The sequence $\sin(n)$

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We consider the sequence

$$x_n := \sin(n), \quad n \in \mathbb{N} \tag{1}$$

The sequence is bounded $-1 \leqslant x_n \leqslant 1$,

Theorem

For any $L \in [-1, 1]$ there is a subsequence $x_{n_k} = \sin(n_k)$ that converges to L.

We will use the following facts without proof

Fact

- *The function $\sin : \mathbb{R} \to [-1, 1]$ is periodic of period 2π, continuous and surjective.*
- *The number π is irrational.*

We recall that

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if for any sequence $x_n \to \alpha$ we have $f(x_n) \to f(\alpha)$.

Preparatory remarks

Since $\sin(z)$ is surjective and periodic, for any $L \in [-1,1]$ there is a unique $\alpha \in [0,2\pi)$ such that $\sin(\alpha) = L$. In order to prove the theorem it is sufficient to show that there is a sequence of pairs of integers n_k, m_k such that

$$\lim_{k\to\infty} |n_k - 2\pi m_k - \alpha| = 0 \tag{2}$$

Indeed if this is the case we have $n_k - 2\pi m_k \to \alpha$ and hence

$$\sin(n_k - 2\pi m_k) \overset{\text{by periodicity}}{=} \sin(n_k) \longrightarrow \sin(\alpha) = L \tag{3}$$

We will prove that the numbers of the form $n - 2\pi m$, $m, n \in \mathbb{N}$ are dense in $[0, 2\pi)$.

The main lemma

Lemma (Main lemma)

We have $\inf_{n,m\in\mathbb{N}} |n - 2\pi m| = 0$.

Proof.

Denote by $< x > = x - \lfloor x \rfloor$ the **fractional part** of the real number x, which is a number in $[0, 1)$. Then I claim that

$$\forall \epsilon > 0 \;\; \exists n_0, m_0 \in \mathbb{N} : \quad |n_0 - 2\pi m_0| < \epsilon. \tag{4}$$

Once this is proved, the lemma also will follow from the properties of inf. To prove the claim consider all the numbers $< 2\pi a >$, $a \in \mathbb{N}$: they all belong to $[0, 1)$ and since they are infinitely many, by the Pigeonhole principle there are two (distinct!) $a, b \in \mathbb{N}$ which are less than ϵ apart from each other:

$$\epsilon > | < 2\pi a > - < 2\pi b > | = \left| 2\pi(a - b) - \lfloor 2\pi a \rfloor + \lfloor 2\pi b \rfloor \right| \tag{5}$$

We can assume $a < b$ (if not, just rename the numbers) and hence we have the claim with $n_0 = \lfloor 2\pi a \rfloor - \lfloor 2\pi b \rfloor$ and $m = b - a$. The lemma now follows from the property of the inf of a set (0 is a lower bound for our set and also an accumulation point!). $\square$

Remark

It is important to note that the number $n_0 - 2\pi m_0$ in (4) **cannot be zero** because 2π is **irrational**.

Lemma (Second Lemma)

The set $K = \left\{n - 2\pi m \ : \ n, m \in \mathbb{N}\right\}$ *is* **dense** *in* $[0, 2\pi)$.

Proof.

Let $\alpha \in [0, 2\pi)$ and let $\epsilon > 0$ be arbitrary. From (4) in the Main Lemma we have that there exist $n_0, m_0 \in \mathbb{N}$ such that

$$-\epsilon < \overbrace{n_0 - 2\pi m_0}^{=:Q} < \epsilon \tag{6}$$

At this point there are two possibilities according to $\epsilon > Q > 0$ or $-\epsilon < Q < 0$. If $Q > 0$ we consider the integer k

$$k = \left\lfloor \frac{\alpha}{Q} \right\rfloor \tag{7}$$

Then

$$\frac{\alpha}{Q} - k < 1 \ \Rightarrow \quad \alpha - kQ = \alpha - (kn_0 - 2\pi k m_0) < Q < \epsilon \tag{8}$$

[....continues...] □

cont'd.

If $-\epsilon < Q < 0$ we set instead

$$k = \left\lfloor \frac{\alpha - 2\pi}{Q} \right\rfloor > 0 \tag{9}$$

and then

$$0 < \frac{\alpha - 2\pi}{Q} - k < 1 \;\Rightarrow\; Q < \alpha - 2\pi - kQ = \alpha - (kn_0 - 2\pi(km_0 - 1)) < 0 \tag{10}$$

In either cases ($Q < 0$ or $Q > 0$) we have found an element of K (in the form $kn_0 - 2\pi m_0$ or $kn_0 - 2\pi m_0 + 2\pi$) that lie in the ϵ–neighborhood of α. Since $\epsilon > 0$ was chosen arbitrarily, this proves that K is dense in $[0, 2\pi)$. □

Proof of the Theorem.

Since $K = \{n - 2\pi m : \; n, m \in \mathbb{N}\}$ is dense in $[0, 2\pi)$ (by the Second Lemma) then for any $\alpha \in [0, 2\pi)$ we can find a sequence $x_k = n_k - 2\pi m_k$ of numbers in K that converges to α. Thus

$$\sin(x_k) = \sin(n_k - 2\pi m_k) \longrightarrow \sin(\alpha). \tag{11}$$

Since sin is surjective on $[-1, 1]$ and 2π–periodic, for any $L \in [-1, 1]$ there is a (unique) $\alpha \in [0, 2\pi)$ such that $\sin(\alpha) = L$ and the proof is complete. □