## Continuity and Uniform Continuity of Real Functions

There are two equivalent definitions of continuity of a function $f:\mathbb{R}\to\mathbb{R}$:

(1) Cauchy (or $\varepsilon-\delta$) definition: $f$ is continuous at point $x_0\in\mathbb{R} \iff$

$$\forall_{\,\varepsilon>0}\ \exists_{\,\delta>0}\ \forall_{\,x\in\mathbb{R}}\ |x-x_0|<\delta \implies |f(x)-f(x_0)|<\varepsilon\,.$$

(2) Heine (or sequential) definition: $f$ is continuous at point $x_0\in\mathbb{R} \iff$
for any sequence $\{x_n\}$ such that $x_n \underset{n\to\infty}{\longrightarrow} x_0$ we have $f(x_n) \underset{n\to\infty}{\longrightarrow} f(x_0)$.

**Theorem 1.** *Definitions (1) and (2) are equivalent.*

*Proof.* $(1)\implies(2)$ : Let $\{x_n\}$ be such that $x_n \underset{n\to\infty}{\longrightarrow} x_0$. We need to prove that $f(x_n) \underset{n\to\infty}{\longrightarrow} f(x_0)$, i.e.,

$$(*)\ \forall_{\,\varepsilon>0}\ \exists_{\,N\geq 1}\ \forall_{\,n\geq N}\ |f(x_n)-f(x_0)|<\varepsilon\,.$$

Let us fix an $\varepsilon>0$. By (1), we can find a $\delta>0$ such that $|x-x_0|<\delta \implies |f(x)-f(x_0)|<\varepsilon$. Since $x_n \underset{n\to\infty}{\longrightarrow} x_0$, we can find an $N\geq 1$ such that for $n\geq N$ we have $|x_n-x_0|<\delta$ and then $|f(x_n)-f(x_0)|<\varepsilon$. $(*)$ has been proved.

$(2)\implies(1)$ : We will prove contrapositive statement $\neg(1)\implies\neg(2)$. Let us assume that (1) does not hold, i.e.,

$$\exists_{\,\varepsilon>0}\ \forall_{\,\delta>0}\ \exists_{\,x\in\mathbb{R}}\ |x-x_0|<\delta \text{ and } |f(x)-f(x_0)|\geq\varepsilon\,.$$

Let $\varepsilon_0>0$ be the $\varepsilon$ whose existence is claimed above. It says "for any $\delta$" so we will use a sequence od $\delta$'s. Let $\delta_n=1/n>0$, $n=1,2,\ldots$. For each $\delta_n$ we can find an $x_n$ such that $|x_n-x_0|<\delta_n$ and $|f(x_n)-f(x_0)|\geq\varepsilon_0$. Thus, the sequence $x_n \underset{n\to\infty}{\longrightarrow} x_0$, but $f(x_n) \underset{n\to\infty}{\not\longrightarrow} f(x_0)$. We proved $\neg(2)$. □

**Example:** We will prove that $f(x)=x^2+3$ is continuous at $x_0=3$. Note that $f(3)=12$.

Using definition (1): Let us fix an $\varepsilon>0$. We have to find $\delta>0$ such that $|x-3|<\delta \implies |f(x)-12|<\varepsilon$. This means $|x^2+3-12|<\varepsilon$ or $|x^2-9|<\varepsilon$ or

$$(**)\ |x-3||x+3|<\varepsilon\,.$$

We have $|x-3|<\delta$. To estimate $|x+3|$ (which is unbounded on real line) we make first assumption on $\delta$: Let $\delta<1$. Then, $|x-3|<\delta$ is $|x-3|<1$ which implies $2<x<4$. This, in turn implies $|x+3|<7$. Inequality (**) becomes $\delta\cdot 7<\varepsilon$. We will satisfy it making second assumption on $\delta$: Let $\delta < \varepsilon/7$. We define

$$\delta = \frac{1}{2}\min\{1, \varepsilon/7\}\,.$$

This $\delta$ satisfies both assumptions. Above we proved that if these assumptions are satisfied and $|x-3| < \delta$, then $|f(x) - 12| < \varepsilon$. This proves that $f$ is continuous at $x_0 = 3$.

Using definition (2): Let $\{x_n\}$ be any sequence such that $x_n \to 3$ as $n \to \infty$. Using the theorems about sums and products of limits we obtain:

$$f(x_n) = x_n^2 + 3 \to 3^2 + 3 = 12 = f(3)\,.$$

We proved that $f$ is continuous at $x_0 = 3$.

## Three "Hard" Theorems about Continuous Functions

**Theorem 2.** ***Continuous Function on a Compact Interval is Bounded:*** *Let $f : [a,b] \to \mathbb{R}$ be continuous on a bounded (compact) interval $[a,b]$. Then, $f$ is bounded, i.e., there exists an $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$.*

*Proof.* Let us assume that function $f$ is not bounded above, i.e., for any $n = 1, 2, 3, \ldots$ we can find a point $x_n \in [a,b]$ such that $f(x_n) \ge n$. The sequence $\{x_n\}$ is bounded so by Bolzano-Weierstrass theorem it contains a convergent subsequence $x_{n_k} \to x_0$, as $k \to \infty$. Then, $x_0 \in [a,b]$. Since $f$ is continuous we have (Heine definition)

$$f(x_{n_k}) \to f(x_0)\ ,\ k \to \infty\,.$$

On the other hand, we have

$$f(x_{n_k}) \ge n_k\ ,\ \text{so } f(x_{n_k}) \to +\infty\ ,\ k \to \infty\,.$$

A contradiction. □

**Theorem 3.** ***Continuous Function on a Compact Interval attains its Extremal Values, the Maximum and the Minimum:*** *Let $f : [a,b] \to \mathbb{R}$ be continuous on a bounded (compact) interval $[a,b]$. Then, there exists a point $x_1 \in [a,b]$ such that*

$$f(x_1) = m = \inf_{x \in [a,b]} f(x)\,,$$

*and, there exists a point* $x_2 \in [a,b]$ *such that*

$$f(x_2) = M = \sup_{x\in[a,b]} f(x)\,.$$

*This also means that* $m = \min_{x\in[a,b]} f(x)$ *and* $M = \max_{x\in[a,b]} f(x)$*.*

*Proof.* We will prove the existence of $x_1$. Since $m = \inf_{x\in[a,b]} f(x)$ for any $n = 1,2,3,\ldots$ we can find a point $x_n \in [a,b]$ such that $m \le f(x_n) \le m + 1/n$. The sequence $\{x_n\}$ is bounded so by Bolzano-Weierstrass theorem it contains a convergent subsequence $x_{n_k} \to x_1$, as $k \to \infty$. Then, $x_1 \in [a,b]$. Since $f$ is continuous we have (Heine definition)

$$f(x_{n_k}) \to f(x_1)\ ,\ k \to \infty\,.$$

We also have

$$m \le f(x_{n_k}) \le m + 1/n_k\ ,\ \text{so } f(x_{n_k}) \to m\ ,\ k \to \infty\,.$$

Thus,

$$f(x_1) = m\,,$$

and $f$ attains its infimum on $[a,b]$. □

**Theorem 4.** ***Intermediate Value Theorem:*** *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on a bounded (compact) interval* $[a,b]$*. If* $f(a) < 0$ *and* $f(b) > 0$*, then there exist a point* $c \in (a,b)$ *such that* $f(c) = 0$*.*

*Proof.* Let $d = b - a$. We will construct approximations of a point $c$ by induction:

**1st step:** Consider the point $t = (a+b)/2$ (middle point between $a$ and $b$).

If $f(t) < 0$, then define $a_1 = t$, $b_1 = b$. Note that $b_1 - a_1 = d/2$.

If $f(t) > 0$, then define $a_1 = a$, $b_1 = t$. Note that also in this case $b_1 - a_1 = d/2$.

**2nd step:** Consider the point $t = (a_1 + b_1)/2$ (middle point between $a_1$ and $b_1$).

If $f(t) < 0$, then define $a_2 = t$, $b_2 = b_1$. Note that $b_2 - a_2 = d/4$.

If $f(t) > 0$, then define $a_2 = a_1$, $b_2 = t$. Note that also in this case $b_2 - a_2 = d/4$.

Assume that we have points $a_n < b_n$ with $f(a_n) < 0 < f(b_n)$ and $b_n - a_n = d/2^n$. (If at any time $f(t) = 0$, then we set $c = t$ and stop the procedure.)

**(n+1)st step:** Consider the point $t = (a_n + b_n)/2$ (middle point between $a_n$ and $b_n$).

If $f(t) < 0$, then define $a_{n+1} = t$, $b_{n+1} = b_n$. Note that $b_{n+1} - a_{n+1} = d/2^{n+1}$.

If $f(t) > 0$, then define $a_{n+1} = a_n$, $b_{n+1} = t$. Note that also in this case $b_{n+1} - a_{n+1} = d/2^{n+1}$.

This way we constructed two sequences: increasing $\{a_n\}$ and decreasing $\{b_n\}$ with $b_n - a_n = d/2^n$. Thus, they converge to the same limit, say $c$:

$$a_n \to c \ , \ b_n \to c \ , \ n \to \infty \, .$$

Moreover, we have $f(a_n) < 0$ and $f(b_n) > 0$ for all $n$. Since, $f$ is continuous we have

$$f(a_n) \to f(c) \ , \ f(b_n) \to f(c) \ , \ n \to \infty \, .$$

Thus, $f(c) \leq 0$ and $f(c) \geq 0$, which means that $f(c) = 0$. □

## Two Definitions of Uniform Continuity

Again, there are two equivalent definitions of the uniform continuity of a function $f : A \to \mathbb{R}$:

(1U) Cauchy (or $\varepsilon - \delta$) definition: $f$ is uniformly continuous on set $A \subset \mathbb{R} \iff$

$$\forall_{\ \varepsilon>0} \ \exists_{\ \delta>0} \ \forall_{\ x,y\in A} \ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \, .$$

(2U) Heine (or sequential) definition: $f$ is uniformly continuous on set $A \subset \mathbb{R} \iff$

for any sequences $\{x_n\}, \{y_n\}$ (contained in $A$) such that $|x_n - y_n| \underset{n\to\infty}{\longrightarrow} 0$ we have $|f(x_n) - f(y_n)| \underset{n\to\infty}{\longrightarrow} 0$.

(We do not make any other assumptions on these sequences, in particular they do not have to be convergent.)

**Theorem 5.** *Definitions (1U) and (2U) are equivalent.*

*Proof.* The proof is quite similar to the proof above.

(1U) $\implies$ (2U) : Let $\{x_n\}$ and $\{y_n\}$ be such that $|x_n - y_n| \underset{n\to\infty}{\longrightarrow} 0$. We need to prove that $|f(x_n) - f(y_n)| \underset{n\to\infty}{\longrightarrow} 0$, i.e.,

$$(*U) \ \forall_{\ \varepsilon>0} \ \exists_{\ N\geq 1} \ \forall_{\ n\geq N} \ |f(x_n) - f(y_n)| < \varepsilon \, .$$

Let us fix an $\varepsilon > 0$. By (1U), we can find a $\delta > 0$ such that $|x_n - y_n| < \delta \implies |f(x_n) - f(y_n)| < \varepsilon$. Since $|x_n - y_n| \underset{n\to\infty}{\longrightarrow} 0$, we can find an $N \geq 1$ such that for $n \geq N$ we have $|x_n - y_n| < \delta$ and then $|f(x_n) - f(y_n)| < \varepsilon$. $(*U)$ has been proved.

(2U) $\implies$ (1U) : We will prove contrapositive statement $\neg(1U) \implies \neg(2U)$. Let us assume that (1U) does not hold, i.e.,

$$\exists_{\ \varepsilon>0} \ \forall_{\ \delta>0} \ \exists_{\ x,y\in A} \ |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon \, .$$

Let $\varepsilon_0 > 0$ be the $\varepsilon$ whose existence is claimed above. It says "for any $\delta$" so we will use a sequence od $\delta$'s. Let $\delta_n = 1/n > 0$, $n = 1, 2, \ldots$. For each $\delta_n$ we can find an $x_n, y_n \in A$ such that $|x_n - y_n| < \delta_n = 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Thus, we have $|x_n - y_n| \underset{n\to\infty}{\longrightarrow} 0$, but $|f(x_n) - f(y_n)| \underset{n\to\infty}{\not\longrightarrow} 0$. We proved $\neg(2U)$. □

**Main Theorems about Uniformly Continuous Functions**

**Theorem 6.** *If $f$ satisfies Lipschitz condition on $A$,*

$$|f(x) - f(y)| \leq L|x - y| \ , \ x, y \in A\,,$$

*then $f$ is uniformly continuous on $A$.*

*Proof.* We use Cauchy definition (1U). Let us fix an $\varepsilon > 0$. Set $\delta = \varepsilon/L$. If $|x-y| < \delta$, then

$$|f(x) - f(y)| \leq L|x - y| < L \cdot \delta = \varepsilon\,.$$

□

**Example** Let $f(x) = 1/x^{2014}$, $x \in [1, +\infty)$. We will show that $f$ is uniformly continuous on $[1, +\infty)$. We will show Lipschitz inequality and invoke Theorem 6. By Mean Value Theorem we have

$$|f(x) - f(y)| = |f'(c)||x - y| = \left|\frac{-2014}{c^{2015}}\right| |x - y| \leq 2014|x - y|\,,$$

since $c \geq 1$. So $f$ satisfies Lipschitz inequality with $L = 2014$.

**Theorem 7.** *If $f$ is continuous on a closed bounded interval $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.*

*Proof.* The proof is immediate if we use Heine definition (2U): Assume that $f$ is not uniformly continuous on $[a, b]$, i.e, There exist sequences $\{x_n\}, \{y_n\}$ (contained in $[a, b]$) such that $|x_n - y_n| \underset{n\to\infty}{\longrightarrow} 0$ and $|f(x_n) - f(y_n)| \underset{n\to\infty}{\not\longrightarrow} 0$.

Since $|f(x_n) - f(y_n)| \underset{n\to\infty}{\not\longrightarrow} 0$ there is a subsequence of natural numbers $\{n_k\}$ such that $|f(x_{n_k}) - f(y_{n_k})| \geq \eta$ for some $\eta > 0$. Both sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are bounded (contained in $[a, b]$) so we can find a common convergent subsequences $\{x_{n_{k_j}}\}$ and $\{y_{n_{k_j}}\}$. Since $|x_n - y_n| \underset{n\to\infty}{\longrightarrow} 0$, they both converge to the same point, say $c \in [a, b]$, i.e., $x_{n_{k_j}} \underset{j\to\infty}{\longrightarrow} c$ and $y_{n_{k_j}} \underset{j\to\infty}{\longrightarrow} c$. Since $f$ is continuous we have $f(x_{n_{k_j}}) \underset{j\to\infty}{\longrightarrow} f(c)$ and $f(y_{n_{k_j}}) \underset{j\to\infty}{\longrightarrow} f(c)$ but this contradicts $|f(x_{n_k}) - f(y_{n_k})| \geq \eta$. □

**Example:** Consider $f(x) = \sqrt{x}$ for $x \in [0,1]$. We know that $f$ is continuous on $[0,1]$ (it follows by inequality $\sqrt{x} - \sqrt{y} \le \sqrt{x-y}$ for $x \ge y$). By Theorem 7 $f$ is uniformly continuous on $[0,1]$. Note that $f$ DOES NOT satisfy Lipschitz inequality on $[0,1]$. For the proof: Assume that it does. Then, in particular, there exist a constant $L$ such that $\sqrt{x} - 0 \le L(x-0)$ or $\frac{1}{\sqrt{x}} \le L$ for all $x \in [0,1]$ which is impossible.

**Example:** Consider a function $f(x) = \frac{x\cos(x^6)}{x^2+1}$, $x \in \mathbb{R}$. We will show that $f$ is uniformly continuous on $\mathbb{R}$. We will use Cauchy definition (1U) and Theorem 7. Let us fix $\varepsilon > 0$. Since cos is bounded we have $\lim_{x\to\pm\infty} f(x) = 0$, i.e., there exists an $M > 0$ such that for any $x \in (M, +\infty)$ we have $|f(x)| < \varepsilon/2$ and also for any $x \in (-\infty, -M)$ we have $|f(x)| < \varepsilon/2$. This means that

$$(\heartsuit) \quad \text{for any } x, y \in (M, +\infty) \quad \text{we have } |f(x) - f(y)| < \varepsilon\,,$$

and also

$$(\clubsuit) \quad \text{for any } x, y \in (-\infty, -M) \quad \text{we have } |f(x) - f(y)| < \varepsilon\,.$$

Consider $f$ on the interval $I = [-M-3, M+3]$. $f$ is continuous on $I$ (as a combination of continuous functions) so by Theorem 7 $f$ is uniformly continuous on $I$. This means that for our $\varepsilon$ we can find a $\delta > 0$ such that

$$(\diamondsuit) \text{ for any } x, y \in I, |x-y| < \delta \quad \text{implies } |f(x) - f(y)| < \varepsilon\,.$$

We can assume that $\delta < 3$. If it is not, then we make it smaller and it will also work.

We will show that this $\delta$ works on the whole $\mathbb{R}$. Let $|x - y| < \delta$. Assuming $x \le y$, there are five possibilities :

(a) $x, y \in (-\infty, -M)$. Then, $|f(x) - f(y)| < \varepsilon$ by ($\clubsuit$);

(b) $x \in (-\infty, -M)$, $y$ outside. Then, since $\delta < 3$ both $x, y \in [-M-3, M+3]$ and $|f(x) - f(y)| < \varepsilon$ by ($\diamondsuit$);

(c) $x, y \in [-M, M]$. Then, both $x, y \in [-M-3, M+3]$ and $|f(x) - f(y)| < \varepsilon$ by ($\diamondsuit$);

(d) $y \in (M, +\infty)$, $x$ outside. Then, since $\delta < 3$ both $x, y \in [-M-3, M+3]$ and $|f(x) - f(y)| < \varepsilon$ by ($\diamondsuit$);

(e) $x, y \in (M, +\infty)$. Then, $|f(x) - f(y)| < \varepsilon$ by ($\heartsuit$).

We proved that $f$ is uniformly continuous on $\mathbb{R}$. Note that

$$f'(x) = \frac{(\cos x^6 - 6x^6 \sin x^6)(x^2+1) - 2x^2 \cos x^6}{(x^2+1)^2}\,,$$

is unbounded on $\mathbb{R}$ so $f$ does not satisfy Lipschitz condition, i. e., Mean Value theorem method would not work to prove uniform continuity of this function.

**Theorem 8.** *If $f$ is uniformly continuous on $A$ and $\{x_n\}$ is a Cauchy sequence contained in $A$, then the sequence $\{f(x_n)\}$ is also Cauchy.*

*Proof.* We want to prove

$$\forall_{\,\varepsilon>0}\ \exists_{\,N\geq 1}\ \forall_{\,n,m\geq N}\ |f(x_n)-f(x_m)|<\varepsilon\,.$$

Let us fix an $\varepsilon>0$. Since $f$ is uniformly continuous, for this $\varepsilon$ we can find a $\delta>0$ such that $|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$. Since $\{x_n\}$ is Cauchy, there exists an $N\geq 1$ such that for any $n,m\geq N$ we have $|x_n-x_m|<\delta$. We see that this $N$ works also for the sequence $\{f(x_n)\}$ and $\varepsilon$: if $n,m\geq N$, then $|x_n-x_m|<\delta$ and then $|f(x_n)-f(x_m)|<\varepsilon$. □

Theorem 8 can be used to show that a function IS NOT uniformly continuous.

**Example:** Show that $f(x)=1/x^{2014}$ is not uniformly continuous on $(0,1]$. We take a sequence $x_n=1/n$ contained in $(0,1]$. It is Cauchy since it converges to 0. The sequence $f(x_n)=n^{2014}$ diverges to $+\infty$ so it is not Cauchy. By Theorem 8 $f$ is not uniformly continuous on $(0,1]$.

Another method to prove that a function is not uniformly continuous is just to use directly the definition. It often requires some ingenuity.

**Example:** Show that $f(x)=\sin x^2$ is not uniformly continuous on $[0,+\infty)$. Looking for a hint we calculate $f'(x)=2x\cos x^2$ and see that the slope of $f$ will be arbitrary large close to points where $\cos x^2=1$ or $x^2=2n\pi$ or $x=\sqrt{2n\pi}$. Let us define two sequences $x_n=\sqrt{2n\pi}$ and $y_n=\sqrt{2n\pi+a}$ for some small $a>0$. We have

$$|x_n-y_n|=|\sqrt{2n\pi}-\sqrt{2n\pi+a}|=\left|\frac{2n\pi-2n\pi-a}{\sqrt{2n\pi}+\sqrt{2n\pi+a}}\right|=\left|\frac{a}{\sqrt{2n\pi}+\sqrt{2n\pi+a}}\right|\to 0\,,$$

as $n\to+\infty$. On the other hand, we have

$$|f(x_n)-f(y_n)|=|\sin(2n\pi)-\sin(2n\pi+a)|=\sin a>0\,.$$

By Heine definition (2U), $f$ is not uniformly continuous on $[0,+\infty)$.

**Example:** Show that $f(x)=x^{16}$ is not uniformly continuous on $[0,+\infty)$. We will use Cauchy definition (1U). Its negation is:

$$\exists_{\,\varepsilon>0}\ \forall_{\,\delta>0}\ \exists_{\,x,y\in A}\ |x-y|<\delta \text{ and } |f(x)-f(y)|\geq\varepsilon\,.$$

Let us consider points $x = z$ and $y = z + \delta/2$. Then, always $|x - y| < \delta$. We have

$$|f(y) - f(x)| = (z + \delta/2)^{16} - z^{16} > (z + \delta/2)z^{15} - z^{16} = z^{15}\delta/2\,.$$

Set $\varepsilon = 1$. For arbitrary $\delta > 0$ we can find $z$ such that $|f(y) - f(x)| > 1$. We proved that $f$ is not uniformly continuous on $[0, +\infty)$.

**Theorem 9.** *Let $f : [a, b] \to \mathbb{R}$. If for any Cauchy sequence $\{x_n\} \subset [a, b]$ the sequence $f(x_n)$ is also Cauchy, then $f$ is uniformly continuous on $[a, b]$.*

*Proof.* We will prove this using contrapositive proof. Let us assume that $f$ is NOT uniformly continuous on $[a, b]$. By definition (2U) we can find an $\varepsilon > 0$ and two sequences $\{x_n\}, \{y_n\} \subset [a, b]$ such that $x_n - y_n \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Sequence $\{x_n\}$ is bounded so by Bolzano-Weierstrass theorem it contains a convergent subsequence $x_{n_k} \to x^*$. Since, $|x_{n_k} - y_{n_k}| \to 0$ the subsequence $y_{n_k}$ also converges to $x^*$. Thus, the sequence $\{x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, y_{n_3}, \ldots, x_{n_k}, y_{n_k}, \ldots\}$ also converges to $x^*$. As a convergent sequence it is Cauchy. At the same time we have $|f(x_{n_k}) - f(y_{n_k}| \geq \varepsilon$ for all $k \geq 1$ so the sequence $\{f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), f(x_{n_3}), f(y_{n_3}), \ldots, f(x_{n_k}), f(y_{n_k}), \ldots\}$ is NOT Cauchy. The theorem is proved. □

**Theorem 10.** *Let $f : [a, b] \to \mathbb{R}$. If $f$ is invertible, then the inverse function is also continuous on its domain.*

We will present two proofs. The first one would use the order of the real line, the second one will not. It will depend on Bolzano-Weierstrass theorem.

*Proof.* We will prove the theorem by contradiction. Let us assume that f is strictly increasing.(Being invertible $f$ must be either strictly increasing or strictly decreasing. Deacreasing case is similar). Then, $f^{-1}$ is also strictly increasing. Let us assume that $f^{-1}$ is NOT continuous at a point $y_0 = f(x_0)$. We know that a monotonic function has one sided limits at any point. We will consider three cases:

(1) $x_0 = a$: Then one sided limit $f^{-1}(y_0^+) = \lim_{y \to y_0^+} f^{-1}(y)$ exist. Since $f^{-1}$ is not continuous at $y_0$ we have $x_0 = f^{-1}(y_0) < f^{-1}(y_0^+)$. This means that the function $f$ is not defined on the interval $(a, f^{-1}(y_0^+))$. Drawing a graph would make this evident.

(2) $a < x_0 < b$: Then both one sided limits $f^{-1}(y_0^-) = \lim_{y \to y_0^-} f^{-1}(y)$ and $f^{-1}(y_0^+) = \lim_{y \to y_0^+} f^{-1}(y)$ exist and $f^{-1}(y_0^-) < f^{-1}(y_0^+)$. The function $f$ is not defined on the interval $(f^{-1}(y_0^-), f^{-1}(y_0^+))$. See Figure 1.

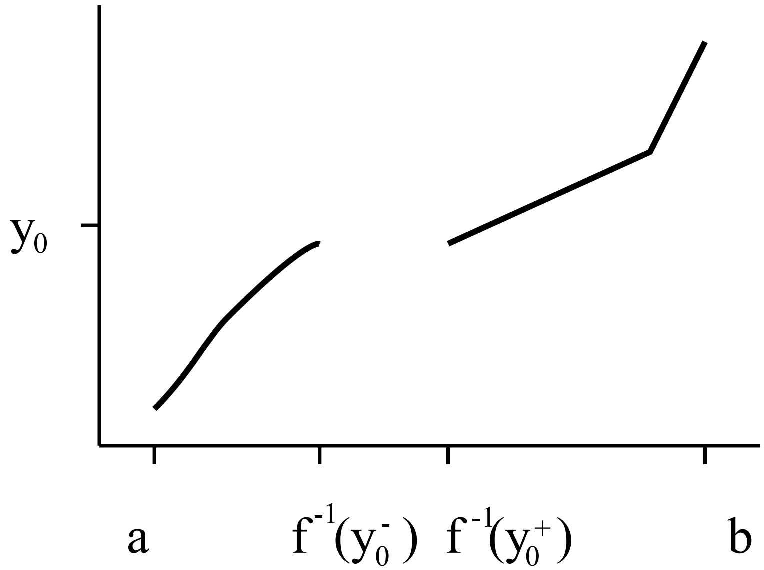


FIGURE 1. Graph for case (2) of Theorem 10

(3) $x_0 = b$: Then one sided limit $f^{-1}(y_0^-) = \lim_{y \to y_0^-} f^{-1}(y)$ exists and $f^{-1}(y_0^-) < f^{-1}(y_0) = x_0$. The function $f$ is not defined on the interval $(f^{-1}(y_0^-), b)$.

The theorem is proved. □

*Proof.* Again, let us assume that $f^{-1}$ is not continuous at $y_0 = f(x_0)$. This means that we can find a sequence $y_n \to y_0$ such that $f^{-1}(y_n) \not\to f^{-1}(y_0) = x_0$. Let $x_n = f^{-1}(y_n)$, $n \geq 1$. The sequence $\{x_n\}$ is bounded and does not converge to $x_0$ so by Bolzano-Weierstrass theorem and some additional reasoning it contains a convergent subsequence $\{x_{n_k}\}$ convergent to some $x^* \neq x_0$. Since $f$ is invertible we have $f(x^*) \neq f(x_0) = y_0$. Since $f$ is continuous $f(x_{n_k}) \to f(x^*)$. By initial assumption $f(x_{n_k}) = y_{n_k} \to y_0 = f(x_0) \neq f(x^*)$. This is a contradiction and the theorem is proved. □