```
> with(plots):Digits:=100:interface(displayprecision=10):with
  (linalg):


> N:=4;
  KK:=1;
  alpha[1]:=0.45;K[1]:=2;
  alpha[2]:=0.5;K[2]:=2;# here we need alpha1 <= alpha2
  alpha[3]:=0.5;K[3]:=4; # order of K1, K2 is arbitrary
  alpha[4]:=0.7;K[4]:=6;
  i:='i':
  beta:=N-KK+sum(alpha[i],i=1..KK);i:='i':
  delta1:=(xw,yw)-> piecewise(xw<=yw,0,1);
  for j from 1 to N do
  a[j]:=j-1-sum((1-alpha[i])*delta1( j,K[i]) ,i=1..KK);
   od; i:='i':
  for j from 1 to N do
  b[j]:=a[j]/beta;
   od;
  b[N+1]:=1;
  for j from 1 to KK do
  c[j]:=b[K[j]+1];
  od;
>
  maa:=a[2]-a[1]:# maximum a[i+1]-a[i]
  for i from 3 to N do
  if (a[i]-a[i-1])>maa then maa:=(a[i]-a[i-1]) fi
  od;#
  maa;
  beta_max:=evalf(1+(a[N]-a[1])/maa);
> if beta> beta_max then print("ERROR") fi;
>
  int_of_x:=x->piecewise(x<=b[2],1,# This function needs additions
  by hand for
                                   # N>9 . Automathic procedure
  causes plotting problems
                                   # but is used in other
  programs
                          x<=b[3],2,
                          x<=b[4],3,
                          x<=b[5],4,
                          x<=b[6],5,
                          x<=b[7],6,
                          x<=b[8],7,
                          x<=b[9],8,
                                9);
  x:='x':
```

```
T:=x->beta*x-a[int_of_x(x)];
for j from 1 to KK do
print(`T(c[`,j,`])  =`,  T(c[j]))
od;
 #plot([int_of_x(x)],x=AA..BB);
plot([T(x),x,0,1,alpha[1],alpha[2]],x=0..1,thickness=[2,1,1,1,1,
1,1]);
```

$$N := 4$$

$$KK := 1$$

$$\alpha_1 := 0.4500000000$$

$$K_1 := 2$$

$$\alpha_2 := 0.5000000000$$

$$K_2 := 2$$

$$\alpha_3 := 0.5000000000$$

$$K_3 := 4$$

$$\alpha_4 := 0.7000000000$$

$$K_4 := 6$$

$$\beta := 3.4500000000$$

$$\delta 1 := (xw, yw) \rightarrow piecewise(xw \le yw, 0, 1)$$

$$a_1 := 0.0000000000$$

$$a_2 := 1.0000000000$$

$$a_3 := 1.4500000000$$

$$a_4 := 2.4500000000$$

$$b_1 := 0.0000000000$$

$$b_2 := 0.2898550725$$

$$b_3 := 0.4202898551$$

$$b_4 := 0.7101449275$$

$$b_5 := 1$$

$$c_1 := 0.4202898551$$

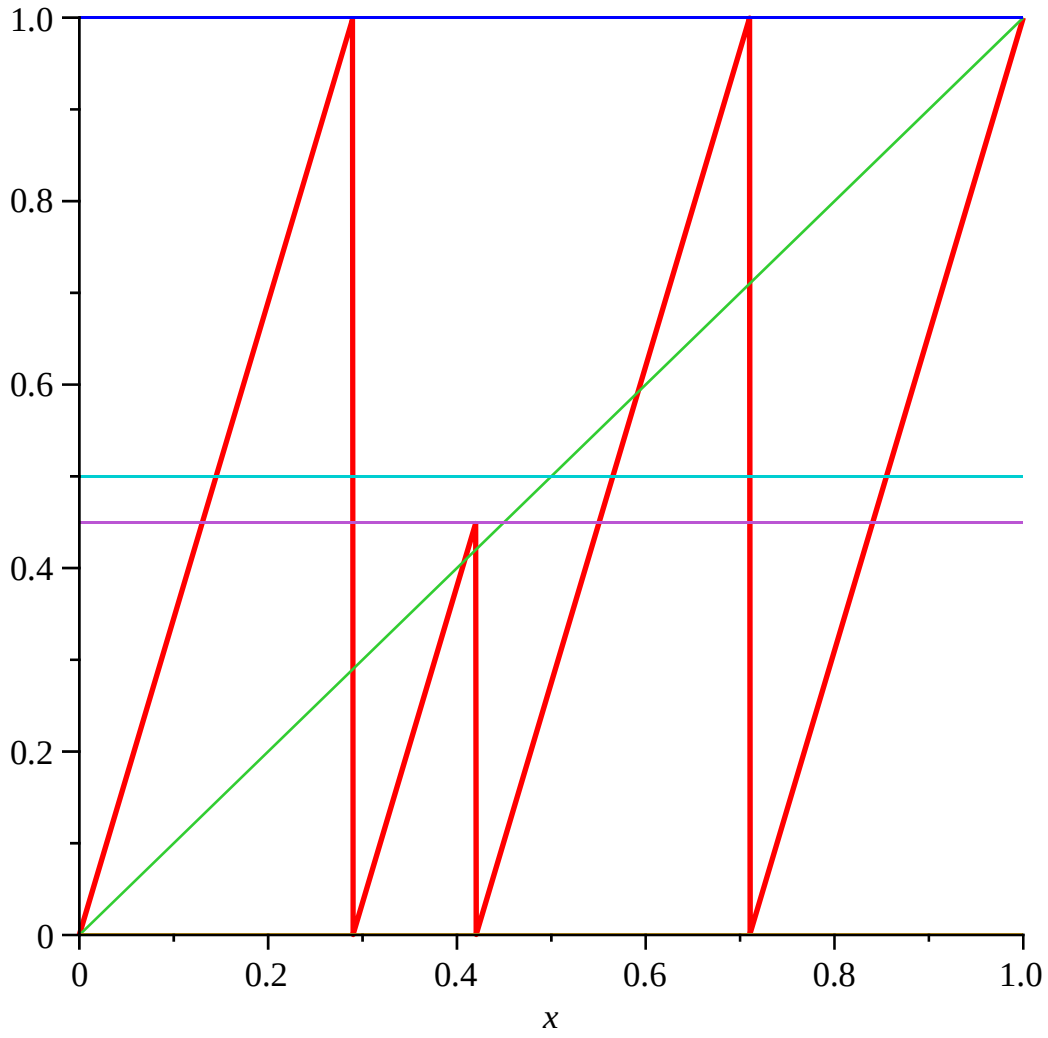
$$1.0000000000$$

$$\beta max := 3.4500000000$$

$$int\_of\_x := x \rightarrow piecewise\left(x \le b_2, 1, x \le b_3, 2, x \le b_4, 3, x \le b_5, 4, x \le b_6, 5, x \le b_7, 6, x \le b_8, 7, x \le b_9, 8, 9\right)$$

$$T := x \rightarrow \beta x - a_{int\_of\_x(x)}$$

$T(c[, 1,]) =, 0.4500000000$




```
>
> d:=vector(50):Digits:=100;NN:=50;
  xx:=evalf(rand()/10^12);
  xxt:=xx:
  for i from 1 to NN do
  d[i]:=a[int_of_x(xxt)];
  xxt:=T(xxt);
  od:
  print(d);
  Is_it_x:=evalf(sum(d[j1]/beta^j1,j1=1..NN));
  err:=xx-Is_it_x;
```

$Digits := 100$

$NN := 50$

$xx := 0.3957188605$

[1.0000000000, 1.0000000000, 0.0000000000, 2.4500000000, 1.4500000000, 2.4500000000, 0.0000000000, 2.4500000000, 0.0000000000, 2.4500000000, 1.4500000000, 0.0000000000, 2.4500000000, 2.4500000000, 0.0000000000, 0.0000000000, 0.0000000000, 0.0000000000, 1.0000000000, 0.0000000000, 0.0000000000, 1.4500000000, 0.0000000000, 1.4500000000, 2.4500000000, 0.0000000000, 1.4500000000, 0.0000000000, 1.4500000000, 1.0000000000, 0.0000000000,

2.4500000000, 0.0000000000, 2.4500000000, 2.4500000000, 2.4500000000,
1.4500000000, 2.4500000000, 2.4500000000, 1.0000000000, 0.0000000000,
2.4500000000, 1.0000000000, 0.0000000000, 1.4500000000, 2.4500000000,
2.4500000000, 1.0000000000, 1.0000000000, 1.0000000000]

$$Is\_it\_x := 0.3957188605$$

$$err := 4.844889400\ 10^{-28} \quad \textbf{(1)}$$

```
>
> NN:=50; chi:=(x1,x2,t)->piecewise(t<x1,0,t<x2,1,0);

  #Expansion of c1, c2 ... and all the S's
  for i from 1 to KK do
  xxt:=c[i];
       for n from 1 to NN+1 do
          dc[i,n]:=a[int_of_x(xxt)];
          ic[i,n]:=int_of_x(xxt)-1;
          for ii from 1 to KK do
              if xxt>c[ii] then cc[i,ii,n]:=1 else cc[i,ii,n]:=0
  fi;
          od;
       valc[i,n]:=xxt;
       xxt:=T(xxt);
       od:
  Is_it_x:=sum(dc[i,j1]/beta^j1,j1=1..NN);
  S[i]:=sum(ic[i,j1+1]/beta^(j1+1),j1=1..NN);
  od;
  for i from 1 to KK do
  for j from 1 to KK do
  SS[i,j]:=sum(cc[i,j,j1+1]/beta^(j1+1),j1=1..NN);
  print(`SS[`,i,j,`] =`,SS[i,j]);
  od;od;
```

$$NN := 50$$

$$\chi := (x1, x2, t) \rightarrow piecewise(t < x1, 0, t < x2, 1, 0)$$

$$xxt := 0.4202898551$$

$$Is\_it\_x := 0.4202898551$$

$$S_1 := 0.1770076556$$

$$SS[, 1, 1,] =, 0.0846779509$$

```
> MM:=matrix(KK,KK,[]):
  for i from 1 to KK do
  for j from 1 to KK do
  MM[j,i]:=-SS[i,j];
  od;od;
  print(MM);
  print(`1/beta =`, 1/beta);
```

```
print(`eigenvalues =`,eigenvalues(MM));

ve:=vector(KK,[]):
for i from 1 to KK do
ve[i]:=1/beta;
MM[i,i]:=MM[i,i]+1/beta;
od:
print(MM);

print(ve);


DD:=linsolve(MM,ve);
```

$$\begin{bmatrix} -0.0846779509 \end{bmatrix}$$

$$1/beta = , 0.2898550725$$

$$eigenvalues = , -0.0846779509$$

$$\begin{bmatrix} 0.2051771216 \end{bmatrix}$$

$$\begin{bmatrix} 0.2898550725 \end{bmatrix}$$

$$DD := \begin{bmatrix} 1.4127065933 \end{bmatrix} \quad \textbf{(2)}$$

```
> density:=proc(t) local j, den;
                    den:=1/beta;
                  for j from 1 to KK do
                    den:=den+ DD[j]*sum((chi(0,valc[j,i1+1],t))
/beta^(i1+1),i1=1..50)
                    od;
                  return den;
            end proc;

#Normalizing factor
NC:=1/beta:
for j from 1 to KK do
 NC:=NC+DD[j]*sum((valc[j,i1+1])/beta^(i1+1),i1=1..50)
 od:
print(`NC = `,NC);
plot([(1/NC)*density(t)],t=0..1,thickness=2);
```

$density := \mathbf{proc}(t)$

$\quad \mathbf{local}\ j, den;$

$\quad den := 1/\mathrm{beta};$

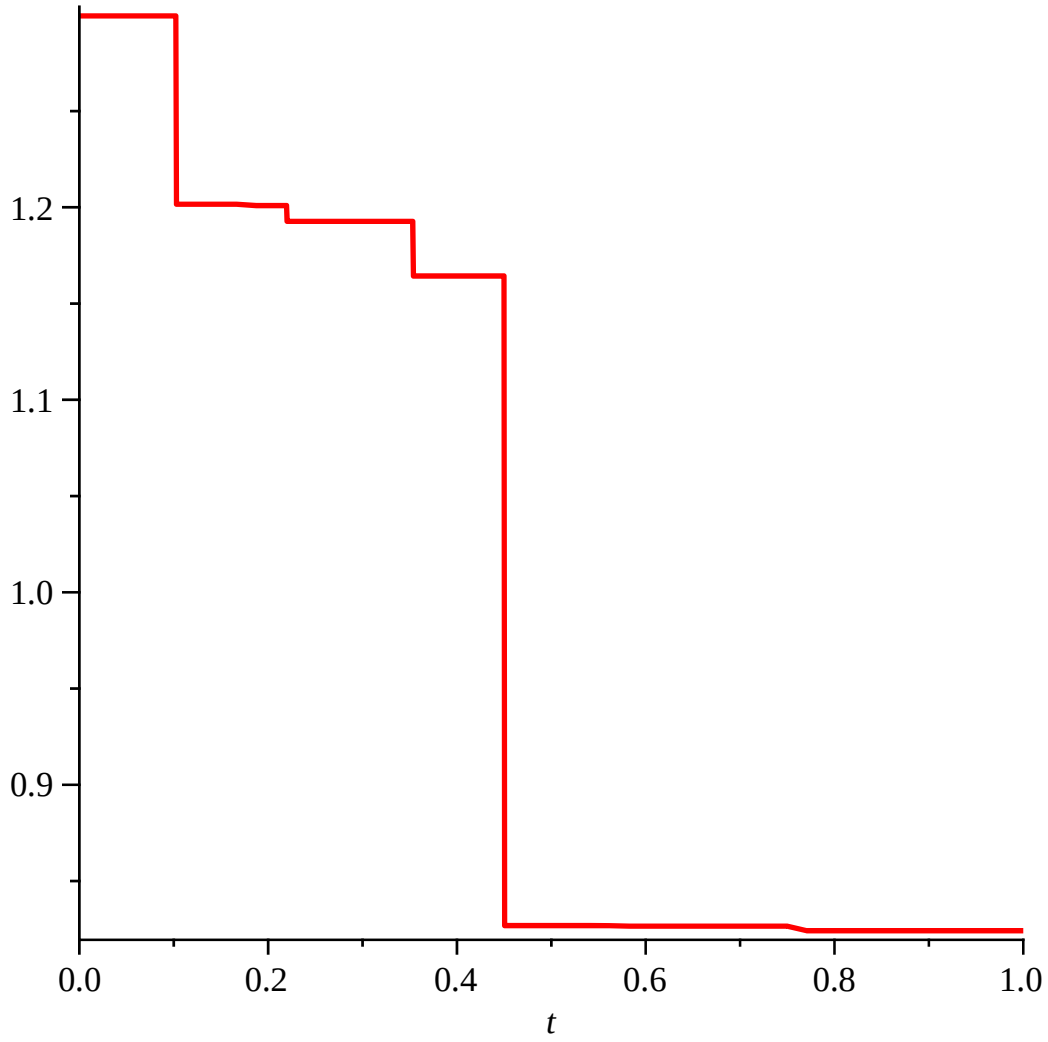
$\quad \mathbf{for}\ j\ \mathbf{to}\ KK\ \mathbf{do}$

$\quad\quad den := den + DD[j] * (sum(\mathrm{chi}(0, valc[j, i1+1], t)/\mathrm{beta}\hat{}(i1+1), i1 = 1..50))$

$\quad \mathbf{end\ do};$

```
    return den
end proc
```

$$NC = , 0.3516867272$$




```
>
> #check density
  #preimages
  for j6 from 1 to KK-1 do
  y[j6]:=alpha[j6]+(alpha[j6+1]-alpha[j6])*rand()/10^12;
  od;
  y[0]:=alpha[1]*rand()/10^12;
  y[KK]:=alpha[KK]+(1-alpha[KK])*rand()/10^12;
  for j6 from 0 to KK do
  for i3 from 1 to N do
  pre[i3]:=(y[j6]+a[i3])/beta;
  od;
  #plot([T(t),0,1,y[j6],T(c[1]),T(c[2])],t=0..1,color=[red,black,
  black,green,yellow,yellow]);
  su:=0:
  for i3 from 1 to N do
  if (pre[i3]>=b[i3] and pre[i3]<=b[i3+1]) then
    su:=su+density(pre[i3])/beta;
```

```
print(i3);
fi;
od;
err[j6]:=density(y[j6])-su;
od;
for j6 from 0 to KK do
print(`err[`,j6,`]=`,err[j6]);
od;
```

$$y_0 := 0.0869129174$$

$$y_1 := 0.4623332938$$

$$su := 0$$

1

2

3

4

$$err_0 := -1.137456360\ 10^{-28}$$

$$su := 0$$

1

3

4

$$err_1 := 3.882080952\ 10^{-29}$$

$$err[,\ 0,\ ]=,\ -1.137456360\ 10^{-28}$$

$$err[,\ 1,\ ]=,\ 3.882080952\ 10^{-29}$$

```
> #check density
#preimages
y:=alpha[1]*rand()/10^12;i3:='i3':
for i3 from 1 to N do
pre[i3]:=(y+a[i3])/beta;
od;i3:='i3':
plot([T(t),0,1,y,T(c[1]),T(c[2])],t=0..1,color=[red,black,black,
green,yellow,yellow]);
su:=0:
for i3 from 1 to N do
if (pre[i3]>=b[i3] and pre[i3]<=b[i3+1]) then
 su:=su+density(pre[i3])/beta;
print(i3);
fi;
od;
err1:=density(y)-su;
#check density
#preimages
y:=alpha[1]+(alpha[2]-alpha[1])*rand()/10^12;
```

```
for i2 from 1 to N do
pre[i2]:=(y+a[i2])/beta;
od;
plot([T(t),0,1,y,T(c[1]),T(c[2])],t=0..1,color=[red,black,black,
green,yellow,yellow]);
su:=0:
for i2 from 1 to N do
if (pre[i2]>=b[i2] and pre[i2]<=b[i2+1]) then
 su:=su+density(pre[i2])/beta;
print(i2);
fi;
od;
err2:=density(y)-su;
#check density
#preimages
y:=alpha[2]+(1-alpha[2])*rand()/10^12;
for i2 from 1 to N do
pre[i2]:=(y+a[i2])/beta;
od;
plot([T(t),0,1,y,T(c[1]),T(c[2])],t=0..1,color=[red,black,black,
green,yellow,yellow]);
su:=0:
for i2 from 1 to N do
if (pre[i2]>=b[i2] and pre[i2]<=b[i2+1]) then
 su:=su+density(pre[i2])/beta;
print(i2);
fi;
od;
err1;err2;
err3:=density(y)-su;
```

$$y := 0.3600843680$$

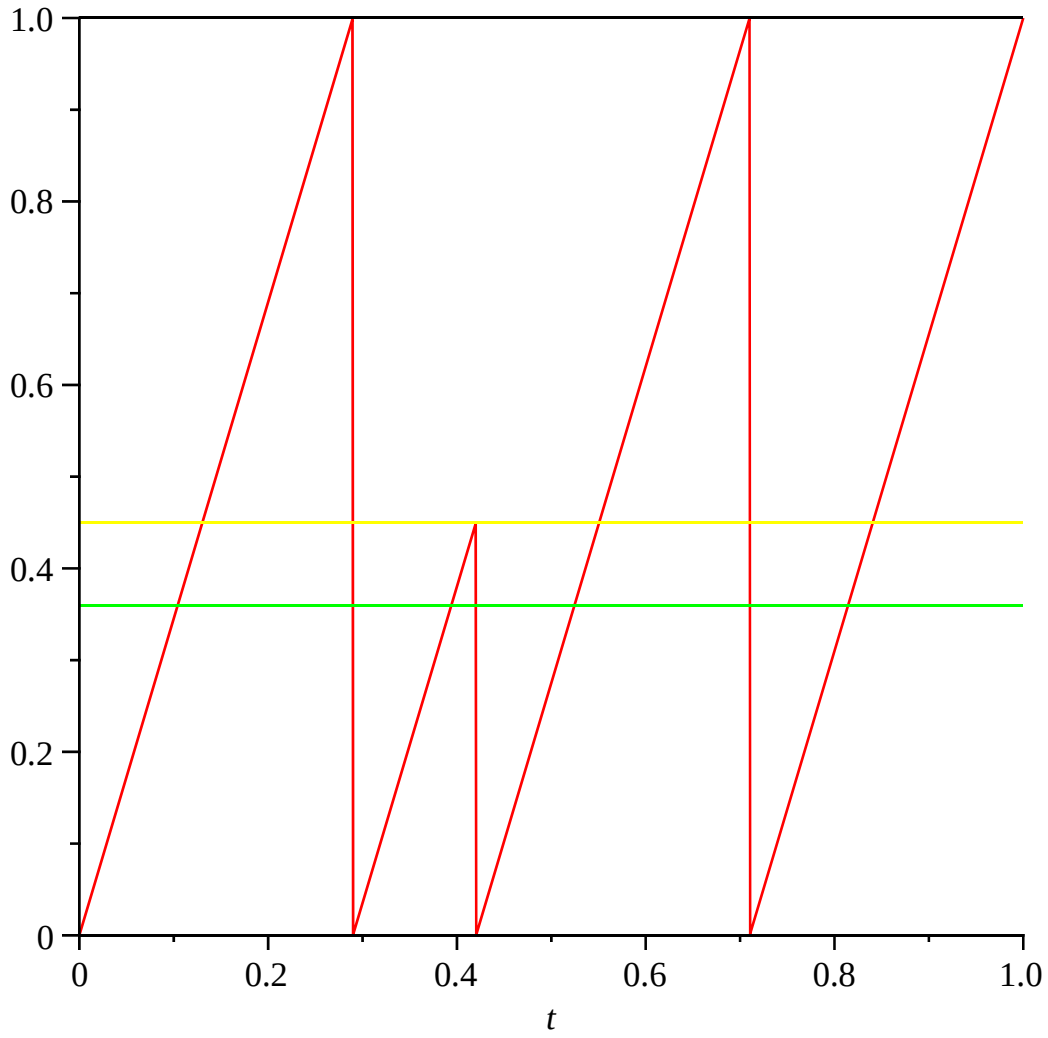
$$pre_1 := 0.1043722806$$

$$pre_2 := 0.3942273530$$

$$pre_3 := 0.5246621357$$

$$pre_4 := 0.8145172081$$

```
Warning, unable to evaluate 1 of the 6 functions to numeric
values in the region; see the plotting command's help page to
ensure the calling sequence is correct
```



1

2

3

4

$err1 := 3.882080952\ 10^{-29}$

$y := 0.4713776028$

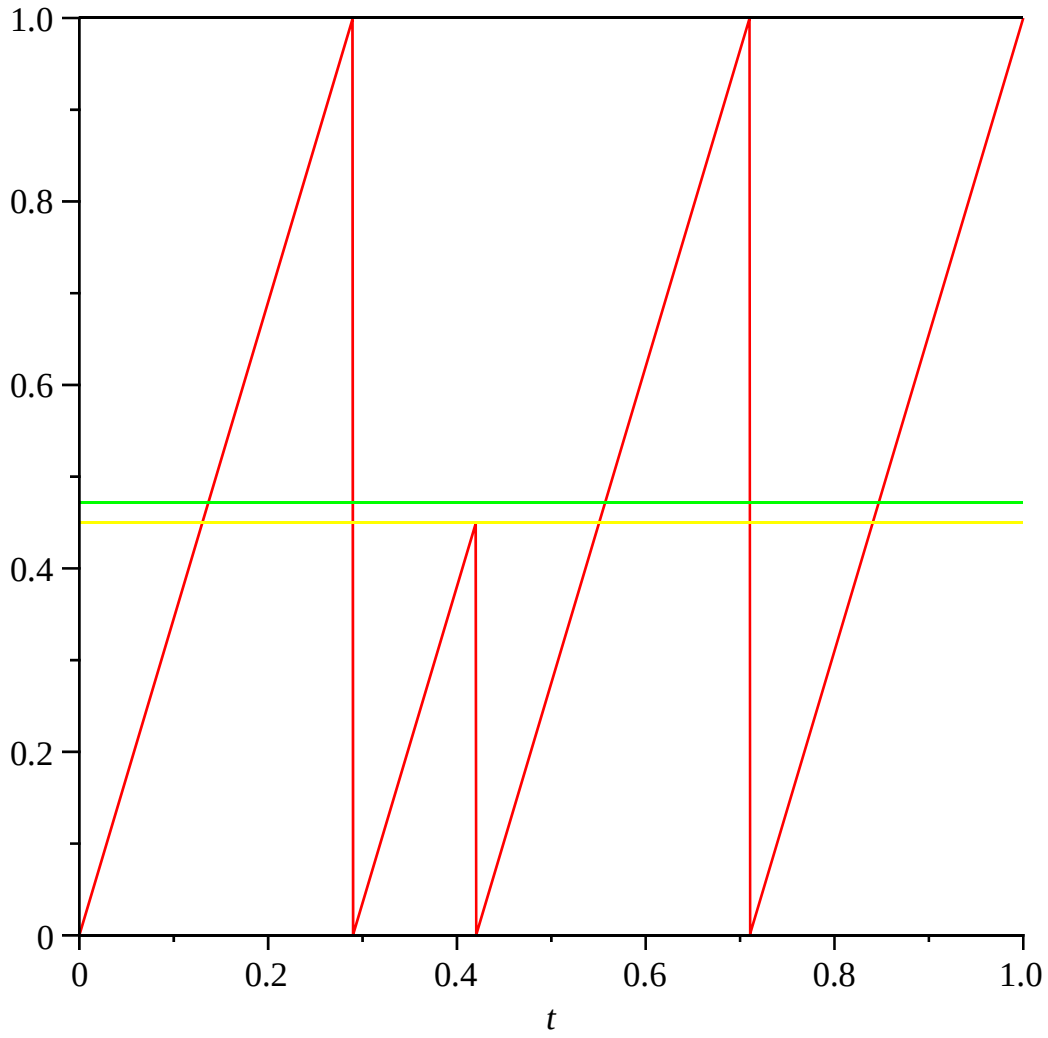
$pre_1 := 0.1366311892$

$pre_2 := 0.4264862617$

$pre_3 := 0.5569210443$

$pre_4 := 0.8467761168$

```
Warning, unable to evaluate 1 of the 6 functions to numeric
values in the region; see the plotting command's help page to
ensure the calling sequence is correct
```



1

3

4

$err2 := 3.882080952\ 10^{-29}$

$y := 0.9213113422$

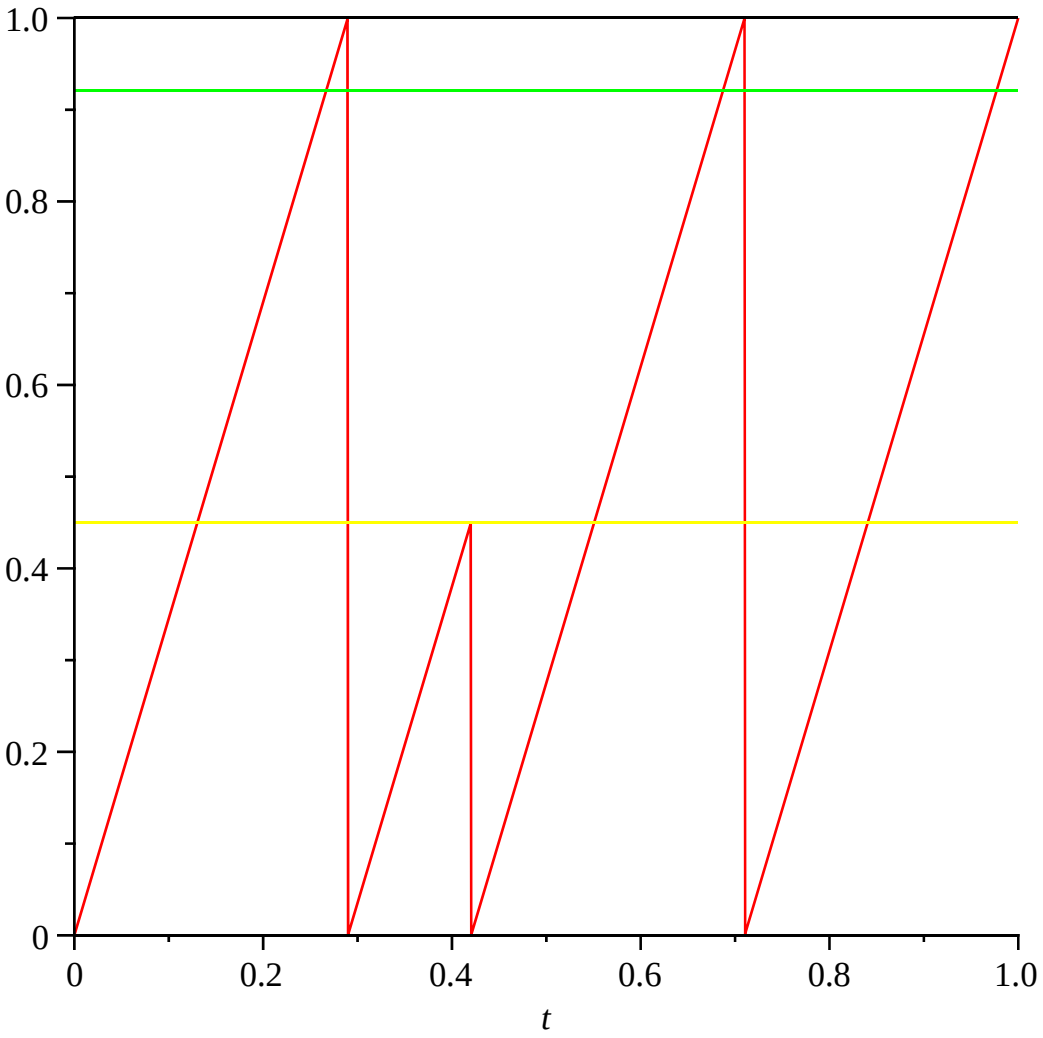
$pre_1 := 0.2670467659$

$pre_2 := 0.5569018383$

$pre_3 := 0.6873366209$

$pre_4 := 0.9771916934$

```
Warning, unable to evaluate 1 of the 6 functions to numeric
values in the region; see the plotting command's help page to
ensure the calling sequence is correct
```



1

3

4

$3.882080952\ 10^{-29}$

$3.882080952\ 10^{-29}$

$err3 := 3.882080952\ 10^{-29}$

>

>