Invariant densities for piecewise linear maps

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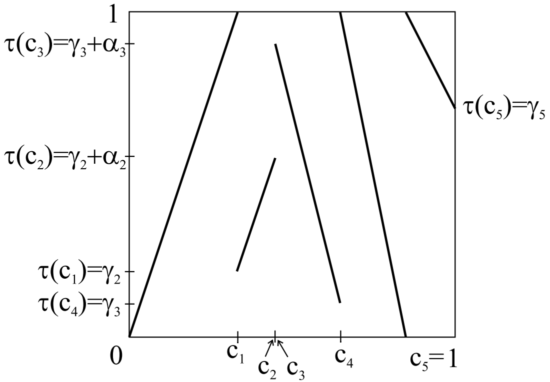
Rediscovery

Contents

Rediscovery, by a different method, of the results of Christoph Kopf (Insbruck)
Invariant measures for piecewise linear transformations of the interval, Applied Mathematics and Computation **39** (1990), issue 2, 123–144.

Piecewise linear map



N branches, three vectors:

1. slopes $\beta = [3, 3, -4, -5, -2]$
2. lengths $\alpha = [1, 0.35, 0.8, 1, 0.3]$
3. heights of lower end $\gamma = [0, 0.2, 0.1, 0, 0.7]$

Digits

Map τ can be conveniently represented using "digits"

$$\begin{aligned} &\text{if } \beta_j > 0, \text{ then } a_j = \beta_j b_j - \gamma_j \,, \\ &\text{if } \beta_j < 0, \text{ then } a_j = \beta_j b_j - (\gamma_j + \alpha_j) \,, \quad j = 1, \ldots, N \,. \end{aligned}$$

Then, map τ is

$$\tau(x) = \beta_j \cdot x - a_j, \quad \text{for} \quad x \in I_j \,, \; j = 1, 2, \ldots, N \,.$$

In the example the digits are:

$$a = \{0, 0.8, -2.7, -4.25, -2.7\}.$$

τ-expansion of numbers

For any $x \in [0,1]$ we define its "index" $j(x)$ and its "digit" $a(x)$:

$$j(x) = j \quad \text{for} \quad x \in I_j \ , \ j = 1, 2, \ldots, N \ ,$$

and

$$a(x) = a_{j(x)} .$$

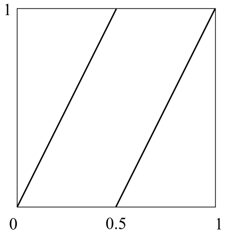
We define the cumulative slopes for iterates of points as follows:

$$\beta(x,1) = \beta_{j(x)} \ ;$$
$$\beta(x,n) = \beta(x,n-1) \cdot \beta_{j(\tau^{n-1}(x))} \ , \quad n \geq 2 \ .$$

Then, the following expansion holds:

$$x = \sum_{n=1}^{\infty} \frac{a(\tau^{n-1}(x))}{\beta(x,n)} \ .$$

Example: Binary expansion



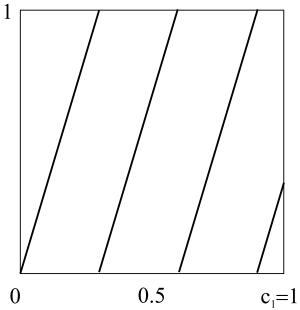
$$\tau(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2; \\ 2x - 1 & \text{if } 1/2 \leq x \leq 1 \,. \end{cases}$$

Example: Binary expansion 2

$$
\begin{aligned}
x &= 0.23, & \tau(x) &= 0.46, & \tau^2(x) &= 0.92,\\
\tau^3(x) &= 0.84, & \tau^4(x) &= 0.68, & \tau^5(x) &= 0.36,\\
\tau^6(x) &= 0.72, & \tau^7(x) &= 0.44, & \tau^8(x) &= 0.88,\\
\tau^9(x) &= 0.76, & \tau^{10}(x) &= 0.52, & \dots
\end{aligned}
$$

$$
\begin{aligned}
x = \quad & \frac{0}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{0}{2^6}\\
& + \frac{1}{2^7} + \frac{0}{2^8} + \frac{1}{2^9} + \frac{1}{2^{10}} + \frac{1}{2^{11}} + \dots
\end{aligned}
$$

Example: Classical β-map , $\beta = 3.3$



$$\tau(x) = \beta \cdot x \mod 1$$

$$x = 0.23, \tau(x) = 0.759, \tau^2(x) = 0.505,$$
$$\tau^3(x) = 0.666, \tau^4(x) = 0.196, \tau^5(x) = 0.674,$$
$$\tau^6(x) = 0.136, \tau^7(x) = 0.450, \tau^8(x) = 0.486,$$
$$\tau^9(x) = 0.603, \tau^{10}(x) = 0.989, \ldots$$

Classical β-map , $\beta = 3.3$

$$x = \frac{0}{3.3} + \frac{2}{3.3^2} + \frac{1}{3.3^3} + \frac{2}{3.3^4} + \frac{0}{3.3^5} + \frac{2}{3.3^6} + \frac{0}{3.3^7} + \frac{1}{3.3^8} + \frac{1}{3.3^9} + \frac{1}{3.3^{10}} + \frac{3}{3.3^{11}} + \dots$$

Parry's invariant density:

$$h(x) = 1 + \sum_{n=1}^{\infty} \chi_{[0,\tau^n(1))} \frac{1}{\beta^n} \, .$$

Classical β-map , $\beta = 3.3$

$$x = \frac{0}{3.3} + \frac{2}{3.3^2} + \frac{1}{3.3^3} + \frac{2}{3.3^4} + \frac{0}{3.3^5} + \frac{2}{3.3^6} + \frac{0}{3.3^7} + \frac{1}{3.3^8} + \frac{1}{3.3^9} + \frac{1}{3.3^{10}} + \frac{3}{3.3^{11}} + \dots$$

Parry's invariant density:

$$h(x) = 1 + \sum_{n=1}^{\infty} \chi_{[0,\tau^n(1))} \frac{1}{\beta^n} .$$

Back to the first example:

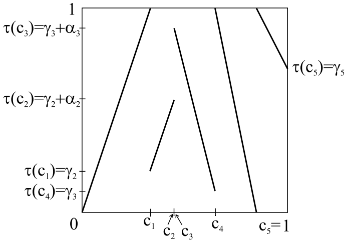
$$\begin{aligned} x &= 0.23, & \tau(x) &= 0.69, & \tau^2(x) &= 0.80, \\ \tau^3(x) &= 0.25, & \tau^4(x) &= 0.75, & \tau^5(x) &= 0.50, \\ \tau^6(x) &= 0.70, & \tau^7(x) &= 0.75, & \tau^8(x) &= 0.50, \\ \tau^9(x) &= 0.70, & \tau^{10}(x) &= 0.75, & \ldots \end{aligned}$$

indices:

$$1, 4, 4, 1, 4, 3, 4, 4, 3, 4, 4, \ldots$$

$$x = \frac{0}{3} + \frac{-4.25}{-15} + \frac{-4.25}{75} + \frac{0}{225} + \frac{-4.25}{-1125} + \frac{-2.7}{4500} + \frac{-4.25}{-22500} + \frac{-4.25}{112500} + \frac{-2.7}{-450000} + \frac{-4.25}{2250000} + \frac{-4.25}{-11250000} + \ldots$$

Special points: c_i, $i = 1, 2, \ldots, K + L$



"Greedy", "lazy" and "hanging" branches. K - number of shorter branches, L - number of hanging branches.

c_i's are endpoints of partition intervals whose image is not 0 or 1. Some of them are duplicated.

c_i's are grouped into "left" U_l and "right" U_r and also into "upper" W_u and "lower" W_l points.

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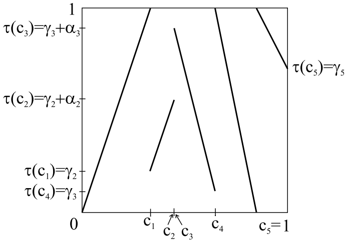

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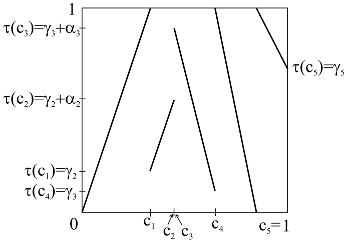

"Greedy", "lazy" and "hanging" branches. K - number of shorter branches, L - number of hanging branches.

c_i's are endpoints of partition intervals whose image is not 0 or 1. Some of them are duplicated.

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Numbers $S_{i,j}$, $1 \leq i, j \leq K + L$, τ increasing

Matrix **S** is constructed in a way somewhat similar to the construction of kneading matrix.

If all branches are increasing: $U_r = W_u$ and $U_l = W_l$.

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_i, n)} \delta(\tau_u^n(c_i) > c_j) \ , \ \text{for } c_i \in W_u \text{ and all } \ c_j \ ,$$

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{\beta(c_i, n)} \delta(\tau_l^n(c_i) < c_j) \ , \ \text{for } c_i \in W_l \text{ and all } \ c_j \ .$$

Numbers $S_{i,j}$, $1 \leq i,j \leq K+L$, τ general

In general:

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i,n)|} \left[\delta(\beta(c_i,n) > 0)\delta(\tau^n(c_i) > c_j) \right.$$
$$\left. + \, \delta(\beta(c_i,n) < 0)\delta(\tau^n(c_i) < c_j) \right] ,$$
$$\text{for} \quad c_i \in U_r \;\text{ and all }\; c_j ,$$

$$S_{i,j} = \sum_{n=1}^{\infty} \frac{1}{|\beta(c_i,n)|} \left[\delta(\beta(c_i,n) < 0)\delta(\tau^n(c_i) > c_j) \right.$$
$$\left. + \, \delta(\beta(c_i,n) > 0)\delta(\tau^n(c_i) < c_j) \right] ,$$
$$\text{for} \quad c_i \in U_l \;\text{ and all }\; c_j .$$

Equation for coefficients $D = [D_1, \ldots, D_{K+L}]$

$$(-\mathbf{S}^T + \mathbf{Id})D = D_0\mathbf{v} \ ,$$

where $\mathbf{v} = [1, 1, \ldots, 1]$.

Parameter D_0 is taken to be 1 if the system is solvable with $D_0 = 1$ and we take $D_0 = 0$ otherwise. The system always has non-vanishing solution with one of the values of the parameter.

Invariant density, τ piecewise increasing

For τ piecewise increasing:

$$h(x) = D_0 + \sum_{i \in W_u} D_i \sum_{n=1}^{\infty} \chi_{[0,\tau^n(c_i)]} \frac{1}{\beta(c_i, n)}$$

$$+ \sum_{i \in W_l} D_i \sum_{n=1}^{\infty} \chi_{[\tau^n(c_i),1]} \frac{1}{\beta(c_i, n)} ,$$

Invariant density, τ general

Let us define:

$$\chi(\beta, x) = \begin{cases} \chi_{[0,x]} \ , & \text{for} \quad \beta > 0 \ , \\ \chi_{[x,1]} \ , & \text{for} \quad \beta < 0 \ . \end{cases}$$

For general τ:

$$h(x) = D_0 + \sum_{c_i \in U_r} D_i \sum_{n=1}^{\infty} \frac{\chi(\beta(c_i, n), \tau^n(c_i))}{|\beta(c_i, n)|} + \sum_{c_i \in U_l} D_i \sum_{n=1}^{\infty} \frac{\chi(-\beta(c_i, n), \tau^n(c_i))}{|\beta(c_i, n)|} \ ,$$

Invariant density, τ general

Let us define:

$$\chi(\beta, x) = \begin{cases} \chi_{[0,x]} \ , & \text{for} \quad \beta > 0 \ , \\ \chi_{[x,1]} \ , & \text{for} \quad \beta < 0 \ . \end{cases}$$

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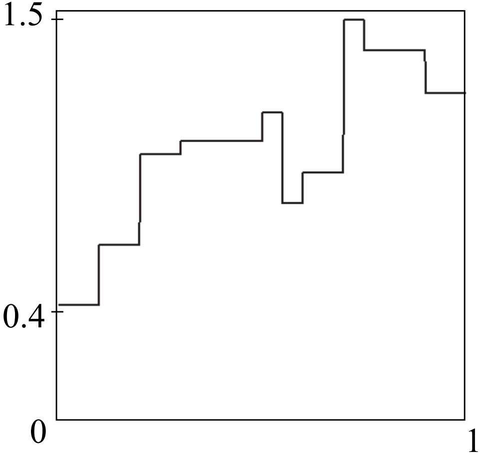
Invariant density for the first example



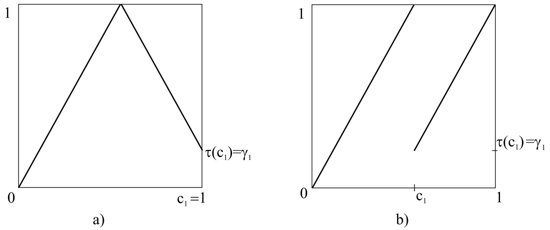
The invariant density of τ of our main example.

Conjecture

Conjecture: Let τ be piecewise linear, piecewise increasing and eventually piecewise expanding map. Then,
1 is not an eigenvalue of matrix $\mathbf{S} \implies$ dynamical system $(\tau, h \cdot m)$ is ergodic on $[0, 1]$.

The conjecture is proved for greedy maps (all shorter branches touch 0). (Thus, it also holds for lazy maps,i.e., maps with all shorter branches touching 1.)

Conjecture fails for maps with decreasing branches



$N = 2,$

$$\alpha = [1, 0.8] \ , \ \beta = [1.8, -1.8] \ , \ \gamma = [0, 0.2] \ .$$

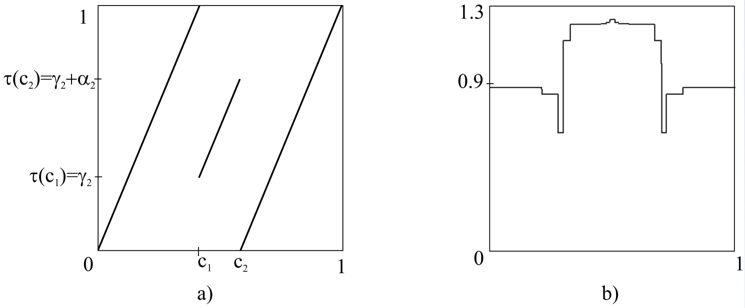
τ is ergodic on a smaller interval $[0.2, 1]$.

Conjecture fails for maps with decreasing branches

Matrix $\mathbf{S} = [S_{1,1}] = [1.125]$ has an eigenvalue 1.125 and system (15) is solvable for $D_0 = 1$. We have $D_1 = -0.8$.

For the corresponding piecewise increasing map, i.e., if we keep the same α's and γ's and change β to $\beta = [1.8, 1.8]$, matrix $\mathbf{S} = [S_{1,1}] = [1]$ has an eigenvalue 1.

Inverse of the Conjecture does not hold



The slope β is constant. Then, τ is ergodic on [0,1] and 1 is an eigenvalue of **S**.

Ergodic properties of τ

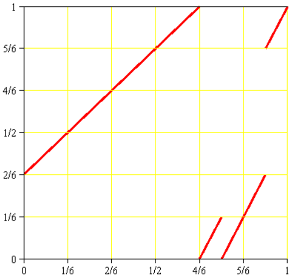
Theorem

Let τ be a piecewise linear and eventually piecewise expanding map which admits an invariant density supported on $[0, 1]$. Then, if at least one branch of τ is onto then τ has at most two ergodic components. If at least two branches are onto, then τ is exact.

A map without hanging branches has at most two ergodic components.

A greedy map with an invariant density supported on [0,1] is exact.

Examples of maps without hanging branches

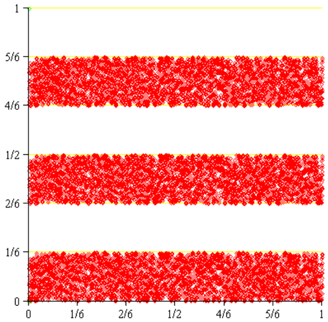


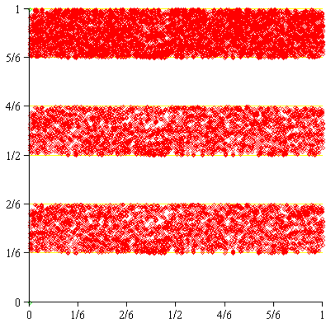
Let $N = 4$ and τ be defined by vectors

$$\alpha = \left[\frac{4}{6}, \frac{1}{6}, \frac{2}{6}, \frac{1}{6}\right] \ , \ \beta = [1, 2, 2, 2] \ , \ \gamma = \left[\frac{2}{6}, 0, 0, \frac{5}{6}\right] \ .$$

τ is eventually expanding and
$C_0 = [0, \frac{1}{6}] \cup [\frac{2}{6}, \frac{3}{6}] \cup [\frac{4}{6}, \frac{5}{6}]$, $C_1 = [\frac{1}{6}, \frac{2}{6}] \cup [\frac{3}{6}, \frac{4}{6}] \cup [\frac{5}{6}, 1]$
are its ergodic components.

Supports of ergodic components





$$C_0 = [0, \frac{1}{6}] \cup [\frac{2}{6}, \frac{3}{6}] \cup [\frac{4}{6}, \frac{5}{6}] \qquad C_1 = [\frac{1}{6}, \frac{2}{6}] \cup [\frac{3}{6}, \frac{4}{6}] \cup [\frac{5}{6}, 1]$$

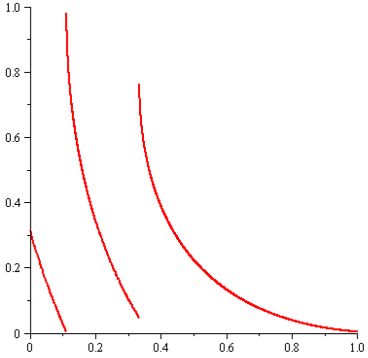
Application: approximation of acim for arbitrary map

Map modeling the movement of rotary drill (A. Lasota and P. Rusek [15], also [3]): τ_Λ depends on Froude number

$$\Lambda = \frac{v^2 M}{FR} .$$

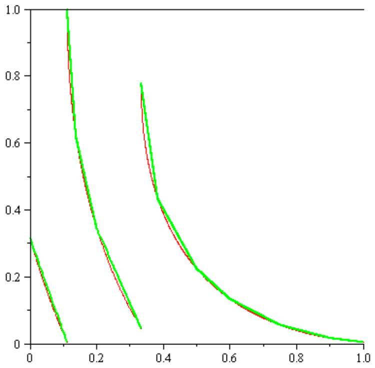
The more uniform is the invariant density of τ_Λ the more efficient is the use of the drill.

Map modeling the movement of rotary drill



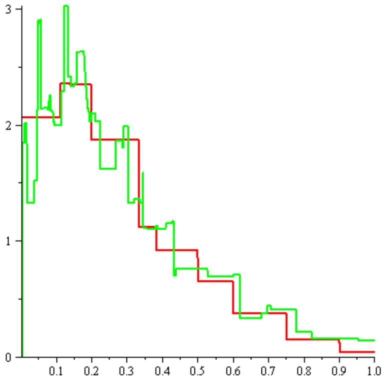
$\tau_{3.5}$ rescaled from $[0, 0.9]$.

Piecewise linear approximation



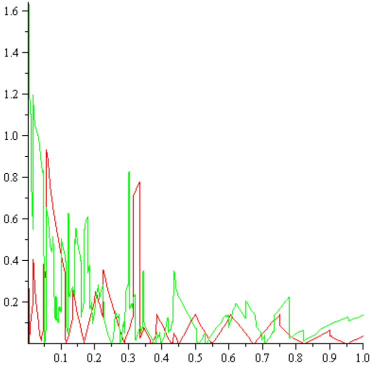
Piecewise linear approximation on the partition $\{0, 0.111, 0.138, 0.2, 0.333, 0.383, 0.5, 0.6, 0.75, 0.9, 1\}$.

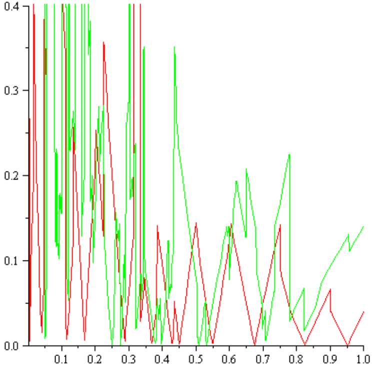
Approximations of the $\tau_{3.5}$-invariant density



Approximations of the $\tau_{3.5}$-invariant density obtained: as invariant density h of the piecewise linear approximation (green);
h_U by Ulam's method on the same partition (red).

Errors of the approximations





Errors: $|P_{\tau_{3.5}} h - h|$ - green
$|P_{\tau_{3.5}} h_U - h_U|$ - red .

Integrals of the errors functions are 0.19 and 0.12, correspondingly.

Ulam's method in a nutshell

Let $\mathcal{P} = \{I_1, I_2, \ldots, I_N\}$ be a partition of $[0, 1]$. Map τ is modeled by a Markov chain with transition matrix

$$\mathbb{P} = \left[\frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_i)} \right] .$$

If $\mathbf{v} = [v_1, v_2, \ldots, v_N]$ is the stationary (left) vector of $\mathbb{P}$, then

$$h_U = \sum_{i=1}^{N} \frac{v_i}{m(I_i)} \chi_{I_i} ,$$

is an approximation of τ-invariant density.

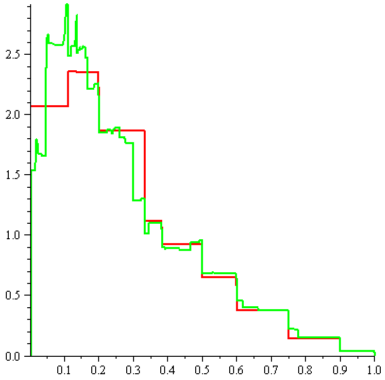
Piecewise linear approximation on actual Ulam's partition

Ulam's method uses Markov linear approximation on finer partition, which can be seen from the transition matrix:

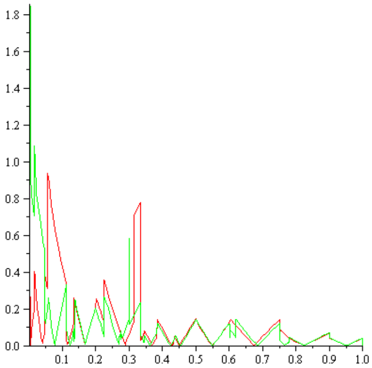
0.397368	0.089818	0.196142	0.316671	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.589628	0.347872	0.062500
0.000000	0.000000	0.000000	0.000000	0.184216	0.467794	0.299714	0.048277	0.000000	0.000000
0.275252	0.102510	0.217319	0.376598	0.028320	0.000000	0.000000	0.000000	0.000000	0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	0.390801	0.382753	0.226446	0.000000	0.000000
0.000000	0.000000	0.000000	0.633478	0.203218	0.163305	0.000000	0.000000	0.000000	0.000000
0.000000	0.058187	0.718809	0.223005	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.771484	0.228516	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

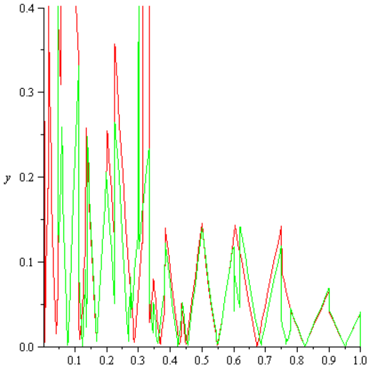
Piecewise linear approximation on actual Ulam's partition

Using this finer partition for our piecewise linear approximation we obtain density h (green)



Errors of the approximations





Errors: $|P_{\tau_{3.5}} h - h|$ - green
$|P_{\tau_{3.5}} h_U - h_U|$ - red .
Integrals of the errors functions are 0.105 and 0.117, correspondingly.

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