ISLANDS SUPPORTING ACIM FOR TWO DIMENSIONAL MAP

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Abstract. With the help of the computer we prove that a two dimensional map, studied before for different ranges of parameters, for some specific values of parameters admits an absolutely continuous invariant measure supported on 175 islands (disjoint small regions of the plane).

1. Introduction

In papers [3] and [4] we considered the two dimensional map

$$G_\alpha(x,y) = [y, \tau(\alpha \cdot y + (1-\alpha)\cdot x)] \,,$$

where $\tau(x) = 1 - 2|x - 1/2|$ is the tent map and $0 < \alpha < 1$, which depending on the parameter α shows different dynamical behaviours. The map G arises from a two-dimensional approach to a one dimensional process (not a map), which we called "map with memory"

$$x_{n+1} = T(x_n) = \tau(\alpha x_n + (1-\alpha)x_{n-1}) \,.$$

T represents the situation when τ on each iteration uses not only current information but also some information from the past. We proved that:

For $0 < \alpha < 0.46$ (more precisely $0 < \alpha <\sim 0.4600595036$), G admits a two dimensional absolutely continuous invariant measure (acim). We conjectured that this acim exists for all $\alpha < 0.5$, but were not able to prove it. As α approaches 0.5 the support of the acims become thinner and thinner.

At $\alpha = 0.5$, all points have period 3 or eventually possess period 3.

For $0.5 < \alpha < 0.75$, G has a global attractor: for all starting points except $(0, 0)$, the orbits are attracted to the fixed point $(2/3, 2/3)$.

At $\alpha = 0.75$, we have slightly more complicated periodic behavior.

For $0.75 < \alpha < 1$, G has a singular SRB measure [4].

In this note we study the behaviour of G in a very narrow window of radius approximately 10^{-6} of parameters around $\alpha = 0.493000007779997$. For these αs the support of the conjectured acim looks very different from typical. It consists of 175 islands (disjoint regions of the plane) which under action of G move by 58 positions in the clockwise direction. Since $3 \cdot 58 = 174$, G^3 moves each island by 1 position in the counterclockwise direction and G^{175} preserves every island.

We observed similar behaviour for $\alpha = 0.4883$ (106 island moving by 35 positions), $\alpha = 0.4943$ (214 islands moving by 71 positions) and $\alpha = 0.4973$ (448 islands moving by 149 positions). Probably there are many other windows of α with similar behaviour.

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Our goal in this note is to prove that for $\alpha = 0.493000007779997$ the map G admits an acim supported on 175 islands. This is an example of a system exhibiting what Freeman Dyson called *weak chaos* [1]. When seen from afar the system looks perfectly periodic, and the islands move in a periodic way. In higher resolutions the system is chaotic, and our system has an acim which definitely shows chaotic behaviour.

In paper [5] the authors consider a two dimensional system very similar to ours and also observe the existence of the islands.

2. Proof

Our proof of the existence of acim is computer aided. The most important problem for which we have to depend on computer calculations is the very existence of the islands, i.e., the fact that the small regions of phase space stay separate.

We used 128-bit floating point calculations, equivalent to approximately 38 decimal digits. The islands stay separated and do not "diffuse" for at least 4 trillion iterations of G.

Once we believe in the existence of the islands, we calculate the the smaller singular values [7] of the derivative matrix $D(G^{175})(x)$ for points of one particular island which we call island 0. It is the island in the upper right corner of the Figure 1. The other islands are numbered by their position in the trajectory of island 0, i.e., island k is G^k(island 0). In Figure 4 island 0 is shown in magnification, with colors indicating the height of the density function (the warmer the color the higher the density). Since G moves islands by 58 positions we can exactly determine the trajectory of island 0. We introduce the *index symbol* of an island. An island has index 1 if it is completely in the region 1, which is below the partition line. An island is of index 2 if it is completely in the region 2, above the partition line, and it has index 3 if it intersects both regions. With this notation, the trajectory of island 0 is

221222221223221222221223221222221223221222221223221222221223221

222221223221222221223221222221223221222221223221222221

2232232222222222222222222222222222222222222222222222222223

There are 12 islands of index 3 in the trajectory: 11, 23, 35, 47,59, 71, 83, 95, 107, 119, 122 and 174, which intersect both regions. This means that for points of island 0 we have 2^{12} possible "region" trajectories. We used Maple 2021 to calculate the matrix $D(G^{175})(x)$ for each such trajectory. If $D(G^{175})(x) = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the singular values of A are square roots of eigenvalues of AA^* or of the roots of the equation $t^2 - (a^2 + b^2 + c^2 + d^2)t + a^2d^2 - 2abcd + b^2c^2 = 0$. We found that the smaller singular value for $D(G^{175})(x)$, for each such trajectory, is larger than 1 (actually greater than 1.1541.) This calculation could, in principle, be performed by hand and the precision we used makes them reliable.

This means that G^{175} is piecewise expanding on a small neighbourhood of the island 0. By Theorem 1 of [6] this implies that the map G^{175} has an acim supported on island 0 (and similarly on any other island). A standard argument shows that G itself has an acim supported on the union of the islands.

As it is discussed in the Appendix to [6], since Frobenius-Perron operator induced by G^{175} is quasi-compact on the space of functions of bounded variation, this also implies a number of strong ergodic properties for G^{175} and its acim on island 0, including exactness and various limit theorems.

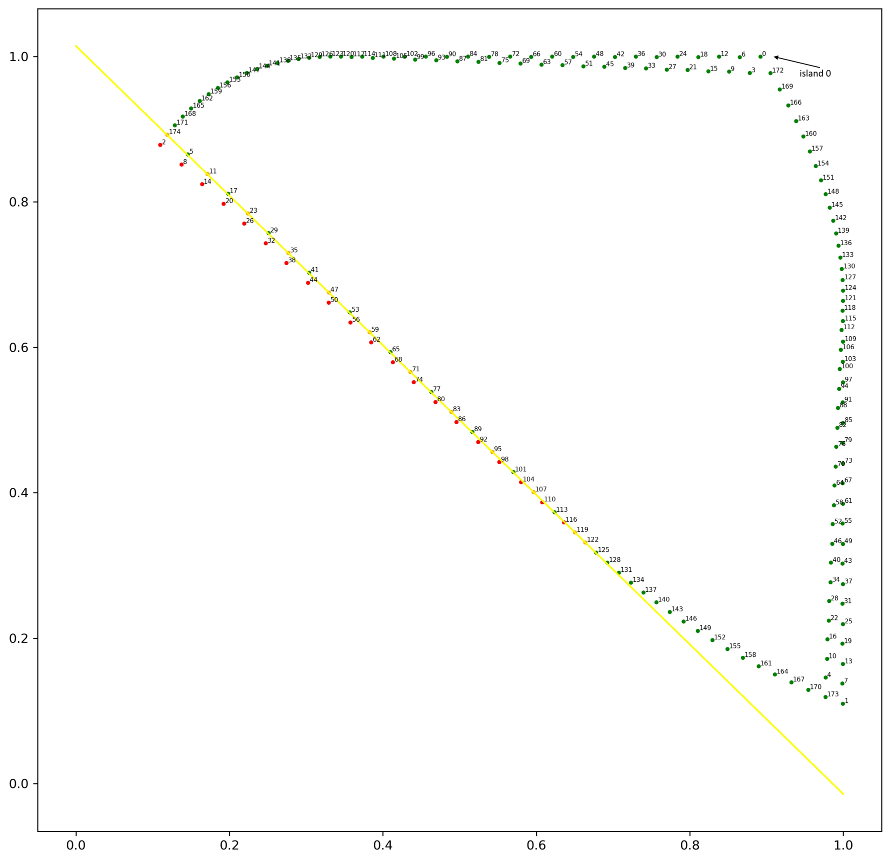

FIGURE 1. All islands with island 0 pointed out by an arrow. The islands are numbered by the position in the trajectory o island 0, i.e., island k is G^k(island 0). The colors indicate the rgion: green - region 2, red -region 1 and orange - an island intersecting both regions.

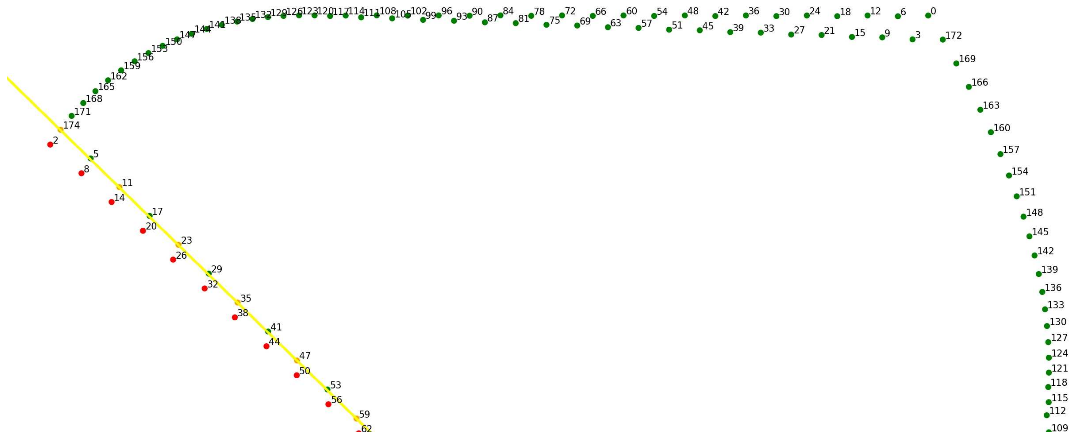

FIGURE 2. Upper half of Figure 1.

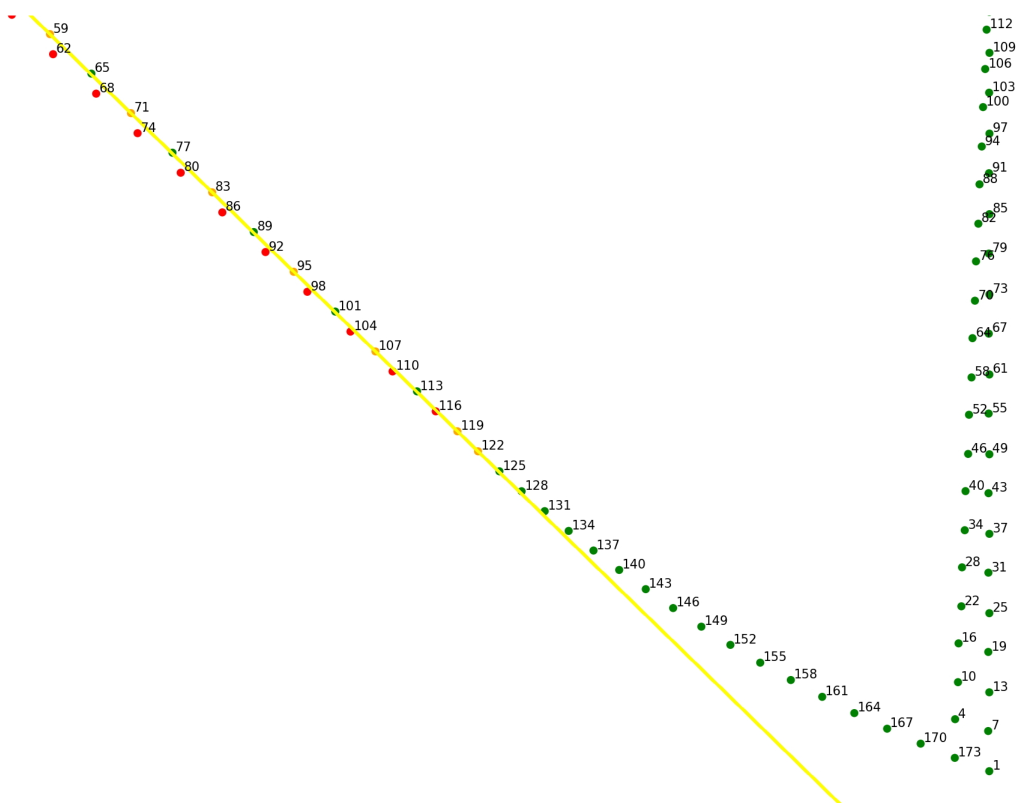

FIGURE 3. Lower half of Figure 1.

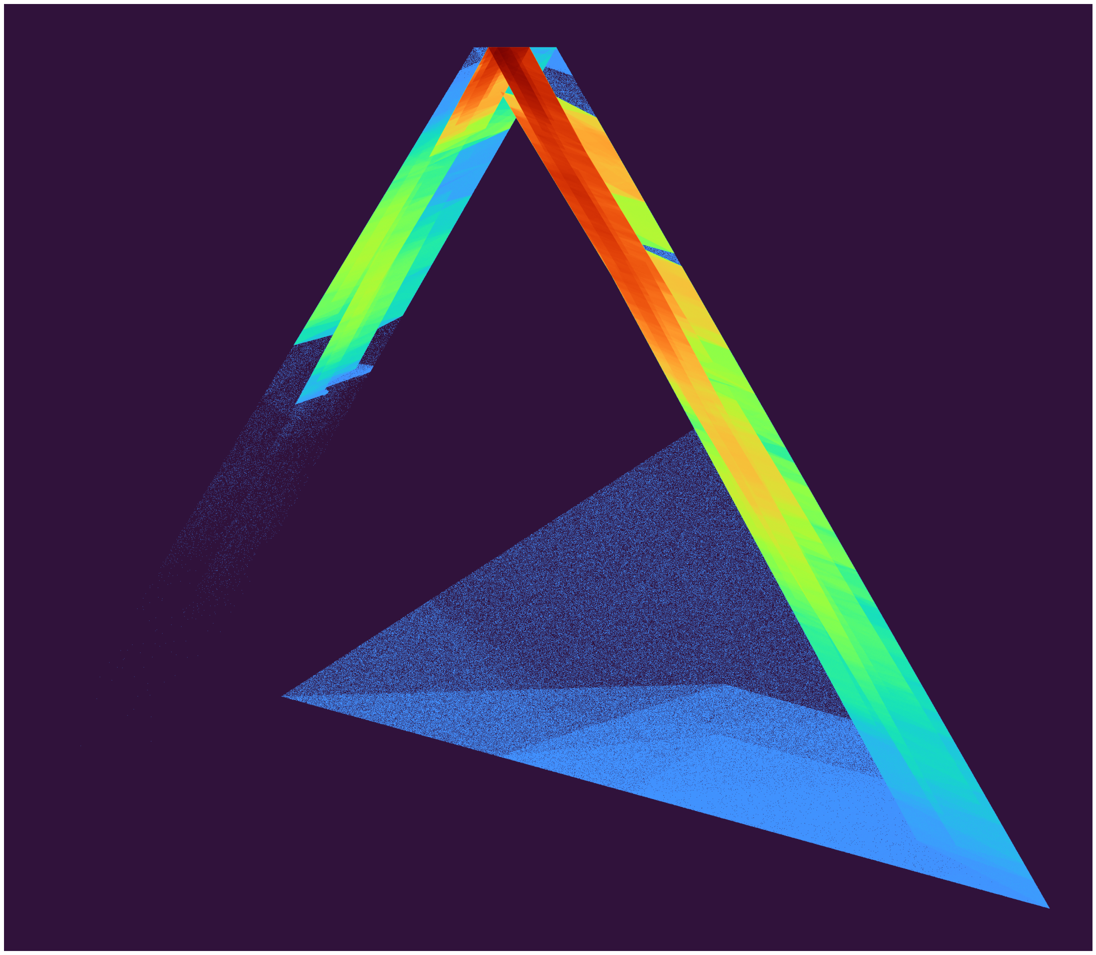

FIGURE 4. Island 0. Colors indicate the hight of the density function (the warmer the color the higher density). Picture shows the 6 billion iterations of G^{175} on a point of the island.

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