

```
> with (plots) : Digits := 100 : interface (displayprecision=10) : with
(linalg) :
```

```
> N := 5;
KK := 3;
```

```
alpha[1] := 0.4; tK[1] := 1;
alpha[2] := 0.5; tK[2] := 3; # here we need alpha1 <= alpha2
alpha[3] := 0.7; tK[3] := 2; # order of K1, K2 is arbitrary
alpha[4] := 0.7; tK[4] := 6;
i := 'i' :
beta := N - KK + sum(alpha[i], i = 1..KK); i := 'i' :
delta := (xw, yw) -> piecewise(xw = yw, 0, 1);
for j from 1 to N do
tb[j] := (j - 1 - sum((1 - alpha[i]) * delta(j, tK[i]), i = 1..KK)) / beta;
od; i := 'i' :
tb[N+1] := 1;
for j from 1 to N do
ta[j] := (j - 1 - sum((1 - alpha[i]) * delta(j, tK[i] - 1), i = 1..KK));
od;
```

```
for j from 1 to N do # alternative formula for a[i]
aa[j] := beta * (tb[j+1] - tb[1]) - 1;
od;
for j from 1 to N do
ta[j] - aa[j];
od;
for j from 1 to KK do
tc[j] := tb[tK[j]];
od;
```

```
>
maa := ta[2] - ta[1]; # maximum a[i+1] - a[i]
for i from 3 to N do
if (ta[i] - ta[i-1]) > maa then maa := (ta[i] - ta[i-1]) fi
od; #
maa;
beta_max := evalf(1 + (ta[N] - ta[1]) / maa);
> if beta > beta_max then print("ERROR") fi;
```

```
##### Conjugated part
for j from 1 to N do
a[j] := ta[N] - ta[N-j+1];
od;
for j from 1 to N do
b[j] := a[j] / beta;
od;
```

```

b[ N+1 ] := 1;
for i from 1 to KK do
K[ i ] := N - t K[ i ] + 1;
c[ i ] := b[ K[ i ] + 1 ];
od;

```

>

```

int_of_x := x -> piecewise( x < b[ 2 ], 1, # This function needs
additions by hand for
# N>9 . Automatic procedure
causes plotting problems
# but is used in other
programs

```

```

x < b[ 3 ], 2,
x < b[ 4 ], 3,
x < b[ 5 ], 4,
x < b[ 6 ], 5,
x < b[ 7 ], 6,
x < b[ 8 ], 7,
x < b[ 9 ], 8,
9);

```

```

int_of_x := x -> piecewise( x <= b[ 2 ], 1, # This function needs additions
by hand for
# N>9 . Automatic procedure
causes plotting problems
# but is used in other
programs

```

```

x <= b[ 3 ], 2,
x <= b[ 4 ], 3,
x <= b[ 5 ], 4,
x <= b[ 6 ], 5,
x <= b[ 7 ], 6,
x <= b[ 8 ], 7,
x <= b[ 9 ], 8,
9);

```

```

x := ' x' :
tT := x -> beta * x - t a[ int_of_x( x ) ];
T := x -> beta * x - a[ int_of_x( x ) ];
for j from 1 to KK do
print( ` tT( t c[ ` , j, ` ] ) = ` , tT( t c[ j ] ) )
od;
for j from 1 to KK do
print( ` T( c[ ` , j, ` ] ) = ` , T( c[ j ] ) )
od;

```

```

plot( [ tT( x ), x, 0, 1, 1 - alpha[ 1 ], 1 - alpha[ 2 ] ], x = 0. . 1, thickness = [ 2, 1,
1, 1, 1, 1, 1 ] );
plot( [ T( x ), x, 0, 1, alpha[ 1 ], alpha[ 2 ] ], x = 0. . 1, thickness = [ 2, 1, 1, 1, 1,

```

1, 1]);

$N := 5$

$KK := 3$

$\alpha_1 := 0.4000000000$

$tK_1 := 1$

$\alpha_2 := 0.5000000000$

$tK_2 := 3$

$\alpha_3 := 0.7000000000$

$tK_3 := 2$

$\alpha_4 := 0.7000000000$

$tK_4 := 6$

$\beta := 3.6000000000$

$\delta l := (xw, yw) \rightarrow \text{piecewise}(xw \leq yw, 0, 1)$

$tb_1 := 0.0000000000$

$tb_2 := 0.1111111111$

$tb_3 := 0.3055555556$

$tb_4 := 0.4444444444$

$tb_5 := 0.7222222222$

$tb_6 := 1$

$ta_1 := -0.6000000000$

$ta_2 := 0.1000000000$

$ta_3 := 0.6000000000$

$ta_4 := 1.6000000000$

$ta_5 := 2.6000000000$

$tc_1 := 0.0000000000$

$tc_2 := 0.3055555556$

$tc_3 := 0.1111111111$

1.0000000000

$\beta_{max} := 4.2000000000$

$a_1 := 0.0000000000$

$a_2 := 1.0000000000$

$a_3 := 2.0000000000$

$$a_4 := 2.5000000000$$

$$a_5 := 3.2000000000$$

$$b_1 := 0.0000000000$$

$$b_2 := 0.2777777778$$

$$b_3 := 0.5555555556$$

$$b_4 := 0.6944444444$$

$$b_5 := 0.8888888889$$

$$b_6 := 1$$

$$K_1 := 5$$

$$c_1 := 1$$

$$K_2 := 3$$

$$c_2 := 0.6944444444$$

$$K_3 := 4$$

$$c_3 := 0.8888888889$$

$$tint_of_x := x \rightarrow \text{piecewise}(x < tb_2, 1, x < tb_3, 2, x < tb_4, 3, x < tb_5, 4, x < tb_6, 5, x < tb_7, 6, x < tb_8, 7, x < tb_9, 8, 9)$$

$$int_of_x := x \rightarrow \text{piecewise}(x \leq b_2, 1, x \leq b_3, 2, x \leq b_4, 3, x \leq b_5, 4, x \leq b_6, 5, x \leq b_7, 6, x \leq b_8, 7, x \leq b_9, 8, 9)$$

$$tT := x \rightarrow \beta x - ta_{int_of_x(x)}$$

$$T := x \rightarrow \beta x - a_{int_of_x(x)}$$

$$tT(tc[1, J]) = 0.6000000000$$

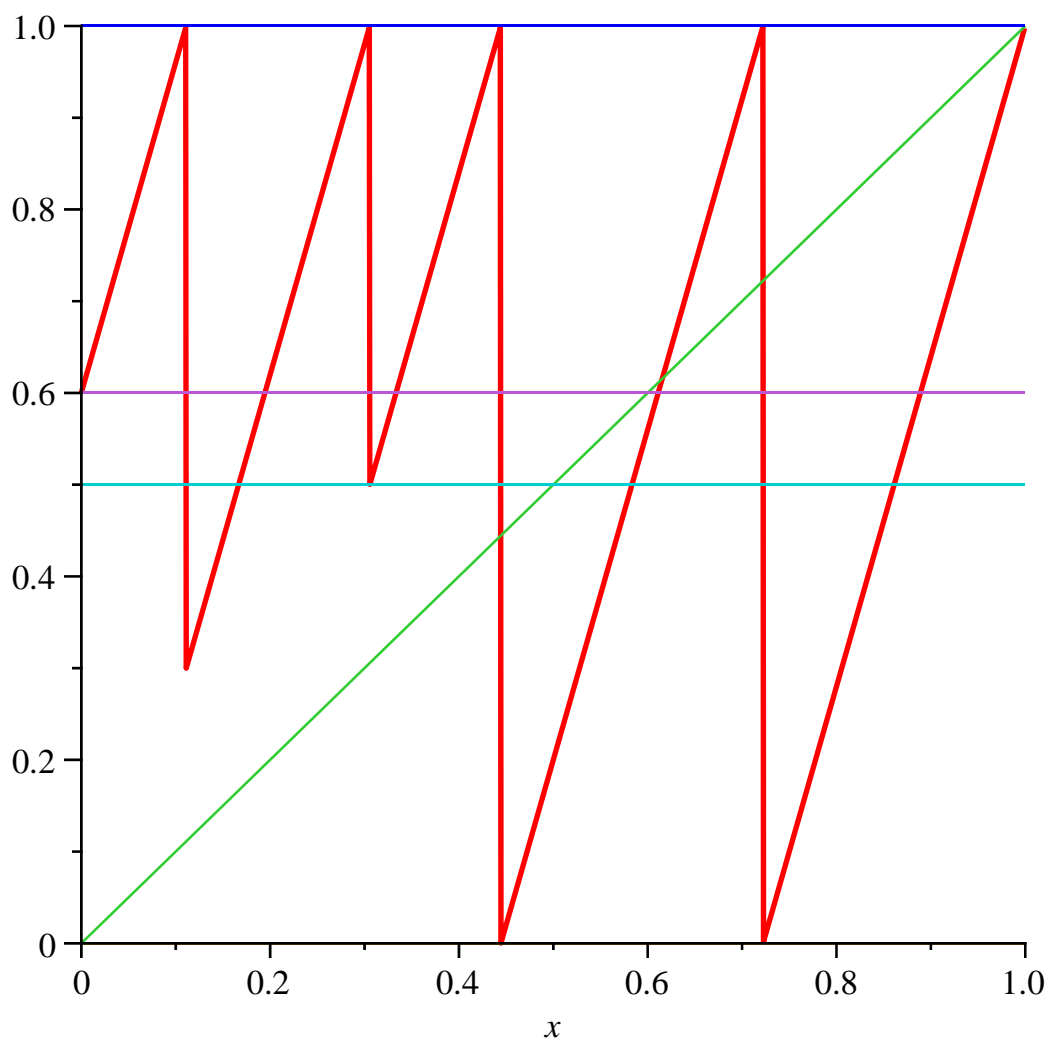
$$tT(tc[2, J]) = 0.5000000000$$

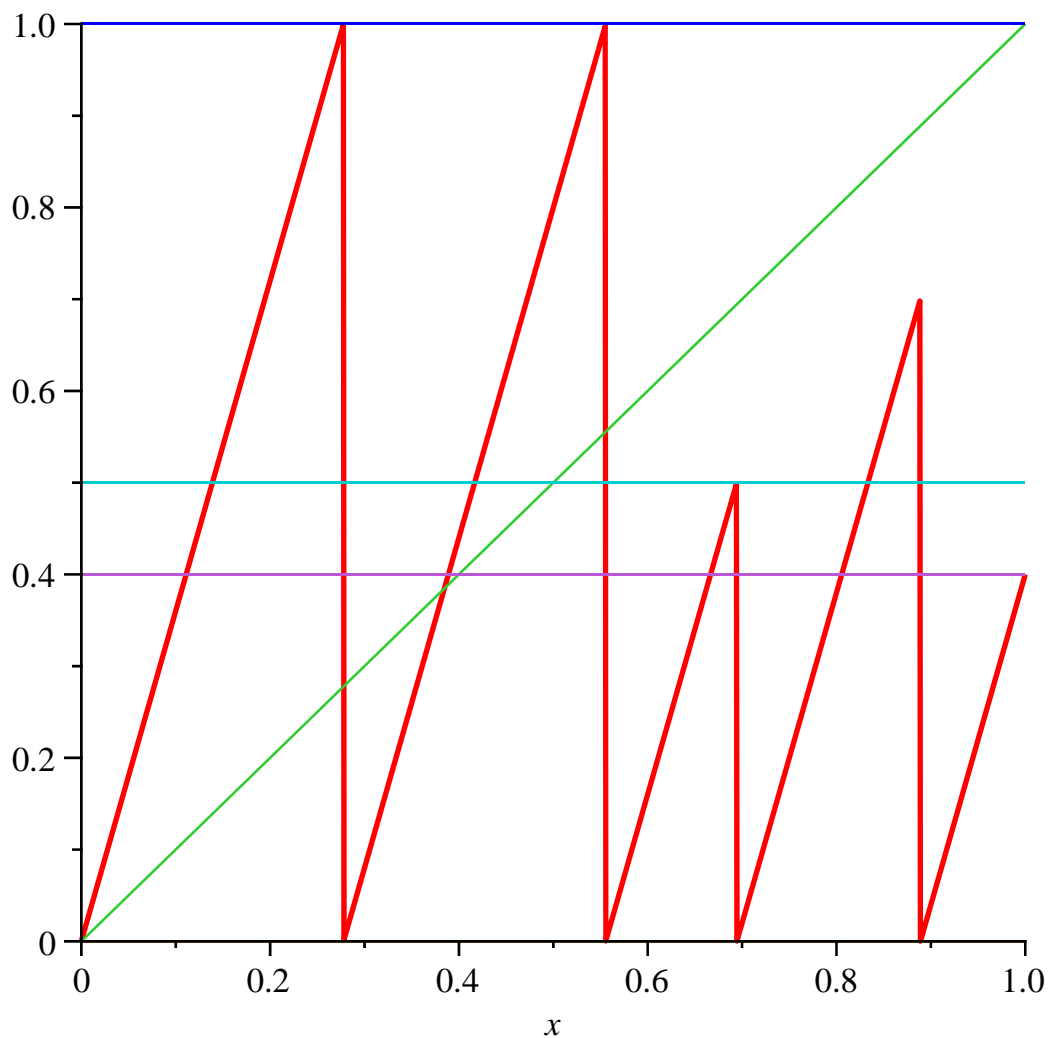
$$tT(tc[3, J]) = 0.3000000000$$

$$T(c[1, J]) = 0.4000000000$$

$$T(c[2, J]) = 0.5000000000$$

$$T(c[3, J]) = 0.7000000000$$





```

>
> t d:=vector(50): Digits:=100; NN:=50;
d:=vector(50):
xx:=evalf(rand()/10^12);
xxt:=xx:
for i from 1 to NN do
t d[i]:=ta[rint_of_x(xxt)];
xxt:=t T(xxt);
od:
xxt:=xx:
for i from 1 to NN do
d[i]:=a[rint_of_x(xxt)];
xxt:=T(xxt);
od:
print(t d);
t l s_i t_x:=evalf(sum(t d[j 1]/beta^j 1, j 1=1..NN));
print(d);
l s_i t_x:=evalf(sum(d[j 1]/beta^j 1, j 1=1..NN));
terr:=xx-t l s_i t_x;
err:=xx-l s_i t_x;

```

Digits := 100

NN := 50

$xx := 0.3957188605$

```
[0.6000000000, 2.6000000000, 0.6000000000, 2.6000000000, -0.6000000000, 1.6000000000,  
2.6000000000, 0.1000000000, 1.6000000000, 0.1000000000, 0.6000000000,  
1.6000000000, 2.6000000000, 2.6000000000, 2.6000000000, 1.6000000000,  
0.6000000000, 2.6000000000, 2.6000000000, 1.6000000000, 0.6000000000,  
2.6000000000, 2.6000000000, 2.6000000000, 2.6000000000, 2.6000000000,  
1.6000000000, 1.6000000000, 2.6000000000, 0.6000000000, 1.6000000000,  
2.6000000000, 2.6000000000, 0.1000000000, 1.6000000000, 2.6000000000,  
2.6000000000, 2.6000000000, 0.1000000000, 2.6000000000, 0.6000000000,  
2.6000000000, 0.1000000000, 2.6000000000, 0.1000000000, 2.6000000000,  
1.6000000000, 1.6000000000, 2.6000000000, 1.6000000000]
```

$tIs_it_x := 0.3957188605$

```
[1.0000000000, 1.0000000000, 1.0000000000, 3.2000000000, 0.0000000000, 0.0000000000,  
2.0000000000, 1.0000000000, 0.0000000000, 1.0000000000, 2.0000000000,  
1.0000000000, 0.0000000000, 0.0000000000, 2.5000000000, 1.0000000000,  
1.0000000000, 3.2000000000, 0.0000000000, 1.0000000000, 2.0000000000,  
1.0000000000, 1.0000000000, 1.0000000000, 1.0000000000, 1.0000000000,  
1.0000000000, 2.0000000000, 1.0000000000, 2.0000000000, 1.0000000000,  
1.0000000000, 2.5000000000, 0.0000000000, 0.0000000000, 0.0000000000,  
0.0000000000, 2.5000000000, 0.0000000000, 1.0000000000, 0.0000000000,  
2.5000000000, 0.0000000000, 1.0000000000, 0.0000000000, 1.0000000000,  
2.5000000000, 1.0000000000, 3.2000000000, 0.0000000000]
```

$Is_it_x := 0.3957188605$

$tterr := 1.420595522 \cdot 10^{-28}$

$err := 1.025994608 \cdot 10^{-28}$

(1)

>

> **NN:=50; chi :=(x1, x2, t) ->piecewise(t <x1, 0, t <x2, 1, 0);**

#Expansion of c1, c2 ... and all the S's

for i from 1 to KK do

xxt:=t c[i];

for n from 1 to NN+1 do

dc[i, n] :=t a[t int_of_x(xxt)];

ic[i, n] :=t int_of_x(xxt) - 1;

for ii from 1 to KK do

if xxt <t c[ii] then cc[i, ii, n] :=1 else cc[i, ii, n] :=0

fi;

od;

t val c[i, n] :=xxt;

xxt :=t T(xxt);

od:

Is_it_x :=sum(dc[i, j 1] / bet a^j 1, j 1=1. . NN);

```

t S[i] := sum( i c[i, j 1+1] / bet a^(j 1+1), j 1=1.. NN);
od;
for i from 1 to KK do
for j from 1 to KK do
t SS[i, j] := sum( cc[i, j, j 1+1] / bet a^(j 1+1), j 1=1.. NN);
#pri nt( `t SS[`, i, j, `] =`, t SS[i, j]);
od; od;
for i from 1 to KK do
xxt := c[i];
  for n from 1 to NN+1 do
    dc[i, n] := a[ i nt_of_x( xxt )];
    i c[i, n] := i nt_of_x( xxt) - 1;
    for ii from 1 to KK do
      if xxt > c[ii] then cc[i, ii, n] := 1 else cc[i, ii, n] := 0
    fi;
  od;
  val c[i, n] := xxt;
  xxt := T( xxt );
  od:
Is_it_x := sum( dc[i, j 1] / bet a^j 1, j 1=1.. NN);
S[i] := sum( i c[i, j 1+1] / bet a^(j 1+1), j 1=1.. NN);
od;
for i from 1 to KK do
for j from 1 to KK do
SS[i, j] := sum( cc[i, j, j 1+1] / bet a^(j 1+1), j 1=1.. NN);
pri nt( `t SS[`, i, j, `] =`, t SS[i, j]);
pri nt( `SS[`, i, j, `] =`, SS[i, j]);
od; od;

```

$N_N := 50$

$\chi := (x_1, x_2, t) \rightarrow \text{piecewise}(t < x_1, 0, t < x_2, 1, 0)$

$x_{xt} := 0.0000000000$

$Is_it_x := -9.327743860 \cdot 10^{-30}$

$tS_1 := 0.3162393162$

$x_{xt} := 0.3055555556$

$Is_it_x := 0.3055555556$

$tS_2 := 0.2777434356$

$x_{xt} := 0.1111111111$

$Is_it_x := 0.1111111111$

$tS_3 := 0.1939578389$

$x_{xt} := 1$

$Is_it_x := 1.0000000000$

$S_1 := 0.1111111111$


```

xxt := 0.69444444444
Is_it_x := 0.69444444444
S2 := 0.1496069917
xxt := 0.88888888889
Is_it_x := 0.88888888889
S3 := 0.2333925884
tSS[, 1, 1, J] =, 0.0000000000
SS[, 1, 1, J] =, 0.0000000000
tSS[, 1, 2, J] =, 2.651734690 10-16
SS[, 1, 2, J] =, 2.651734690 10-16
tSS[, 1, 3, J] =, 4.251796410 10-29
SS[, 1, 3, J] =, 4.251796410 10-29
tSS[, 2, 1, J] =, 0.0000000000
SS[, 2, 1, J] =, 0.0000000000
tSS[, 2, 2, J] =, 0.0214362056
SS[, 2, 2, J] =, 0.0214362056
tSS[, 2, 3, J] =, 7.653545582 10-12
SS[, 2, 3, J] =, 7.653545582 10-12
tSS[, 3, 1, J] =, 0.0000000000
SS[, 3, 1, J] =, 0.0000000000
tSS[, 3, 2, J] =, 0.0776206518
SS[, 3, 2, J] =, 0.0776206518
tSS[, 3, 3, J] =, 0.0004593937
SS[, 3, 3, J] =, 0.0004593937

```

```

> tMM =matrix(KK, KK, []):
MM =matrix(KK, KK, []):
for i from 1 to KK do
for j from 1 to KK do
tMM[j, i] := -tSS[i, j];
MM[j, i] := -SS[i, j];
od; od;
print(`tMM = `, tMM); print(`MM = `, MM);
print(`1/beta = `, 1/beta);
print(`eigenvalues tMM = `, eigenvalues(tMM));
print(`eigenvalues MM = `, eigenvalues(MM));

```

```

ve :=vector(KK, []):
for i from 1 to KK do
ve[i] := 1/beta;
tMM[i, i] := tMM[i, i] + 1/beta;
MM[i, i] := MM[i, i] + 1/beta;

```

```

od:
pr i nt ( t MM ) ;
pr i nt ( MM ) ;
pr i nt ( ve ) ;

```

```

t DD:=l i nsol ve( t MM, ve) ;
DD:=l i nsol ve( MM, ve) ;

```

$$tMM = \begin{bmatrix} -0.0000000000 & -0.0000000000 & -0.0000000000 \\ -2.651734690 \cdot 10^{-16} & -0.0214362056 & -0.0776206518 \\ -4.251796410 \cdot 10^{-29} & -7.653545582 \cdot 10^{-12} & -0.0004593937 \end{bmatrix}$$

$$MM = \begin{bmatrix} -0.0000000000 & -0.0000000000 & -0.0000000000 \\ -2.651734690 \cdot 10^{-16} & -0.0214362056 & -0.0776206518 \\ -4.251796410 \cdot 10^{-29} & -7.653545582 \cdot 10^{-12} & -0.0004593937 \end{bmatrix}$$

$$1/\text{beta} =, 0.2777777778$$

eigenvalues tMM =, -0.0004593936, -0.0214362057, -0.0000000000

eigenvalues MM =, -0.0004593936, -0.0214362057, -0.0000000000

$$\begin{bmatrix} 0.2777777778 & -0.0000000000 & -0.0000000000 \\ -2.651734690 \cdot 10^{-16} & 0.2563415722 & -0.0776206518 \\ -4.251796410 \cdot 10^{-29} & -7.653545582 \cdot 10^{-12} & 0.2773183841 \end{bmatrix}$$

$$\begin{bmatrix} 0.2777777778 & -0.0000000000 & -0.0000000000 \\ -2.651734690 \cdot 10^{-16} & 0.2563415722 & -0.0776206518 \\ -4.251796410 \cdot 10^{-29} & -7.653545582 \cdot 10^{-12} & 0.2773183841 \end{bmatrix}$$

$$\begin{bmatrix} 0.2777777778 & 0.2777777778 & 0.2777777778 \end{bmatrix}$$

$$tDD := \begin{bmatrix} 1.0000000000 & 1.3869268632 & 1.0016565568 \end{bmatrix}$$

$$DD := \begin{bmatrix} 1.0000000000 & 1.3869268632 & 1.0016565568 \end{bmatrix}$$

(2)

```

> t densi ty:=proc(t) l ocal j, den;
    den:=1/ bet a;
    for j from 1 to KK do
        den:=den+ t DD[ j ] * sum( ( chi ( t val c[ j , i 1+1] , 1, t ) )
/ bet a^( i 1+1) , i 1=1. . 50)
    od;
    r et urn den;
end proc;
densi ty:=proc(t) l ocal j, den;
    den:=1/ bet a;

```

```

        for j from 1 to KK do
            den: =den+ DD[j] * sum( ( chi ( 0, val c[j, i 1+1], t) )
/ bet a^(i 1+1), i 1=1.. 50)
            od;
        return den;
    end proc;
#Normalizing factor
tNC: =1/ bet a:
for j from 1 to KK do
    tNC: =tNC+t DD[j] * sum( ( 1- t val c[j, i 1+1] ) / bet a^(i 1+1), i 1=1.. 50)
    od:
NC: =1/ bet a:
for j from 1 to KK do
    NC: =NC+DD[j] * sum( ( val c[j, i 1+1] ) / bet a^(i 1+1), i 1=1.. 50)
    od:
print ( ` tNC = ` , tNC);

print ( ` NC = ` , NC);
plot ( [ ( 1/ tNC) * t density(t) ], t=0.. 1- 0. 00001, title="lazy density",
thickness=2);
plot ( [ ( 1/ NC) * density(t) ], t=0.. 1, title="greedy density",
thickness=2);

```

tdensity := **proc**(*t*)

local *j, den*;

den := 1/beta;

for *j* to *KK* do

den := *den* + *t*DD[*j*] * (sum(chi(*t*valc[*j*, *il* + 1], 1, *t*) /beta^(*il* + 1), *il* = 1..50))

end do;

return *den*

end proc

density := **proc**(*t*)

local *j, den*;

den := 1/beta;

for *j* to *KK* do

den := *den* + DD[*j*] * (sum(chi(0, valc[*j*, *il* + 1], *t*) /beta^(*il* + 1), *il* = 1..50))

end do;

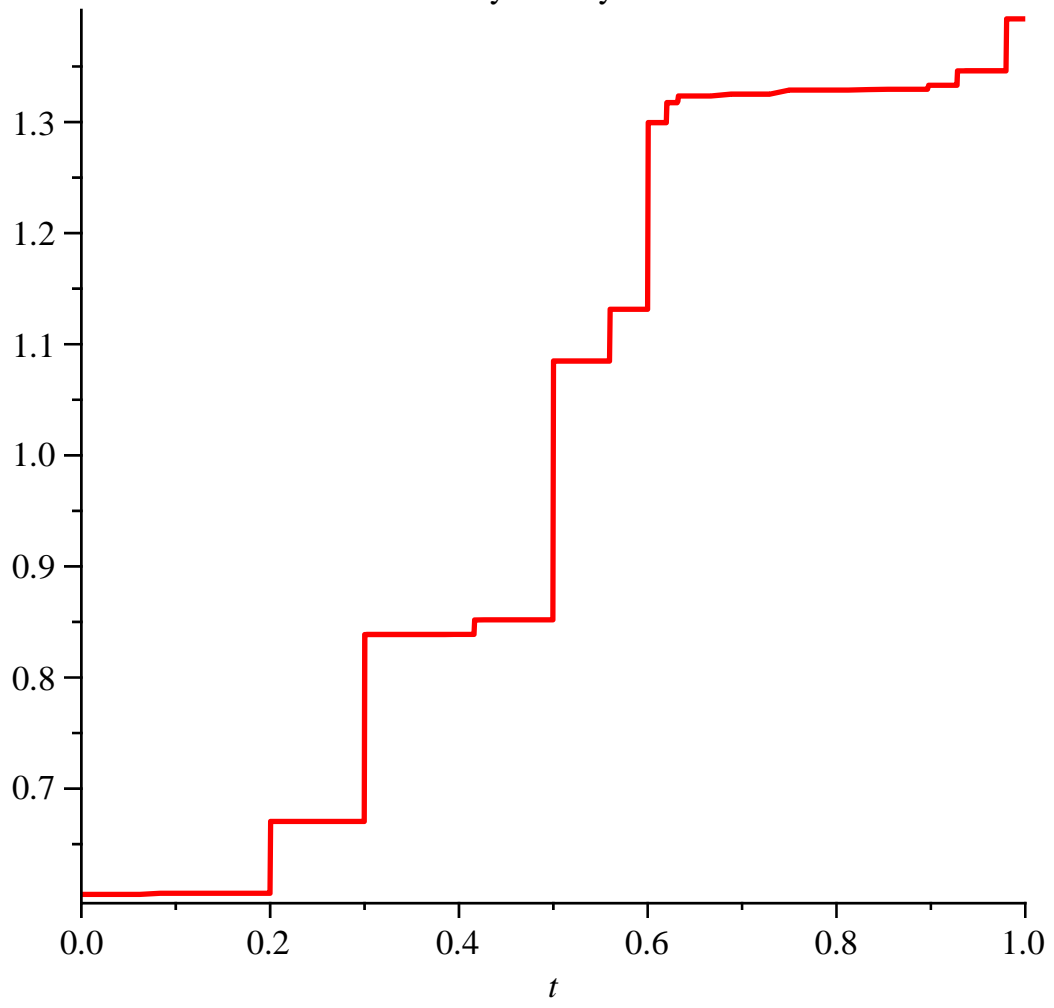
return *den*

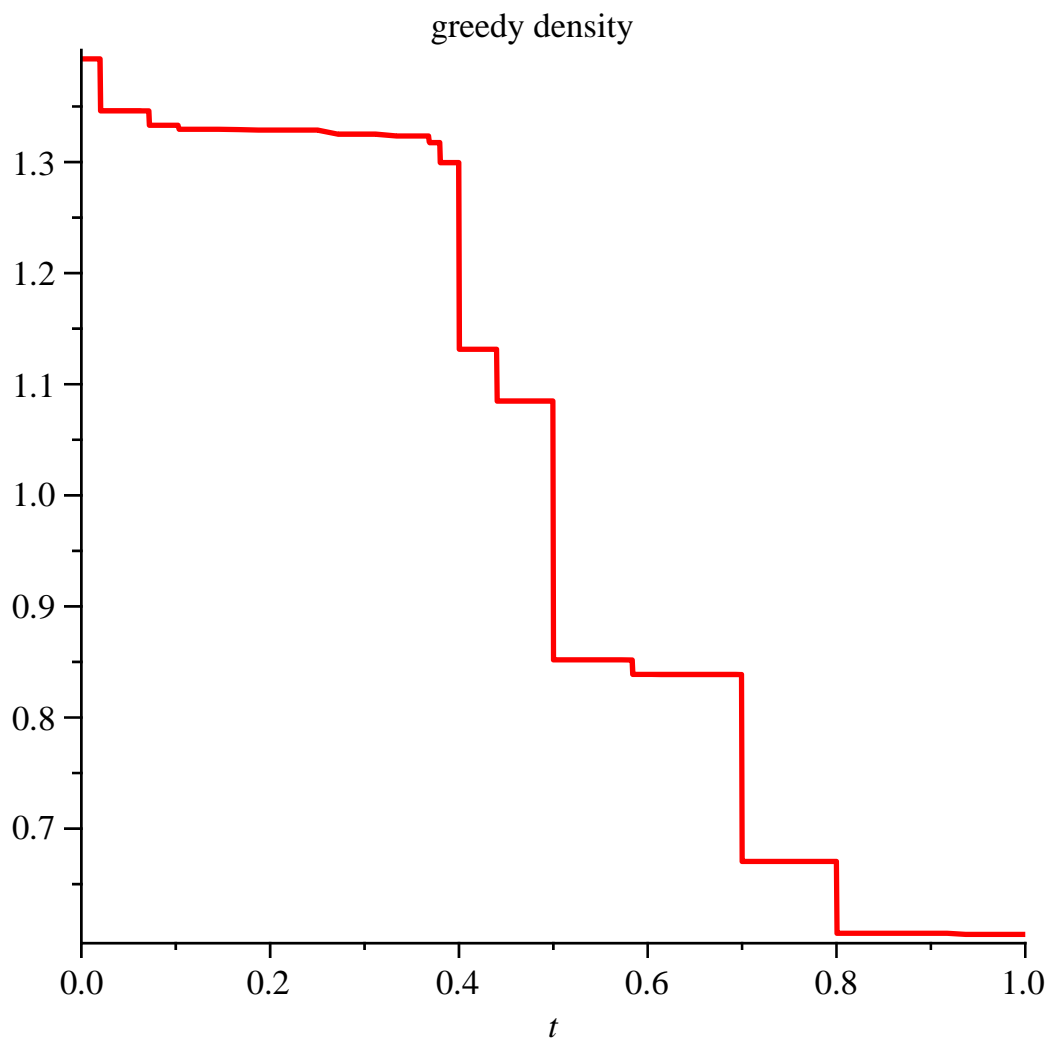
end proc

tNC = , 0.4593491897

NC = , 0.4593491897

lazy density





>

> #check density lazy

#preimages

for j6 from 1 to KK-1 do

y[j6] := al pha[j6] + (al pha[j6+1] - al pha[j6]) * rand() / 10^12;

od;

y[0] := al pha[1] * rand() / 10^12;

y[KK] := al pha[KK] + (1 - al pha[KK]) * rand() / 10^12;

for j6 from 0 to KK do

for i3 from 1 to N do

pre[i3] := (y[j6] + ta[i3]) / bet a;

od;

#plot ([T(t), 0, 1, y[j6], tT(tc[1]), tT(tc[2])], t=0..1, color=[red, black, black, green, yellow, yellow]);

su: =0:

for i3 from 1 to N do

if (pre[i3] >= tb[i3] and pre[i3] <= tb[i3+1]) then

su: =su + t density(pre[i3]) / bet a;

print(i3);

fi;

od;

terr[j6] := t density(y[j6]) - su;

00000000000000000000000000000000

su := 0

- 1
- 2
- 3
- 4
- 5

*err*₀ :=

-3.20447061907561761608813370993015379255188021018668807187977640853050910
10⁻²⁹

su := 0

- 1
- 2
- 3
- 4

*err*₁ :=

7.9763127512436236453074164479747231066026419230420572105992634092176338
10⁻³⁰

su := 0

- 1
- 2
- 4

*err*₂ :=

7.9763127512436236453074164479747231066026419230420572105992634092176337
10⁻³⁰

su := 0

- 1
- 2

*err*₃ :=

7.9763127512436236453074164479747231066026419230420572105992634092176338
10⁻³⁰

terr[, 0,] =,

-3.20447061907561761608813370993015379255188021018668807187977640853050910
10⁻²⁹

terr[, 1,] =,

7.9763127512436236453074164479747231066026419230420572105992634092176338
10⁻³⁰

terr[, 2,] =,

7.9763127512436236453074164479747231066026419230420572105992634092176337

10^{-30}

terr[, 3,]=,

7.9763127512436236453074164479747231066026419230420572105992634092176338

10^{-30}

err[, 0,]=,

-3.20447061907561761608813370993015379255188021018668807187977640853050910

10^{-29}

err[, 1,]=,

7.9763127512436236453074164479747231066026419230420572105992634092176338

10^{-30}

err[, 2,]=,

7.9763127512436236453074164479747231066026419230420572105992634092176337

10^{-30}

err[, 3,]=,

7.9763127512436236453074164479747231066026419230420572105992634092176338

10^{-30}

> **#check density lazy**

#preimages

y: =alpha[1] * rand() / 10^12; i3: =' i 3' :

for i3 from 1 to N do

pre[i3] := (y + ta[i3]) / beta;

od; i3: =' i 3' :

plot ([T(t), 0, 1, y, T(tc[1]), T(tc[2])], t=0..1, color=[red, black, black, green, yellow, yellow]);

su: =0:

for i3 from 1 to N do

if (pre[i3] >= tb[i3] and pre[i3] <= tb[i3+1]) then

su: =su + t density(pre[i3]) / beta;

print(i3);

fi;

od;

err1: =t density(y) - su;

print(`-----`);

#check density greedy

#preimages

y: =alpha[1] + (alpha[2] - alpha[1]) * rand() / 10^12;

for i2 from 1 to N do

pre[i2] := (y + a[i2]) / beta;

od;

plot ([T(t), 0, 1, y, T(c[1]), T(c[2])], t=0..1, color=[red, black, black, green, yellow, yellow]);

su: =0:

