Cardinality I

Definition 1.1

Two sets X, Y are sait to have the same **cardinality** if there exists a bijection $\varphi : X \to Y$. In this case we write $Card(X) = Card(Y)$.

Exercise 1

Show that having the same cardinality is an equivalence relation.

Definition 1.2

The subsets $[1 : n] = \{1, 2, \ldots n\}$ of $\mathbb{N}$ are called "initial segments". We say that they have cardinality n, finite. A set X has finite cardinality n if there is $n \in \mathbb{N}$ for which $Card(X) = Card([1 : n])$. The empty set has zero cardinality. X is infinite (infinite cardinality) if it is not finite.

Proposition 1.1 (Exercise)

1. If A is (in)finite and $Card(A) = Card(B)$ then B is (in)finite;
2. If A is finite and $B \subset A$ then B is finite;
3. If B is infinite and $B \subset A$ then A is infinite;

Definition 1.3

A **sequence** with values in a set X is a function $f : \mathbb{N} \to X$. The notation is $f(n) = x_n$ and $(x_n)_{n \in \mathbb{N}}$.

Proposition 1.2

The set X is infinite if and only if there is an injective map $\phi : X \to U$ where U is a **proper** subset of X.

Proof. In class. ■

Proposition 1.3 (Exercise)

1. If A is finite and there is $\phi : A \to B$, surjective, then B is finite.
2. If A is infinite and there is $\phi : A \to B$, injective, then B is infinite.

Definition 1.4

A set X is **countable** if $Card(X) = Card(\mathbb{N})$, i.e. whenever there is a bijective sequence. It is **uncountable** if it is infinite and $Card(X) \neq Card(\mathbb{N})$.
On occasion the term **at most countable** means (finite or countable).

Proposition 1.4

1. Let $Card(A) = Card(B)$. A is (un)countable if and only if B is (un)countable;
2. if A is countable and $B \subset A$ then B is at most countable;
3. if B is uncountable and $B \subset A$ then A is uncountable.

Proof. We prove only the second point, the other being left as exercises. If B is finite, there is nothing to prove. Suppose B is infinite. Since the elements of A can be enumerated $(a_n)_{n\in\mathbb{N}}$ then the elements of B can be obtained by induction as follows; $B = \{a_{n_1}, a_{n_2}, \dots, \}$ where $n_j < n_{j+1}$. The map $\psi(j) = b_j = a_{n_j}$ is the required bijection. ■

Similar to Prop. 1.4 is the following

Proposition 1.5

1. Suppose $f : A \to B$ is injective; if B is countable, then A is countable. (**Contrapositive:** if A is uncountable, then B is uncountable).
2. Suppose $f : A \to B$ is surjective; if B is uncountable then A is uncountable. (**Contrapositive:** if A is countable, then B is countable).

Proof. [1] Since f is injective, then $f : A \to Ran(f)$ is a bijection; we can construct a surjective partial inverse $h : Ran(f) \to A$ as follows. For any $b \in Ran(f)$, define $h(b) = a$ where a is the unique element in $a \in f^{-1}(\{b\})$. h is injective. Thus, A has the same cardinality as $Ran(f)$; since $Ran(f)$ is a subset of the countable set B, then (by Prop. $1.4_{(2)}$) $Ran(f)$ is countable and thus so is A.

[2] Since f is surjective, there is a partial inverse $h : B \to A$ as follows; for each $b \in B$ define $h(b) = a$ where a is any chosen (once and for all) element $a \in f^{-1}(\{b\})$ (the Axiom of Choice guarantees that it is possible to do so). Then h is injective and it gives a bijection between B and $Ran(h) \subset A$. If B is uncountable, then also $Ran(h)$ is uncountable and thus also A by Prop. $1.4_{(3)}$. ■

Lemma 1

The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof The map $\phi(n, m) = 2^n 3^m$ is injective by the unique factorization theorem. Hence $\mathbb{N} \times \mathbb{N}$ has the cardinality of a subset of $\mathbb{N}$. Thus $\mathbb{N} \times \mathbb{N}$ is countable by Prop. $1.4_{(2)}$ (here $A = \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$). ■

Corollary 1.1

A **finite** Cartesian product of countable sets is countable.

Proof. Let us show that $\mathbb{N}^r$ is countable, where r is a fixed finite integer.
Let $p_1, \ldots, p_r$ be the first r prime numbers $\in \{2, 3, 5, 7, 11, 13, \ldots\}$. Then the map $\Phi : \mathbb{N}^r \to \mathbb{N}$

$$\Phi(n_1, \ldots, n_r) = p_1^{n_1} \cdots p_r^{n_r}. \tag{1}$$

is injective. Then we argue as in the lemma that $\mathbb{N}^r$ is countable. The general statement about countable sets is left as exercise. ■

Remark 1.1

The above corollary is **not true** if we lift the assumption that the Cartesian product is finite.

Proposition 1.6

The countable union of (at most)countable sets is (at most) countable.

Proof. We construct a surjective map $\Phi : \mathbb{N} \times \mathbb{N} \to \bigcup_{\lambda \in \Lambda} A_\lambda$ so that we can use the contrapositive form of Prop. $1.4_{(2)}$. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a surjective sequence in Λ. For each λ_n let $g_n : \mathbb{N} \to A_{\lambda_n}$ be a surjective sequence with values in A_{λ_n}. Then $\Phi(n, j) = g_n(j)$ is the required surjection. ■

(Give informal proof with zig-zag picture for clarity).

Theorem 1.1 (Schröder–Bernstein)

Suppose A, B *are sets and there are* **injective** *maps* $f : A \to B$ *and* $g : B \to A$. *Then* $Card(A) = Card(B)$.

We skip the proof (can be found in Larson). It is worth remarking that the proof is constructive.

Definition 1.5

We say that $Card(A) \leqslant Card(B)$ if there is a injective map $\phi : A \to B$. We say that $Card(A) \geqslant Card(B)$ if there is a surjective map $\phi : A \to B$.

Remark 1.2

The Schröder–Bernstein theorem is then saying that
$(Card(A) \leqslant Card(B)) \wedge (Card(B) \leqslant Card(A)) \Rightarrow (Card(A) = Card(B))$.

Definition 1.6

The **power set** of a set A is the set of all subsets of A and it is denoted by $\mathcal{P}(A)$ or 2^A.

Remark 1.3

The notation 2^A is a convention based on the following observation; if $Card(A) = n$ (finite) then the number of elements of $\mathcal{P}(A)$ is $2^{Card(A)}$. This is seen by assigning a word of n bits to each subset S of A, where a 1 in the k-th place indicates that the k-th element belongs to S.

Proposition 1.7

The cardinality of the power set $\mathcal{P}(A)$ is strictly greater than the cardinality of A.

Proof. If $A = \varnothing$ then $\mathcal{P}(A) = \{\varnothing\}$ and thus $Card(A) = 0$, $Card(\mathcal{P}(A)) = 1$.
Let A be not empty. Then note that the map $\mathbf{Sing} : A \to \mathcal{P}(A)$ that associates the **singletons**

$$\mathbf{Sing}(a) = \{a\} \in \mathcal{P}(A) \tag{2}$$

is clearly an injection. Thus $Card(A) \leqslant Card(\mathcal{P}(A))$. We need to show that there is no possibility of a bijection. We proceed by contradiction; suppose that there is a bijection

$$\Phi : A \to \mathcal{P}(A). \tag{3}$$

Let $T = \{a \in A : \quad a \notin \Phi(a)\}$. Since Φ is a bijection and $T \in \mathcal{P}(A)$, there must be a $b \in A$ such that $T = \Phi(b)$.
We now reach a contradiction; either $b \in T$ or not. if $b \in T$ then $b \notin \Phi(b)$. But $\Phi(b) = T$ and hence $(b \in T) \wedge (b \notin T)$. If $b \notin T$ then $b \in \Phi(b)$. But $\Phi(b) = T$ and hence the same contradiction arises. Thus there is no bijection. ∎

Remark 1.4

It is worth noticing that this proof is a genuine proof by contradiction; we need to prove $\alpha =$ "A is a set" implies $\beta =$ "There is no bijection between A and $\mathcal{P}(A)$". We have assumed $\alpha \wedge \neg\beta$ and we reached the statement $\gamma \wedge \neg\gamma$, where $\gamma = (b \in T)$. The statement γ is a statement that is not just a reformulation of α or β.

Example 1.1

1. $\mathbb{Q}$ is countable;
2. $\mathbb{Z}$ is countable;
3. The set of polynomials with integer (or rational) coefficients is countable.
4. The set of points in $\mathbb{R}^n$ with rational coordinates is countable.
5. $\mathcal{P}(\mathbb{N})$ is uncountable. This set is bijectively equivalent to the set of infinite binary sequences, by the remark 1.3.

We now address the issue of finding an uncountable set; for this purpose we will assume a working knowledge of the real numbers $\mathbb{R}$.

Proposition 1.8

The interval $J = (0, 1) \subset \mathbb{R}$ is uncountable (and hence also $\mathbb{R}$ itself is).

Proof. The proof is important also on a general conceptual level because it uses "Cantor's diagonal trick". We proceed by contradiction. Suppose J is countable. Assume that $(x_n)_{n\in\mathbb{N}}$ is a surjective and injective sequence of real numbers. We write the numbers in their decimal expansion

$$x_n = 0, d_1^{(n)} d_2^{(n)} \dots \tag{4}$$

where $d_j^{(n)}$ is the j-th digit in the decimal expansion of the n-th number. We now construct a number that does not belong to the list of existing numbers, thus proving the contradiction (since the sequence was supposed surjective). Let x_0 be the number such that the j-th digit is different from 9, $d_j^{(j)}$. Thus $x_0 \neq x_j$ for all $j \in \mathbb{N}$.
The exclusion of 9 prevents infinite expansion of $\overline{9}$, which is equal to another number with a terminating expansion (i.e. with a tail of zeroes). ∎

Corollary 1.2

The countable Cartesian product of set containing at least two elements is uncountable.

The proof is an exercise using a similar incarnation of the diagonal trick.

Remark 1.5

The countable Cartesian product of sets A_n is simply the set of sequences $(a_1, a_2, \ldots,)$ where each $a_j \in A_j$.