Proofs by contraposition and by contradiction:

The following sentences are theorems, i.e., are true for all choices of sentences α, β, γ.

This one is the pattern of the proof by contraposition:

$$(a) \qquad (\alpha \Rightarrow \beta) \Longleftrightarrow (\neg\beta \Rightarrow \neg\alpha),$$

This one is the pattern of the proof by contradiction:

$$(b) \qquad [(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta).$$

Now we will use these patterns to prove the simple theorem: If n is divisible by 6, then n is divisible by 2.

(a) First by contraposition: the theorem is written as an implication: If n is divisible by 6, then n is divisible by 2. We can denote "n is divisible by 6" by α and "n is divisible by 2" by β. Then the theorem is $\alpha \Rightarrow \beta$. Using pattern (a) we see that instead we can prove $\neg\beta \Rightarrow \neg\alpha$. This means we need to prove "If n is not divisible by 2, then n is not divisible by 6".

If n is not divisible by 2, then $n = 2k + 1$. Now, k is of one of the three forms: $k = 3s$, $k = 3s + 1$ or $k = 3s + 2$ (depending on the remainder when k is divided by 3). Then, we have $n = 2 \cdot 3s + 1$, $n = 2 \cdot 3s + 2 + 1$, or $n = 2 \cdot 3s + 4 + 1$. In other words $n = 6s + 1$, $n = 6s+3$, or $n = 6s+5$. Thus, n is not divisible by 6. We proved $\neg\beta \Rightarrow \neg\alpha$. Using pattern (a) this implies $\alpha \Rightarrow \beta$. We proved the theorem by contraposition.

(b) Now, by contradiction: we again see that the theorem is written as an implication and denote "n is divisible by 6" by α and "n is divisible by 2" by β. Then the theorem is $\alpha \Rightarrow \beta$. Using pattern (b) we write the negation of the theorem $\alpha \wedge \neg\beta$, i.e., "n is divisible by 6 and n is not divisible by 2".

Now, we will get a contradiction: since n is divisible by 6, we have $n = 6k = 2(3k)$ and we see that n is divisible by 2. At the same time our assumption says " and n is not divisible by 2". If we denote by γ the sentence "n is divisible by 2", we proved $\gamma \wedge \neg\gamma$. According to pattern (b) this proves the theorem.

I understand that this was an unnecessary complication of the trivial proof of a trivial theorem but we did this to present an example of how the patterns (a) and (b) are used. We will use them many times in the future for much more challenging proofs.