 - \frac{S_{1,1}}{\beta_1} - \frac{S_{1,3}}{\beta_2} & S_2 - \frac{S_{2,1}}{\beta_1} - \frac{S_{2,3}}{\beta_2} - \frac{1}{\beta_2} & S_3 - \frac{S_{3,1}}{\beta_1} - \frac{S_{3,3}}{\beta_2} & S_4 - \frac{S_{4,1}}{\beta_1} - \frac{S_{4,3}}{\beta_2} - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_3} - \frac{1}{\beta_4}
\end{array}
$$

System ES is simplified to equivalent system $EQS \cup \{EQ_{K+L+1}\}$: $EQ_1 = E_4 - E_3$, $EQ_2 = E_0 - E_1$, $EQ_3 = E_2 - E_1$ and $EQ_4 = E_3 - E_2$. The fifth equation can be obtained as $EQ_5 = E_3 - EQ_3$.

$$
\begin{array}{lllll}
-\frac{S_{1,1}}{\beta_1} + \frac{1}{\beta_1} & -\frac{S_{2,1}}{\beta_1} & -\frac{S_{3,1}}{\beta_1} & -\frac{S_{4,1}}{\beta_1} & \frac{1}{\beta_1} \\
-\frac{S_{1,2}}{\beta_2} & -\frac{S_{2,2}}{\beta_2} + \frac{1}{\beta_2} & -\frac{S_{3,2}}{\beta_2} & -\frac{S_{4,2}}{\beta_2} & \frac{1}{\beta_2} \\
-\frac{S_{1,3}}{\beta_2} & -\frac{S_{2,3}}{\beta_2} & -\frac{S_{3,3}}{\beta_2} + \frac{1}{\beta_2} & -\frac{S_{4,3}}{\beta_2} & \frac{1}{\beta_2} \\
-\frac{S_{1,4}}{\beta_4} & -\frac{S_{2,4}}{\beta_4} & -\frac{S_{3,4}}{\beta_4} & -\frac{S_{4,4}}{\beta_4} + \frac{1}{\beta_4} & \frac{1}{\beta_4} \\
S_1 - \frac{1}{\beta_1} & S_2 - \frac{1}{\beta_2} & S_3 - \frac{1}{\beta_2} & S_4 - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} - \frac{1}{\beta_4}
\end{array}
$$

For $D_0 = 1$ the solution of system (4) is $D \simeq [-0.876, -0.876, -0.883, -16.539]$. The normalizing constant is $\simeq -7.812$. The normalized τ-invariant density is shown in Figure 2 a).

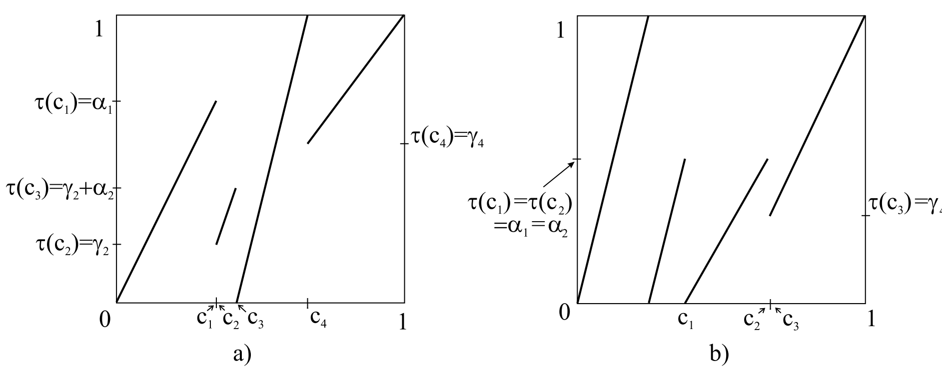

FIGURE 1. Maps τ of a) Example 1 and b) Example 2.

Example 2: Here, we have $\tau(c_1) = \tau(c_2)$. Let $N = 4$ and let τ be defined by the vectors

$$\alpha = [1, 0.5, 0.5, 0.7] \ , \ \ \beta = [4, 3, 2, 2.1] \ , \ \ \gamma = [0, 0, 0, 0.3] \ .$$

We have $K = 3$, $L = 0$. The graph of τ is shown in Figure 1 b). The digits are $\{0, 0.75, 0.833\ldots, 1.1\}$. The first branch of τ is onto, the second and third are greedy and the last one is lazy. The points c_i are $c_1 = 0.4166\ldots$, $c_2 = 0.66\cdots = (0.66\ldots, 3)$, $c_3 = 0.66\cdots = (0.66\ldots, 4)$. $c_1, c_2 \in W_u$ and $c_3 \in W_l$.

We have $0 < \tau(c_3) < \tau(c_1) = \tau(c_2) < 1$ and taking the points $x_0 < x_1 < x_2$ between them we obtain the system ES: (again, we show only the coefficients)

$$\begin{array}{llll}
S_1 - \frac{S_{1,3}}{\beta_4} - \frac{1}{\beta_2} & S_2 - \frac{S_{2,3}}{\beta_4} - \frac{1}{\beta_3} & S_3 - \frac{S_{3,3}}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} \\
S_1 - \frac{1}{\beta_2} & S_2 - \frac{1}{\beta_3} & S_3 - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} - \frac{1}{\beta_4} \\
S_1 - \frac{S_{1,1}}{\beta_2} - \frac{S_{1,2}}{\beta_3} & S_2 - \frac{S_{2,1}}{\beta_2} - \frac{S_{2,2}}{\beta_3} & S_3 - \frac{S_{3,1}}{\beta_2} - \frac{S_{3,2}}{\beta_3} - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_4}
\end{array}$$

Again, system ES is simplified to equivalent system $EQS \cup \{EQ_{K+L+1}\}$: $EQ_1 = EQ_2 = E_2 - E_1$, $EQ_3 = E_0 - E_1$. The third (or formally the fourth) equation can be obtained as $EQ_4 = E_1$.

$$\begin{array}{llll}
-\frac{S_{1,1}}{\beta_2} - \frac{S_{1,2}}{\beta_3} + \frac{1}{\beta_2} & -\frac{S_{2,1}}{\beta_2} - \frac{S_{2,2}}{\beta_3} + \frac{1}{\beta_3} & -\frac{S_{3,1}}{\beta_2} - \frac{S_{3,2}}{\beta_3} & \frac{1}{\beta_2} + \frac{1}{\beta_3} \\
-\frac{S_{1,3}}{\beta_4} & S_2 - \frac{S_{2,3}}{\beta_4} & -\frac{S_{3,3}}{\beta_4} + \frac{1}{\beta_4} & \frac{1}{\beta_4} \\
S_1 - \frac{1}{\beta_2} & S_2 - \frac{1}{\beta_3} & S_3 - \frac{1}{\beta_4} & 1 - \frac{1}{\beta_1} - \frac{1}{\beta_2} - \frac{1}{\beta_3} - \frac{1}{\beta_4}
\end{array} \qquad (21)$$

The solution of system (4), for $D_0 = 1$, is $D \simeq [8.794, 3.382, 3.382]$. System (21) is not equivalent to (4), but solution of (4) satisfies also (21). System (21) has infinitely many solutions $D^{(t)} \simeq [t, 9.2447 - 0.6667t, 3.382]$. We have $D = D^{(t)}$ for $t = D_1$. The functions

$$h_1 = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]} \frac{1}{\beta(c_1, n)} \quad \text{and} \quad h_2 = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_2)]} \frac{1}{\beta(c_2, n)}$$

are proportional, $\beta_2 h_1 = \beta_3 h_2$, and the invariant density h stays the same whether we use constants D_1, D_2, D_3 or $D_1^{(t)}, D_2^{(t)}, D_3^{(t)}$ for arbitrary t. The normalizing constant is $\simeq 5.989$. The normalized τ-invariant density is shown in Figure 2 b).

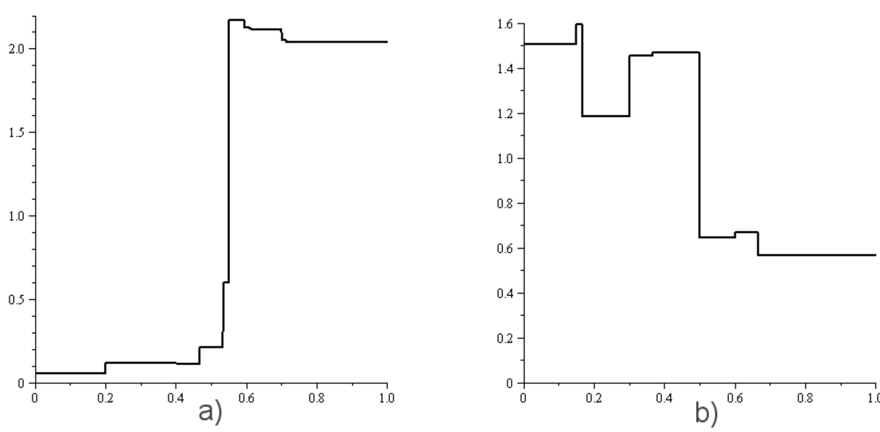

FIGURE 2. Invariant densities for maps of a) Example 1 and b) Example 2.

In the next example we show a map τ which is not ergodic. Matrix $\mathbf{S}$ has an eigenvalue 1. The system (4) with $D_0 = 1$ is solvable (non-uniquely). Both methods, i.e., using $D_0 = 1$ or $D_0 = 0$, of finding τ-invariant density agree.

Example 3: Let $N = 8$ and τ be defined by the constant slope $\beta = 3$ and the vectors

$$\alpha = [0.5, 0.25, 0.25, 0.5, 0.5, 0.25, 0.25, 0.5] \quad , \quad \gamma = [0, 0, 0.1, 0, 0.5, 0.65, 0.75, 0.5] \, .$$

The graph of τ is shown in Figure 3 a). The matrix

$$\mathbf{S} \simeq \begin{bmatrix} 0.5 & 0.5 & 0.34654 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.34654 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.34654 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0.35 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.487037 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.45 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.45 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0.45 & 0.116667 & 0.5 & 0.5 \end{bmatrix} .$$

For $D_0 = 1$ system (4) has solutions

$$D(t) \simeq \Big[t,\ \frac{2t}{3},\ \frac{2t}{3},\ 0.769231t,\ 0,$$
$$0,\ -0.691358t - 2.074074,\ -\frac{2t}{3} - 2,\ -\frac{2t}{3} - 2,\ -0.77778t - \frac{7}{3}\Big] \, .$$

The eigenvector of $\mathbf{S}$ corresponding to the eigenvalue 1 is

$$D_v \simeq [-0.943423,\ -0.628949,\ -0.628949,\ -0.725710,\ 0,$$
$$0,\ 0.652243,\ 0.628949,\ 0.628949,\ 0.733774] \, .$$

The τ-invariant densities are shown in Figure 3 b). Density for $D_0 = 1$ and constants $D(-0.5)$ is shown in black, density for $D_0 = 1$ and constants $D(-1.9)$ is shown in gray, and density for $D_0 = 0$ and constants D_v is shown in gray dash line. The last one happens to be a combination of negative density for one ergodic component and a positive density for the other one.

3. *Ergodic properties of piecewise linear, piecewise increasing maps*

In this section we discuss the ergodic implication of having invariant density with full support. In particular, this applies to any τ satisfying the assumptions of Theorem 2 with $D_0 = 1$.

Theorem 4. *Let τ be a piecewise linear, piecewise increasing and eventually piecewise expanding map which admits an invariant density supported on* $[0, 1]$*. Then, if at least one branch of τ is onto then τ has at most two ergodic components. If at least two branches are onto, then τ is exact.*

Proof: It follows from the general theory (for example [2, Chapter 8]) that τ has finite number of ergodic components and the support of each ergodic component consists of a finite number of intervals. To prove exactness of an ergodic component it is enough to show that the images of arbitrarily small interval in the component grow to cover the whole domain of the component.

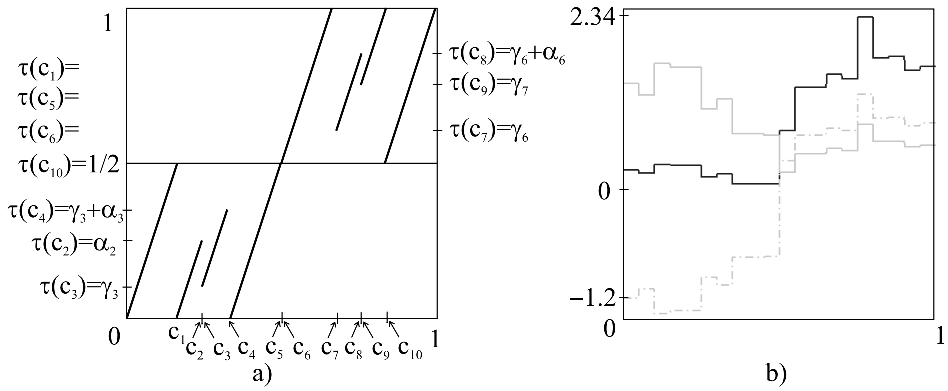

FIGURE 3. Map τ of Example 3 and three versions of its invariant density.

If τ has an onto branch, then let x_0 be a fixed point in the domain of this branch. There are two possibilities:

a) Some neighborhood J of x_0 is contained in one ergodic component of τ. Then, the images $\tau^n(J)$ grow to cover the whole $[0,1]$ and τ has one exact component.

b) τ has at least two ergodic components and some intervals J_1 of one component and J_2 of the second component touch x_0. Let $J_1 \subset [0,x_0)$. Then, the images $\tau^n(J_1)$ grow to cover $[0,x_0)$ and the images $\tau^n(J_2)$ grow to cover $(x_0,1]$. τ has two ergodic components.

If τ has at least two onto branches, then the fixed points in these branches, x_0 and x_1 are different. Each of intervals $[0,x_0]$, $[0,x_1]$, $[x_0,1]$, $[x_1,1]$, is completely contained in a support of an ergodic component. Thus, we have at most one ergodic component. Since arbitrary neighborhood of any of these fixed points grows under iteration to cover the whole $[0,1]$ the system is exact. □

COROLLARY 5. *If 1 is not an eigenvalue of* $\mathbf{S}$ *and* τ *has at least two onto branches, then the system* $(\tau, h\cdot m)$ *is exact.*

In Example 4 we show that the inverse of Corollary 5 is not always true.

Example 4: Let τ be as in Figure 4 a). The slope β is constant, the first and the third branches are onto, the second is hanging. Let $\alpha=\alpha_2<1$ and $\gamma=\gamma_2=\frac{1-\alpha}{2}$. Then, $\beta = 2+\alpha$. The digits are $\{0,(1+\alpha)/2, 1+\alpha\} = \{0\cdot d, 1\cdot d, 2\cdot d\}$, where $d=(1+\alpha)/2$. Using the symmetry of map τ and definition (9) in this special case we obtain

$$\begin{aligned} S_1 &= \sum_{n=1}^{\infty} \frac{N - j(\tau^n(c_1))}{\beta^{n+1}} = \sum_{n=1}^{\infty} \frac{j(\tau^n(c_2))-1}{\beta^{n+1}} \\ &= \frac{1}{d\beta}\sum_{n=1}^{\infty}\frac{(j(\tau^n(c_2))-1)\cdot d}{\beta^n} = \frac{\tau(c_2)}{d\beta} = \frac{1}{\beta}\ . \end{aligned} \tag{22}$$

By the symmetry of τ we have $S_{1,1}=S_{2,2}$ and $S_{1,2}=S_{2,1}$. We will show that $S_{1,1}+S_{2,1}=1$ (and also $S_{1,2}+S_{2,2}=1$). In the proof of Theorem 2 we showed

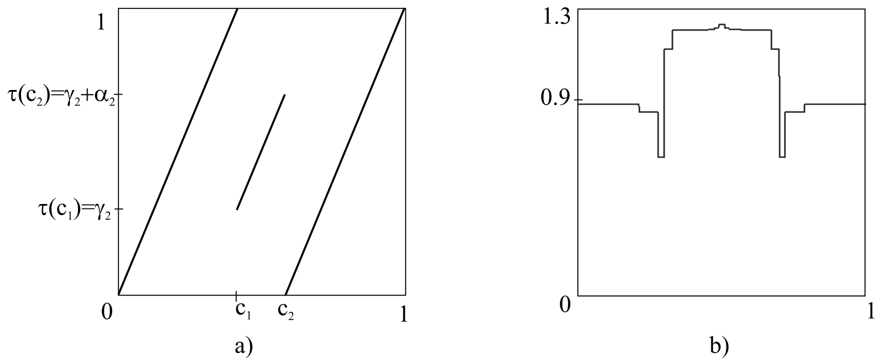

FIGURE 4. Map of Example 4 and its invariant density.

that

$$\frac{-S_{1,1}+1}{\beta}\gamma_2 + \frac{-S_{1,2}}{\beta}(1-\gamma_2-\alpha_2) = S_1 - \frac{1}{\beta}\ .$$

In our case we have $\gamma_2 = 1-\gamma_2-\alpha_2 = (1-\alpha)/2$ so equality (22) implies

$$-S_{1,1}+1-S_{1,2} = 0\ ,$$

which in turn gives $S_{1,1}+S_{2,1} = 1$ and $S_{1,2}+S_{2,2} = 1$. This shows that the matrix $\mathbf{S}$ has eigenvalue 1. At the same time τ is exact and has unique absolutely continuous invariant measure supported on $[0,1]$. For $D_0 = 1$ the system (4) is contradictory and does not have any solutions. For $D_0 = 0$ it is solvable and $D_1 = D_2 = 1$ is one of the solutions. Thus, τ-invariant density is

$$h = \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_2)]}\frac{1}{\beta^n} + \sum_{n=1}^{\infty} \chi_{[\tau^n(c_1),1]}\frac{1}{\beta^n}\ .$$

It is shown in Figure 4 b).

Let us note that the smallest change from the symmetry of this example results in a solvable system (4) with $D_0 = 1$ and the invariant density for τ can be obtained as a limit of densities for perturbed maps with perturbations converging to zero.

Another example with the same properties is given by τ^2. It preserves the same density h. □

In the following example we show that τ with one ergodic component is not necessarily exact.

Example 5: Let $N = 4$ and let τ be defined by the vectors

$$\alpha = [0.5, 0.5, 0.5, 0.5]\ \ ,\ \ \beta = [2,2,2,2]\ \ ,\ \ \gamma = [0.5, 0.5, 0, 0]\ .$$

We have $K = 4$ and $L = 0$. τ is obviously ergodic and τ^2 has two exact components. System (4) with $D_0 = 1$ is solvable, $D_1 = D_4 = -0.5$, $D_2 = D_3 = -1$ and normalizing factor is -1. $h \equiv 1$.

Example 6 shows a non-ergodic map τ. Matrix $\mathbf{S}$ has 1 as an eigenvalue, although $h \equiv 1$ is a τ-invariant density.

Example 6: Let $N=3$, and let τ be defined by the vectors

$$\alpha=[0.5,1,0.5]\ \ ,\ \ \beta=[2,2,2]\ \ ,\ \ \gamma=[0,0,0.5]\,.$$

We have $K=2$ and $L=0$. τ obviously has two exact components and $h\equiv 1$ is a τ-invariant density. Matrix $\mathbf{S}$ has an eigenvalue 1 and system (4) is not solvable for $D_0=1$. For $D_0=0$, any pair D_1, D_2 satisfies system (4) which agrees with the fact that

$$h_1=D_1\sum_{n=1}^{\infty}\chi_{[0,\tau^n(c_1)]}\frac{1}{2^n}\quad\text{ and }\quad h_2=D_2\sum_{n=1}^{\infty}\chi_{[\tau^n(c_2),1]}\frac{1}{2^n}\ ,$$

are invariant densities for the ergodic components of τ.

4. *Special case: Greedy maps*

In this section we discuss maps related to the greedy expansion with deleted digits [5, 18], i.e., piecewise linear, piecewise increasing maps for which all shorter branches touch 0. They are called greedy since the digits are the largest possible for given α's and β's.

Absolutely continuous invariant measures for such maps with constant slope were investigated in [6, 7] by other methods.

Our definition of a greedy map is a little more general than the one usually used. We give the standard definition for reference. It is assumed that the last branch is onto and the slope is constant $\beta>1$. Under these conditions the digits define the map τ. Let the digits be $\{a_1,a_2,\dots,a_N\}$. We want to define τ on $[0,1]$ so we will make some unrestrictive assumptions: $a_1=0$ and

$$M_a=\max_{1\le j\le N-1}(a_{j+1}-a_j)=1\,. \tag{23}$$

Any set of digits can be shifted and scaled to satisfy these assumptions. The maps for both sets are linearly conjugated. Now, we set $\beta=a_N+1$ and define $b_i=a_i/\beta$, $i=1,\dots,N$, $b_{N+1}=1$. We have $\alpha_i=\frac{b_{i+1}-b_i}{\beta}$, for $i=1,\dots,N$. All γ's are 0 by assumption.

We return to our, slightly more general, setting. For greedy maps we have $\gamma_i=0$ for all $i=1,\dots,N$. We assume that at least one branch is onto as otherwise τ should be considered on a different interval. Since the set W_l is in this case empty, we have

$$h=D_0+\sum_{i=1}^{K}D_i\cdot\sum_{n=1}^{\infty}\chi_{[0,\tau^n(c_i)]}\frac{1}{\beta(c_i,n)}\ .$$

We will prove a number of results specific to the greedy maps.

THEOREM 6. *Let us assume that τ is a greedy map. If the system (4) is solvable, then h is a non-normalized τ-invariant density. If the system (4) is solvable for $D_0=1$, then the system $(\tau,h\cdot m)$ is exact.*

In particular, system (4) is uniquely solvable (1 is not an eigenvalue of $\mathbf{S}$) if $\min_{1\le j\le N}\beta_j>K+1$. If the last branch is greedy, then the condition $\min_{1\le j\le N}\beta_j>K$ is sufficient and the coefficient $D_K=1$.

Proof: Most of the claims of the Theorem follow by Theorem 2. We will prove exactness. Since we assume that τ has at least one onto branch, τ has at most two ergodic components by Theorem 4. From general theory (for example [2, Chapter 8]), we know that the support of each ergodic component contains a neighborhood J of some inner partition point. Then, the image $\tau(J)$ touches 0. This proves there is only one ergodic component. To show exactness, note that for arbitrarily small neighborhood J_1 of the fixed point on the onto branch its images $\tau^n(J_1)$ grow to cover the whole $[0,1]$.

If the last branch is shorter, then $c_K = 1$. We have $S_{K,i} = 0$ for all $i = 1,\dots,K$ and Perron-Frobenius estimate on the modulus of eigenvalues of $\mathbf{S}$ is $\frac{K-1}{\beta-1}$. Thus, $\beta > K$ is sufficient in this case. The last equation in system (4) is then $D_K \cdot 1 = 1$ and $D_K = 1$. □

In a very special case of greedy map with only one shorter branch, $K = 1$, and constant slope β we have the following

Proposition 7. *Let τ be a greedy map with $K = 1$ and constant slope $\beta = \beta_i$, $i = 1,\dots,N$. If the first branch is onto, then the non-normalized τ-invariant density h is given by the formula*

$$h = 1 + D_1 \cdot \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_1)]} \frac{1}{\beta(c_1,n)} ,$$

where $D_1 = \frac{1}{1-S_{1,1}}$, and the system $(\tau, h\cdot m)$ is exact.

If the first branch is not onto, then the support of τ-invariant absolutely continuous measure is the interval $[0,\alpha_1]$. The restricted map $\tau_{|_{[0,\alpha_1]}}$ is again a greedy map with one shorter branch.

Proof: The second part of the claim holds since $\tau(c_1) < x_0$, where x_0 is the fixed point on the second branch. This, in turn, is true since $\beta_1 = \beta_2$.

To show the first part, we only have to show that $S_{1,1} \neq 1$. If τ has at least two onto branches, then $\beta > 2$ and $S_{1,1} \le \frac{1}{\beta-1} < 1$.

If $\beta \le 2$, then τ has two branches (as $K = 1$) and we consider two cases:

a) The second branch is shorter: Then, $c_1 = 1$ and $\delta(\tau^n(c_1) > c_1) = 0$ for all $n \ge 1$. Thus, $S_{1,1} = 0$ and $D_1 = 1$. We obtained classical Parry's formula [16].

b) The first branch is shorter: The τ-invariant absolutely continuous measure is supported on $[0,\alpha_1]$. τ restricted to this interval is a map from case a). □

We have proved

Proposition 8. *If τ is a greedy map with $K = 1$ and constant slope β, then τ is ergodic on $[0,1]$ if and only if $S_{1,1} \neq 1$.*

Example 7: Let τ be a greedy map with $K = 1$ and the first branch shorter. If $\beta_2 = \min_{2\le j\le N} \beta_j$ and $\alpha_1 = \tau(c_1) > x_0$, where x_0 is the fixed point on the second branch, then $S_{1,1} \neq 1$ and the claims of Proposition 7 hold.

The second branch of τ is $\tau(x)=\beta_2 x-\beta_2\frac{\alpha_1}{\beta_1}$. Thus, $x_0=\frac{\alpha_1\beta_2}{\beta_1(\beta_2-1)}$ and $\tau(c_1)>x_0$ gives $\frac{\beta_2}{\beta_1(\beta_2-1)}<1$. At the same time, we have

$$S_{1,1}\le\sum_{n=1}^{\infty}\frac{1}{\beta_1\beta_2^{n-1}}=\frac{\beta_2}{\beta_1(\beta_2-1)}<1\ .$$

□

Now, we consider greedy maps τ with constant slope $\beta>1$ with $K=2$ shorter branches satisfying $\beta\le 3$, or $\beta\le 2$ if the last branch is shorter.

We will first consider cases when τ has two shorter branches, $\beta\le 2$ and the last branch is shorter. This means that τ has 3 branches.

(A) The first branch is onto: Then, τ is exact, which can be proved as in Theorem 6. Since $c_2=1$ we have $S_{1,2}=S_{2,2}=0$ and $D_2=1$. D_1 has to satisfy $D_1(-S_{1,1}+1)=1+S_{2,1}$. We will show that

$$S_{1,1}<1\ . \tag{24}$$

Let us assume that $\tau(c_1)=\alpha_2\ge\alpha_3=\tau(c_2)$. We have $\beta=1+\alpha_1+\alpha_2\le 2$ so $\alpha_2<\beta-1$. The fixed point on the second branch would be x_0 such that $\beta x_0-1=x_0$ which gives $x_0=\frac{1}{\beta-1}\ge 1$. Thus, the second branch is always below the diagonal. In particular, $\alpha_2<c_1$. Also, whenever $\tau^n(c_1)>c_1$ then $\tau^{n+1}(c_1)\le\alpha_3<c_1$. Thus, $S_{1,1}<\frac{1}{\beta^2-1}$ and (24) is shown at least for $\beta>\beta^{(1)}=\sqrt{2}$ such that $(\beta^{(1)})^2-1=1$.

Assume that $\beta\le\beta^{(1)}$. Then, $(\beta+1)(\beta-1)\le 1$ or $\beta-1<\frac{1}{\beta+1}$. Since $\alpha_2<\beta-1$ this means that $\alpha_2<\frac{1}{\beta}$ and $\tau^2(c_1)=\beta\alpha_2\le\frac{\beta}{\beta+1}\frac{\beta}{\beta}\le\frac{2}{\beta+1}\frac{1}{\beta}<\frac{1}{\beta}<c_1$. Thus, $\tau(c_1)<c_1$ and $\tau^2(c_1)<c_1$. Moreover, whenever $\tau^n(c_1)>c_1$ then the next two iterates are smaller then $\frac{1}{\beta}$. Thus, $S_{1,1}<\frac{1}{\beta^3-1}$ and (24) is shown at least for $\beta>\beta^{(2)}=\sqrt[3]{2}$ such that $(\beta^{(2)})^3-1=1$.

Assume again that $\beta\le\beta^{(2)}$. Then, $(\beta^2+\beta+1)(\beta-1)\le 1$ or $\alpha_2<\frac{1}{\beta^2+\beta+1}$ which means that $\tau^k(c_1)<c_1$ for $k=1,2,3,4$. Moreover, whenever $\tau^n(c_1)>c_1$ then the next four iterates are smaller then $\frac{1}{\beta}$. Thus, $S_{1,1}<\frac{1}{\beta^5-1}$ and (24) is shown at least for $\beta>\beta^{(3)}=\sqrt[5]{2}$ such that $(\beta^{(3)})^5-1=1$.

Since the roots $\sqrt[n]{2}$ converge to 1 as n converges to infinity, repeating the above reasoning inductively, we can prove (24) for all $\beta>1$.

Now, let us assume that $\tau(c_1)=\alpha_2<\alpha_3=\tau(c_2)$. The proof is similar. Again, $\tau(c_1)\le c_1$ which gives $S_{1,1}\le\frac{1}{\beta(\beta-1)}$. Thus, (24) is shown at least for $\beta>\beta^{(0)}=(1+\sqrt{5})/2\simeq 1.618$ such that $\beta^{(0)}(\beta^{(0)}-1)=1$.

Assume that $\beta\le\beta^{(0)}$. Then, $\beta(\beta-1)\le 1$ or $\beta-1<\frac{1}{\beta}$. Since $\alpha_2<\alpha_3<\beta-1$, we have $\tau(c_1)\le c_1$ and whenever $\tau^n(c_1)>c_1$ then $\tau^{n+1}(c_1)\le\alpha_3<\frac{1}{\beta}$. This gives $S_{1,1}\le\frac{1}{(\beta^2-1)}$. Thus, (24) is shown at least for $\beta>\beta^{(1)}=\sqrt{2}$ such that $(\beta^{(1)})^2-1=1$. Then, the proof proceeds as in the previous case.

Example 8: τ considered in case (A) gives an example of maps for which invariant density h exists although β can be arbitrarily close to 1.

(B) The first branch is shorter. Then, the fixed point in the middle onto branch is $x_0 = \alpha_1/(\beta - 1)$ and $x_0 \geq \alpha_1$. The support of absolutely continuous invariant measure is the interval $[0, \alpha_1]$ and τ restricted to this interval is classical β-map.

Now, we consider situation where the last branch is onto and $\beta \leq 3$. This means that τ has 3 or 4 branches.

3 branches case: Since the last branch of τ is onto, the first and the second branch are shorter.

(C) $\alpha_1 \leq \alpha_2$: There are two possibilities:

(Ca) α_1 is below the fixed point on the second branch (or this fix point does not exist). Then, map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_1]$. τ restricted to this interval is a classical β-map and the invariant density can be found by Parry's formula (or our formula after rescaling).

(Cb) The image of the first branch covers the fixed point on the second branch. Then, map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first and the last branches shorter. This situation is considered in (B).

(D) $\alpha_1 > \alpha_2$: Map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_1]$. τ restricted to this interval has the first branch onto. This situation is considered in (A).

4 branches case: The last branch of τ is onto.

(E) The first branch is onto. $2 < \beta \leq 3$. τ is exact. We will prove that 1 is not an eigenvalue of $\mathbf{S}$.

First, we will show that it is not possible for both α_2, α_3 to be above the point $c_1 = \frac{1+\alpha_2}{\beta}$.

Assume $\alpha_2 \leq \alpha_3$. Since $\beta = 2 + \alpha_2 + \alpha_3 \leq 3$ we have $\alpha_2 \leq \frac{1}{2}$. Then, if $\alpha_2 > \frac{1+\alpha_2}{\beta}$, we would have $\beta > \frac{1+\alpha_2}{\alpha_2} \geq 3$, a contradiction.

Assume $\alpha_2 > \alpha_3$. Now, we have $\alpha_3 \leq \frac{1}{2}$. If $\alpha_3 > \frac{1+\alpha_2}{\beta} > \frac{1+\alpha_3}{\beta}$, we would have $\beta > \frac{1+\alpha_3}{\alpha_3} \geq 3$, again a contradiction.

Thus, at least one of the images $\tau(c_i)$, $i = 1, 2$ is below both points c_1, c_2. This makes Perron-Frobenius estimate on eigenvalues of $\mathbf{S}$ (or $\mathbf{S}^T$) equal to $\frac{1}{\beta-1} + \frac{1}{\beta(\beta-1)}$. Let $\beta^{(1)}$ be the positive solution of

$$\frac{1}{\beta - 1} + \frac{1}{\beta(\beta - 1)} = 1 \ .$$

We proved that 1 is not an eigenvalue of $\mathbf{S}$ for $\beta > \beta^{(1)} = \sqrt{2} + 1$.

Now, we assume that $\beta \leq \beta^{(1)}$. We have $\alpha_2 + \alpha_3 \leq \beta^{(1)} - 2$. We will show that both α_2, α_3 are below the point $c_1 > \frac{1}{\beta}$. The worst case scenario is when the smaller of α's is almost 0 and and the other one is almost $\beta^{(1)} - 2$. Since $\frac{1}{\beta^{(1)}} = \beta^{(1)} - 2$, inequality $\beta^{(1)} - 2 \leq \frac{1}{\beta}$ is satisfied for all $2 < \beta \leq \beta^{(1)}$. We proved that both images $\tau(c_i)$, $i = 1, 2$ are below both points c_1, c_2. Now, Perron-Frobenius estimate becomes $\frac{2}{\beta(\beta-1)}$. Since

$$\frac{2}{\beta(\beta - 1)} < 1 \ ,$$

for all $\beta > 2$ we completed the proof.

(F) The two first branches are shorter. $2 < \beta \leq 3$.

Assume first $\alpha_1 \leq \alpha_2$: Since the fixed point in the second branch is $x_0 = \frac{\alpha_1}{\beta-1} < \alpha_1$ the image of the first branch covers it. There are two cases:

(Fa) If α_2 is above the fixed point in the third, onto branch, then τ is exact. The third branch is $\tau(x) = \beta x - (\alpha_1 + \alpha_2)$ so this fixed point is $x_0 = \frac{\alpha_1+\alpha_2}{\beta-1}$. Conditions $\alpha_2 > x_0$ and $\alpha_1 + \alpha_2 < 1$ lead to inequality

$$\alpha_1 < \min\{1 - \alpha_2, \frac{\alpha_2^2}{1-\alpha_2}\} .$$

(Fb) If α_2 is below the fixed point in the third, onto branch, then map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_2]$. τ restricted to this interval has the first and the last branches shorter. This situation is considered in (B).

Now, assume $\alpha_1 > \alpha_2$: Again, there are two cases:

(Fc) If α_1 is above the fixed point in the third, onto branch, then τ is exact. This fixed point is again $x_0 = \frac{\alpha_1+\alpha_2}{\beta-1}$. Conditions $\alpha_1 > x_0$ and $\alpha_1 + \alpha_2 < 1$ lead to inequality

$$\alpha_2 < \min\{1 - \alpha_1, \frac{\alpha_1^2}{1-\alpha_1}\} .$$

(Fd) If α_1 is below the fixed point in the third, onto branch, then map τ has unique absolutely continuous invariant measure supported on $[0, \alpha_1]$. τ restricted to this interval has the second and the third (the last) branches shorter. This situation is considered in (A).

(G) The first and the third branches are shorter. $2 < \beta \leq 3$. Since again the image of the first branch covers the fixed point in the second onto branch, map τ is exact.

We have $c_2 = 1 - \frac{1}{\beta}$. We will find when both α_1 and α_3 are below the point c_2. Let $\alpha = \max\{\alpha_1, \alpha_3\}$. We need $\alpha \leq c_2$. Since $\alpha < \beta - 2$ it is enough to have $\beta - 2 \leq 1 - \frac{1}{\beta}$. Let $\beta^{(2)} = (3 + \sqrt{5})/2 \simeq 2.618$ be the larger solution of equation $\beta - 2 = 1 - \frac{1}{\beta}$. For $\beta \leq \beta^{(2)}$ Perron-Frobenius estimate on eigenvalues of $\mathbf{S}$ is $\frac{1}{\beta-1} + \frac{1}{\beta(\beta-1)}$. For $\beta > \beta^{(1)} \simeq 2.414$ of case (E), this implies that 1 is not an eigenvalue of $\mathbf{S}$. Thus, this holds in our case for $\beta^{(1)} < \beta \leq \beta^{(2)}$ or $2.414 < \beta \leq 2.618$.

We have proved the following

PROPOSITION 9. *If τ is a greedy map with $K = 2$ and constant slope β and τ satisfies assumptions of case (A), (E) or (G) with $2.414 < \beta \leq 2.618$ then τ is ergodic on $[0, 1]$ if and only if 1 is not an eigenvalue of $\mathbf{S}$. For cases (B), (C), (D), (Fb) and (Fd) analogous statement is true for τ restricted to a smaller interval. Cases (Fa), (Fc) and (G) outside the mentioned interval of β's are open to further investigation.*

In all computer experiments we performed during the work on this paper, matrices $\mathbf{S}$ for greedy ergodic maps never had an eigenvalue 1. Therefore we state the following conjecture.

Conjecture 2: Let τ be a greedy map, i.e., a piecewise linear, piecewise increasing map with shorter branches touching 0. Then, 1 is not an eigenvalue of matrix $\mathbf{S}$ $\Longleftrightarrow$ dynamical system $(\tau, h \cdot m)$ is exact on $[0,1]$.

5. *Special case: Lazy maps.*

In this section we consider piecewise linear maps of an interval $[0,1]$ with all branches increasing and such that the images of shorter branches touch 1. This means that $\alpha_i + \gamma_i = 1$ for all $i = 1, \ldots, N$. Such maps are related to so called "lazy expansions with deleted digits" [5]. They are called lazy since the digits are the smallest possible for the given α's and β's.

We will show that any lazy map is conjugated by a linear map to a corresponding greedy map so all results proven in the previous section hold, after necessary changes, for lazy maps as well.

Let $\tilde{\tau}$ be a lazy map. Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma} = 1 - \tilde{\alpha}$ denote vectors of α's, β's and γ's defining $\tilde{\tau}$. The partition points are defined, as in the general case, by

$$\tilde{b}_1 = 0 \quad , \quad \tilde{b}_j = \sum_{i=1}^{j-1} \frac{\tilde{\alpha}_i}{\tilde{\beta}_i} \ , \quad j = 2 \ldots, N+1 \ .$$

Note, that $\tilde{b}_{N+1} = 1$. Let $\tilde{I}_j = (\tilde{b}_j, \tilde{b}_{j+1})$, $j = 1 \ldots, N$. The digits $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N\}$, are as before defined by

$$\tilde{a}_j = \tilde{\beta}_j \tilde{b}_j - \tilde{\gamma}_j = \tilde{\beta}_j \tilde{b}_{j+1} - 1 \ , \quad j = 1, \ldots, N \ .$$

We will now show that lazy map $\tilde{\tau}$ is conjugated to some greedy map τ by diffeomorphism $f(x) = 1 - x$ on $[0,1]$. First we define "conjugated" vectors α, β and γ by

$$\begin{aligned} \alpha_j &= \tilde{\alpha}_{N-j+1} \ , \\ \beta_j &= \tilde{\beta}_{N-j+1} \ , \qquad j = 1, 2, \ldots, N \ , \\ \gamma_j &= 0 \ . \end{aligned}$$

This defines the "conjugated" partition points

$$b_1 = 0 \ , \quad b_j = \sum_{i=1}^{j-1} \frac{\alpha_i}{\beta_i} = \sum_{i=1}^{j-1} \frac{\tilde{\alpha}_{N-i+1}}{\tilde{\beta}_{N-i+1}} = 1 - \tilde{b}_{N-j+2} \ , \quad j = 2 \ldots, N+1 \ .$$

This defines also the conjugated set of digits $A = \{a_1, a_2, \ldots, a_N\}$ with

$$a_j = \beta_j b_j = \tilde{\beta}_{N-j+1}(1 - \tilde{b}_{N-j+2}) = \tilde{\beta}_{N-j+1} - 1 - \tilde{a}_{N-j+1} \ , \quad j = 1, 2, \ldots, N \ .$$

In particular, $a_1 = 0$. For standard greedy and lazy maps this reduces to $a_j = \tilde{a}_N - \tilde{a}_{N-j+1}$, $j = 1, 2, \ldots, N$. The lengths of intervals $\tilde{I}_j$ and I_{N-j+1} are equal since $b_{N-j+2} - b_{N-j+1} = (1 - \tilde{b}_j) - (1 - \tilde{b}_{j+1}) = \tilde{b}_{j+1} - \tilde{b}_j$, $j = 1, 2, \ldots, N$.

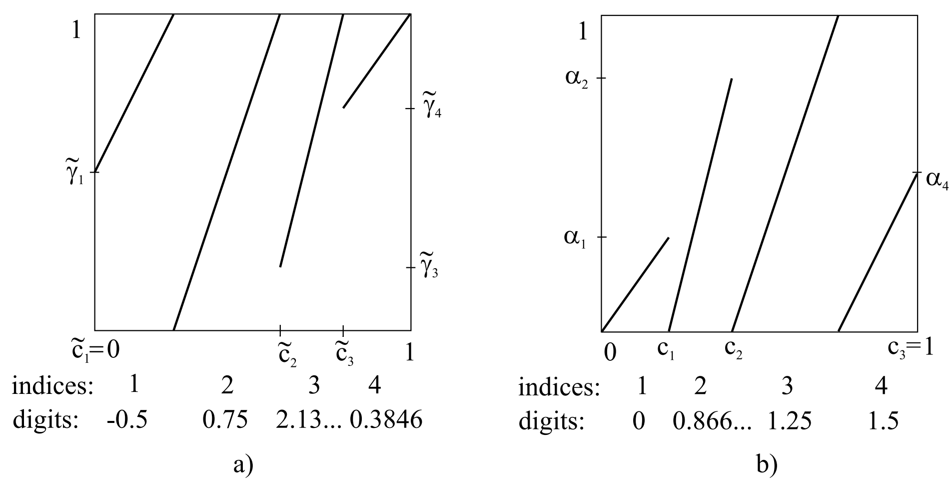

FIGURE 5. Graphs of a) lazy map and b) greedy map of Example 9.

THEOREM 10. *The maps $\tilde{\tau}$ and τ are conjugated by the diffeomorphism $f(x) = 1-x$. If h is a τ-invariant density, then the density $\tilde{h}(x) = h(1-x)$ is $\tilde{\tau}$-invariant. We have*

$$\tilde{h}(x) = D_0 + \sum_{i=1}^{K} \tilde{D}_i \sum_{n=1}^{\infty} \chi_{[\tilde{\tau}^n(\tilde{c}_i),1]} \frac{1}{\tilde{\beta}(\tilde{c}_i, n)} \ ,$$

where constants $\tilde{D}_i = D_{K-i+1}$, $i = 1, \dots, K$, satisfy the system (4) (for $\tilde{\tau}$), and points $\tilde{c}_i = 1 - c_i$, $i = 1, \dots, K$ are the special points for $\tilde{\tau}$.

Proof: Both τ and $f \circ \tilde{\tau} \circ f^{-1}$ are piecewise linear, piecewise increasing maps and the images of shorter intervals touch 0. The equality of the lengths of the intervals I_j and $\tilde{I}_{N-j+1}$ and of the slopes $\beta_j = \tilde{\beta}_{N-j+1}$, $j = 1, 2, \dots, N$, proves that they are identical. Then, $\tilde{h}(x) = h(1-x)$ since $|f'| = 1$. The formula for $\tilde{h}$ follows by the general Theorem 2. □

Example 9: Let the lazy map $\tilde{\tau}$ be defined by $N = 4$, $K = 3$ and

$$\tilde{\alpha} = [0.5, 1, 0.8, 0.3] \ , \ \tilde{\beta} = [2, 3, 4, 1.3846] \ , \ \tilde{\gamma} = [0.5, 1, 0.2, 0.7] \, .$$

The digits are $\tilde{A} = \{-0.5, 0.75, 2.13\dots, 0.3846\}$. The graph of $\tilde{\tau}$ is shown in Figure 5 a). The conjugated greedy map τ is defined by

$$\alpha = [0.3, 0.8, 1, 0.5] \ , \ \beta = [1.3846, 4, 3, 2] \ , \ \gamma = [0, 0, 0, 0] \, .$$

The digits are $A = \{0, 0.866\dots, 1.25, 1.5\}$. The graph of map τ is shown in Figure 5 b). Using Maple 11 we calculated, for $D_0 = 1$, $\tilde{D}_1 = 1$, $\tilde{D}_2 \simeq 7.9992$, $\tilde{D}_3 \simeq 99.671$. We have $D_i = \tilde{D}_{K-i+1}$, $i = 1, \dots, K$. The normalizing constant of the density is $\simeq 33.7996$. The graph of $\tilde{\tau}$-invariant density is shown in Figure 6 a)and the graph of τ-invariant density is shown in Figure 6 b) .

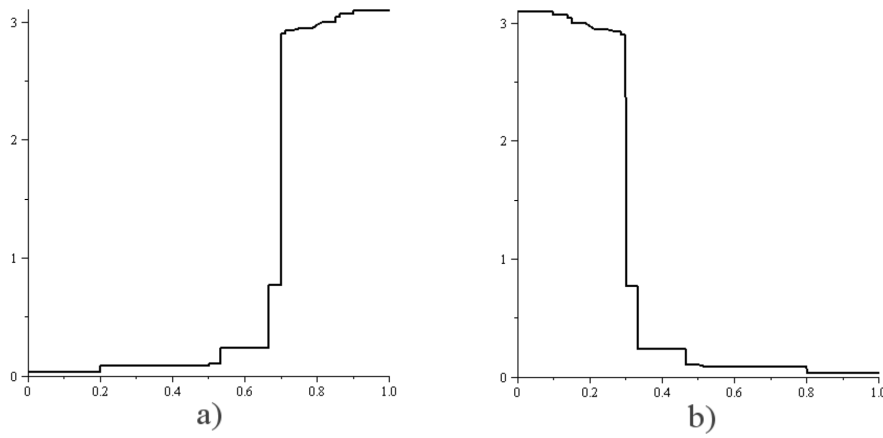

FIGURE 6. Invariant densities of a) lazy map and b) greedy map of Example 9.

6. *Special case: mixed greedy-lazy maps.*

In this section we consider maps with some shorter branches touching 0 and others touching 1. We do not assume that there is at least one onto branch.

We prove some results which are specific for mixed type maps.

THEOREM 11. *Let τ be an eventually piecewise expanding map of mixed type. If there exist an invariant density h with full support, in particular if the system (4) is solvable with $D_0 = 1$, then the dynamical system $\{\tau, h \cdot m\}$ can have at most two ergodic components. If τ has at least two onto branches, then it is exact.*

Proof: It follows from the general theory that the support of each ergodic component contains neighborhood of some inner endpoint of the partition. Since the image of each branch touches either 0 or 1, there can be at most two ergodic components. The second statement was proved in general in Theorem 4. □

Example 6 shows that mixed type map can actually have two ergodic components. In this specific case system (4) is not solvable for $D_0 = 1$.

We will describe the situation in the case of two ergodic components in more detail.

Let τ be a mixed type map with an invariant density h with support equal to $[0, 1]$. Let us assume there are two ergodic components. Since 0 belongs to one component and 1 belongs to the other component we will denote the supports of the components by C_0 and C_1 respectively. There are two possibilities:

(C1): there exists $x_0 \in [0, 1]$ such that $C_0 = [0, x_0]$ and $C_1 = [x_0, 1]$. Let $\tau_0 = \tau_{|C_0}$ and $\tau_1 = \tau_{|C_1}$. For example, this happens if τ has at least one onto branch.

We have $\tau^n(c_k) \le c_j$ for all $n \ge 1$ and all $c_k \in C_0$, $c_j \in C_1$ and $\tau^n(c_k) \ge c_j$ for all $n \geq 1$ and all $c_k \in J_1$, $c_j \in J_0$. Thus, matrix $\mathbf{S}$ is a block matrix

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_0 = (S_{i,j})_{1 \leq i,j \leq M} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1 = (S_{i,j})_{M+1 \leq i,\, j \leq K+L} \end{pmatrix},$$

where $c_1, \dots, c_M \in C_0$ and $c_{M+1}, \dots, c_{K+L} \in C_1$.

The image of at least one $c_{i_0} \in C_0$ and at least one $c_{i_1} \in C_1$ is equal to x_0 as otherwise there would be a hole in the support of h. Even if x_0 is a fixed point in a common onto branch od τ, there must exist such points.

Since h has full support, each of the systems $(\tau_0, h \cdot m_{|C_0})$, $(\tau_1, h \cdot m_{|C_1})$ is exact by Theorem 6. Each can be considered separately and the invariant densities can be combined.

(C2): Each component C_0 and C_1 consists of some number of disjoint subintervals separated by the subintervals of the other component. A map τ with each C_i consisting of 2 subintervals is given in Example 10 and a map where each C_i has 3 subintervals is given in Example 11. Examples with more subintervals in each C_i can be constructed in analogous way.

Example 10: Let $N = 4$ and τ be defined by vectors

$$\alpha = \left[\frac{2}{4}, \frac{1}{4}, \frac{2}{4}, \frac{1}{4}\right] \quad , \quad \beta = [1, 2, 2, 2] \quad , \quad \gamma = \left[\frac{2}{4}, 0, 0, \frac{3}{4}\right] \quad .$$

τ is eventually expanding and $C_0 = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$, $C_1 = [\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1]$.

Example 11: Let $N = 4$ and τ be defined by vectors

$$\alpha = \left[\frac{4}{6}, \frac{1}{6}, \frac{2}{6}, \frac{1}{6}\right] \quad , \quad \beta = [1, 2, 2, 2] \quad , \quad \gamma = \left[\frac{2}{6}, 0, 0, \frac{5}{6}\right] \quad .$$

τ is eventually expanding and $C_0 = [0, \frac{1}{6}] \cup [\frac{2}{6}, \frac{3}{6}] \cup [\frac{4}{6}, \frac{5}{6}]$, $C_1 = [\frac{1}{6}, \frac{2}{6}] \cup [\frac{3}{6}, \frac{4}{6}] \cup [\frac{5}{6}, 1]$.

References

[1] Alves, J. F., Fachada, J. L., Sousa Ramos, J., *Detecting topological transitivity of piecewise monotone interval maps*, Topology Appl. **153** (2005), no. 5-6, 680–697, MR2201481 (2007k:37046).

[2] Boyarsky, Abraham; Góra, Paweł, *Laws of chaos. Invariant measures and dynamical systems in one dimension*, Probability and its Applications, Birkhäuser Boston, Inc., Boston, MA, 1997, MR1461536 (99a:58102).

[3] Dajani, Karma; Hartono, Yusuf; Kraaikamp, Cor, *Mixing properties of (α, β)-expansions*, preprint.

[4] Dajani, Karma; Kraaikamp, Cor, *Ergodic theory of numbers*, Carus Mathematical Monographs, **29**, Mathematical Association of America, Washington, DC, 2002, MR1917322 (2003f:37014).

[5] Dajani, Karma; Kalle, Charlene *Random β-expansions with deleted digits*, Discrete Contin. Dyn. Syst. 18 (2007), no. 1, 199–217, MR2276494 (2007m:37016).

[6] Dajani, Karma; Kalle, Charlene *A note on the greedy β-transformations with deleted digits*, to appear in SMF Sminaires et Congres, Number 19, 2008.

[7] Dajani, Karma; Kalle, Charlene *A natural extension for the greedy β-transformation with three deleted digits*, preprint arXiv:0802.3571.

[8] Gelfond, A. O., *A common property of number systems* (Russian), Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 809–814, MR0109817 (22 #702).

[9] Góra, P., *Invariant densities for generalized β-transformations*, Ergodic Th. and Dynamical Systems 27, Issue 05, October 2007, 1583–1598.

[10] Islam, Shafiqul, *Absolutely continuous invariant measures of linear interval maps*, Int. J. Pure Appl. Math. **27** (2006), no. 4, 449–464, MR2223985 (2006k:37100).

[11] Lasota, Andrzej; Mackey, Michael C., *Chaos, fractals, and noise. Stochastic aspects of dynamics*, Second edition, Applied Mathematical Sciences **97**, Springer-Verlag, New York, 1994, MR1244104 (94j:58102).

[12] A. Lasota; J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations* , Trans. Amer. Math. Soc. **186** (1973), 481–488 (1974); MR0335758 (49 #538).

[13] T. Y. Li; J. A. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. **235** (1978), 183–192; MR0457679 (56 #15883).

[14] Milnor, John; Thurston, William, *On iterated maps of the interval*, Dynamical systems (College Park, MD, 1986–87), 465–563, Lecture Notes in Math., 1342, Springer, Berlin, 1988, MR0970571 (90a:58083).

[15] Henryk Minc, *Nonnegative matrices*, John Wiley& Sons, New York, 1988.

[16] Parry, W., *On the β-expansions of real numbers,* Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416, MR0142719 (26 #288).

[17] Parry, W., *Representations for real numbers*, Acta Math. Acad. Sci. Hungar. **15** (1964), 95–105, MR0166332 (29 #3609).

[18] Pedicini, Marco, *Greedy expansions and sets with deleted digits*, Theoret. Comput. Sci. **332** (2005), no. 1-3, 313–336, MR2122508 (2005k:11013).

[19] Rényi, A., *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493, MR0097374 (20 #3843).