Short Introduction to The Absolute Basics of Math Logic

1. Introduction

Logic is useful for everybody. For mathematicians it is the basic tool used in their every professional activity. What is presented here is the very minimum of mathematical logic every math student should know and enjoy using.

2. Elementary sentences and calculus of sentences

A "sentence" is a primary notion, i.e., something which is not defined but everybody knows what it is. It is a notion like a point in geometry or a horse in an old Polish encyclopedia: "Horse, what it is, everybody knows". For a sentence to be a mathematical sentence we need to be able to say (at least in principle) whether it is true or false. Thus, "The sky is blue" is a mathematical sentence, while "Let's go to the movies" is not. It does not matter if the sentence is false or true, as long as its "logical value" can be assigned it is a mathematical sentence.

Other examples of mathematical sentences:

There is 23 students in this class.

It will be raining tomorrow.

I have 231456 hairs on my head.

Some English sentences are not mathematical sentences:

Is this blackboard white?

Why is it raining right now?

This sentence is a lie.

The last example "This sentence is a lie" is particularly interesting. It seems that it says something so it should be either true or false. If we assume that it is true, then it is a lie, i.e., false and we obtain a contradiction. If we assume that it is false, then it is not a lie, i.e., it is true and again we obtain a contradiction. Thus, it is not true neither false, and for this reason it is not a mathematical sentence. Sentences like this one are called self referencing sentences and often cause problems.

The sentences considered above were "simple" sentences. Using them and logical operators we can build more complex sentences. Most popular logical operators are: "and", "or", "not" and "implies". There are many others, and everybody can define her/his own new operators. Traditionally, we denote them by

and $\wedge$;

or $\vee$;

not $\neg$;

implies $\Rightarrow$.

Note, that they are similar to the set operation symbols: intersection $\cap$, union $\cup$, set subtraction $\setminus$. The symbol for implication should be similar for the inclusion symbol $\subset$ but it is not.

We define logical operators by giving their value tables, i.e., for all possible logical values of sentences α and β we specify the value of the sentence αoperatorβ. We will denote value of "true" by 1 and value of "false" by 0. (Some computer languages use opposite notation.) Here are the tables:

α	β	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\neg\alpha$	$\alpha \Rightarrow \beta$
0	0	0	0	1	1
0	1	0	1	1	1
1	0	0	1	0	0
1	1	1	1	0	1

For example, for $\alpha = 1$ and $\beta = 1$ we have $(\alpha \vee \beta) = 1$. This is slightly different from everyday use of "or". Also, for $\alpha = 0$ and $\beta = 1$ we have $(\alpha \Rightarrow \beta) = 1$. This may seem incorrect but we will understand it better later.

A sentence which is always true is called a theorem (or a tautology). The simplest way to check if a given sentence is a theorem is just to check its logical value for all possible combinations of arguments values. Let us consider a sentence

$$\neg(\alpha \wedge \beta) \Longleftrightarrow (\neg\alpha \vee \neg\beta)\,. \tag{1}$$

$\gamma \Longleftrightarrow \delta$ means that γ and δ have the same logical value, i.e., are equivalent. It is the same as $(\gamma \Rightarrow \delta) \wedge (\delta \Rightarrow \gamma)$ (You can prove this). We have

α	β	$\alpha \wedge \beta$	$\neg(\alpha \wedge \beta)$	**LHS**	$\neg\alpha$	$\neg\beta$	$\neg\alpha \vee \neg\beta$	**RHS**
0	0	0	1	**1**	1	1	1	**1**
0	1	0	1	**1**	1	0	1	**1**
1	0	0	1	**1**	0	1	1	**1**
1	1	1	0	**0**	0	0	0	**0**

Thus, the sentences on both sides of (1) are equivalent and sentence (1) is a theorem. With its sibling

$$\neg(\alpha \vee \beta) \Longleftrightarrow (\neg\alpha \wedge \neg\beta)\,, \tag{2}$$

they are called De Morgan's laws, named after Augustus De Morgan (1806-71), British mathematician.

Let us consider a sentence $\neg(\alpha \Rightarrow \beta)$. By common sense it should be true only when α is true and β is false (Do you agree?). Let us check if the following sentence is a theorem:

$$(3) \qquad \neg(\alpha \Rightarrow \beta) \Longleftrightarrow (\alpha \wedge \neg\beta)\,.$$

α	β	$\alpha \Rightarrow \beta$	$\neg(\alpha \Rightarrow \beta)$	**LHS**	$\neg\beta$	$\alpha \wedge \neg\beta$	**RHS**
0	0	1	0	**0**	1	0	**0**
0	1	1	0	**0**	0	0	**0**
1	0	0	1	**1**	1	1	**1**
1	1	1	0	**0**	0	0	**0**

Thus, the sentences on both sides of (3) are equivalent and sentence (3) is a theorem. In particular, we can see that for $\alpha = 0$ and $\beta = 1$ we have $(\alpha \Rightarrow \beta) = 1$, which in a way justifies the definition of implication. (The definitions do not need justification, but it is nicer to have them reasonable.)

3. Proofs by contraposition and by contradiction:

The following sentences are theorems, i.e., are true for all choices of sentences α, β, γ. You can prove this using the truth tables.

This one is the pattern of the proof by contraposition:

$$(a) \qquad (\alpha \Rightarrow \beta) \Longleftrightarrow (\neg\beta \Rightarrow \neg\alpha),$$

This one is the pattern of the proof by contradiction:

$$(b) \qquad [(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta).$$

Now we will use these patterns to prove the simple theorem: If n is divisible by 6, then n is divisible by 2.

(a) First by contraposition: the theorem is written as an implication: If n is divisible by 6, then n is divisible by 2. We can denote "n is divisible by 6" by α and "n is divisible by 2" by β. Then the theorem is $\alpha \Rightarrow \beta$. Using pattern (a) we see that instead we can prove $\neg\beta \Rightarrow \neg\alpha$. This means we need to prove "If n is not divisible by 2, then n is not divisible by 6".

If n is not divisible by 2, then $n = 2k + 1$. Now, k is of one of the three forms: $k = 3s$, $k = 3s + 1$ or $k = 3s + 2$ (depending on the remainder when k is divided by 3). Then, we have $n = 2 \cdot 3s + 1$, $n = 2 \cdot 3s + 2 + 1$, or $n = 2 \cdot 3s + 4 + 1$. In other words $n = 6s + 1$, $n = 6s + 3$, or $n = 6s + 5$. Thus, n is not divisible by 6. We proved $\neg\beta \Rightarrow \neg\alpha$. Using pattern (a) this implies $\alpha \Rightarrow \beta$. We proved the theorem by contraposition.

(b) Now, by contradiction: we again see that the theorem is written as an implication and denote "n is divisible by 6" by α and "n is divisible by 2" by β. Then the theorem is $\alpha \Rightarrow \beta$. Using pattern (b) we write the negation of the theorem $\alpha \wedge \neg\beta$, i.e., "n is divisible by 6 and n is not divisible by 2".

Now, we will get a contradiction: since n is divisible by 6, we have $n = 6k = 2(3k)$ and we see that n is divisible by 2. At the same time our assumption says " and n is not divisible by 2". If we denote by γ the sentence "n is divisible by 2", we proved $\gamma \wedge \neg\gamma$. According to pattern (b) this proves the theorem.

I understand that this was an unnecessary complication of the trivial proof of a trivial theorem but we did this to present an example of how the patterns (a) and (b) are used. We will use them many times in the future for much more challenging proofs.

4. Sentences and set operations

Again, we do not define the notion of a set. It has to be understood. Let X denote out space, i.e., the set which contains all sets we are going to consider. Considering sets "in general", i.e., without deciding on some space containing all of them, leads to contradictions. The basic operations on sets are defined using sentences:

$A \cap B = \{x \in X : (x \in A) \wedge (x \in B)\}$, intersection of A and B;

$A \cup B = \{x \in X : (x \in A) \vee (x \in B)\}$, union of A and B;

$A^c = X \setminus A = \{x \in X : \neg(x \in A)\}$, complement of A.

We say that $A \subset B$ (A is contained in B, or A is a subset of B) if and only if $(x \in A) \Rightarrow (x \in B)$.

We can use this correspondence to prove theorems about sets. For example De Morgan's laws for set operations are

$(A \cap B)^c = A^c \cup B^c$, corresponding to (1) and

$(A \cup B)^c = A^c \cap B^c$, corresponding to (2).

Other set identities also can be "translated" into sentences equivalences and proved this way. For example:

$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ corresponds to

$(\alpha \vee \beta) \wedge \neg\gamma \iff (\alpha \wedge \neg\gamma) \vee (\beta \wedge \neg\gamma)$ and can be proved using a table of logical values.

5. Quantifiers

Let $P(x)$ be a sentence depending on variable x (say, $x^2 \geq 0$), which is true for all x in some domain X. Instead of saying: $P(x_1)$ and $P(x_2)$ and $P(x_3)$ and ... we just say: for all $x \in X$ $P(x)$, or symbolically

$$\forall_{x\in X}\ P(x)\,.$$

Quantifier $\forall$ is called a general quantifier. For example, sentence $\forall_{x\in\mathbb{R}}\ x^2 \geq 0$ is true and the sentence $\forall_{x\in\mathbb{R}}\ x \geq 3$ is false.

Similarly, the multiple (possibly infinite) or are expressed through existential quantifier "there exists", for example

$$\exists_{x\in\mathbb{R}}\ x > 3\,,$$

is a true sentence.

In some books "for all" is denoted by $\bigwedge$ and "there exists" by $\bigvee$ to highlight the fact that they are generalizations of "and" and "or", respectively.

The negations of sentences with quantifiers are obtained using de Morgan's Laws:

$$\neg\forall_{x\in X}\ P(x) \Longleftrightarrow \exists_{x\in X}\ \neg P(x)\,,$$

$$\neg\exists_{x\in X}\ P(x) \Longleftrightarrow \forall_{x\in X}\ \neg P(x)\,,$$

which seems to actually be in agreement with our common sense.

Example: below we write the definition of a continuous function $f : \mathbb{R} \to \mathbb{R}$:

$$\forall_{x\in\mathbb{R}}\ \forall_{y\in\mathbb{R}}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ |y-x| < \varepsilon \Rightarrow |f(y) - f(x)| < \varepsilon.$$

To obtain a definition of the function which is not continuous we will negate the above sentence and simplify it step by step using de Morgan's laws:

$$\neg\left[\forall_{x\in\mathbb{R}}\ \forall_{y\in\mathbb{R}}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ |y-x| < \varepsilon \Rightarrow |f(y) - f(x)| < \varepsilon\right] \Longleftrightarrow$$

$$\exists_{x\in\mathbb{R}}\ \neg\left[\forall_{y\in\mathbb{R}}\ \forall_{\varepsilon>0}\ \exists_{\delta>0}\ |y-x| < \varepsilon \Rightarrow |f(y) - f(x)| < \varepsilon\right] \Longleftrightarrow$$

$$\exists_{x\in\mathbb{R}}\ \exists_{y\in\mathbb{R}}\ \neg\left[\forall_{\varepsilon>0}\ \exists_{\delta>0}\ |y-x| < \varepsilon \Rightarrow |f(y) - f(x)| < \varepsilon\right] \Longleftrightarrow$$

$$\exists_{x\in\mathbb{R}}\ \exists_{y\in\mathbb{R}}\ \exists_{\varepsilon>0}\ \neg\left[\exists_{\delta>0}\ |y-x| < \varepsilon \Rightarrow |f(y) - f(x)| < \varepsilon\right] \Longleftrightarrow$$

$$\exists_{x\in\mathbb{R}}\ \exists_{y\in\mathbb{R}}\ \exists_{\varepsilon>0}\ \forall_{\delta>0}\ \neg\left[|y-x| < \varepsilon \Rightarrow |f(y) - f(x)| < \varepsilon\right] \Longleftrightarrow$$

$$\exists_{x\in\mathbb{R}}\ \exists_{y\in\mathbb{R}}\ \exists_{\varepsilon>0}\ \forall_{\delta>0}\ |y-x| < \varepsilon \wedge |f(y) - f(x)| \geq \varepsilon\,.$$

We used the fact that $\neg(\alpha \Rightarrow \beta) \Longleftrightarrow\ \alpha \wedge \neg\beta$.

Quantifiers correspond to generalized (possibly infinite) operations on sets: if $\{A_s\}_{s\in S}$ are subsets of some space X, then we define their intersection and union as follows:

$$\bigcap_{s\in S} A_s = \{x \in X : \forall_{s\in S}\ x \in A_s\}\,,$$

$$\bigcup_{s\in S} A_s = \{x \in X : \exists_{s\in S}\ x \in A_s\}\,.$$

The Morgan's laws "translate" for set operations as

$$\left(\bigcap_{s\in S} A_s\right)^c = \bigcup_{s\in S}(A_s)^c\,,$$

$$\left(\bigcup_{s\in S} A_s\right)^c = \bigcap_{s\in S}(A_s)^c\,.$$

6. Representation of numbers in different bases

We usually use number written in decimal notation, which means that

$$2345678 = 2\cdot 10^6 + 3\cdot 10^5 + 4\cdot 10^4 + 5\cdot 10^3 + 6\cdot 10^2 + 7\cdot 10^1 + 8\cdot 10^0\,,$$

and

$$0.2345678 = 2\cdot 10^{-1} + 3\cdot 10^{-2} + 4\cdot 10^{-3} + 5\cdot 10^{-4} + 6\cdot 10^{-5} + 7\cdot 10^{-6} + 8\cdot 10^{-7}\,.$$

This notation allows to conveniently express arbitrarily large and arbitrarily small numbers using 10 digits, 0,1,2,3,4,5,6,7,8 and 9. We say that the base of this system is 10. Popularity of number ten comes probably from the fact that an average person has ten fingers naturally used to count objects. Other civilizations used (or may be using) different representations of numbers and/or different bases. You can view examples following the link on the course page (http://www.mathstat.concordia.ca/faculty/pgora/m364AA/Number representations.html).

More generally, let us assume that our base is an integer $n > 1$. We use digits $0, 1, 2, \dots, n-1$. We have, for $0 \le a_k, a_{k-1}, a_{k-2}, \dots, a_4, a_3, a_2, a_1, a_0 \le n-1$,

$$a_k a_{k-1} a_{k-2} \dots a_4 a_3 a_2 a_1 a_0 = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \dots + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n^1 + a_0 n^0\,,$$

and

$$0.a_1 a_2 a_3 \dots a_{k-2} a_{k-1} a_k = a_1 n^{-1} + a_2 n^{-2} + a_3 n^{-3} + \dots + a_{k-2} n^{-(k-2)} + a_{k-1} n^{-(k-1)} + a_k n^{-k}\,.$$

Examples: Base is denoted by the number in brackets at the bottom.

$$457_{(8)} = 4\cdot 8^2 + 5\cdot 8 + 7 = 111_{(10)}\ ,\ 10101_{(2)} = 1\cdot 2^4 + 1\cdot 2^2 + 1 = 21_{(10)}\,.$$

$$0.121_{(3)} = 1/3 + 2/9 + 1/27 = 16/27\ ,\ 0.1001_{(2)} = 1/2 + 1/16 = 7/16\,,$$

$$0.AF_{(16)} = 10/16 + 15/256 = 175/256\,.$$

The last example is in hexadecimal system (base 16) using A=10, B=11, C=12, D=13, E=14 and F=15 as additional digits. This system finds a variety of computer related uses (check at Wikipedia). As everybody knows (?) computers translate everything into binary (base 2) representations to operate on them.

The formulas above show how to change numbers represented in other bases into decimal numbers. How to perform an inverse operation, i.e., represent a decimal number in say base 3? First we have to think how we really make decimal representation of numbers. (We are so used to it that we never think about this process.) We do it performing consecutive division by 10 and writing down the remainders:

$$\begin{aligned} 736 : 10 &= 73 \text{ remainder } 6 \\ 73 : 10 &= 7 \text{ remainder } \ 3 \\ 7 : 10 &= 0 \text{ remainder } \ 7\,. \end{aligned}$$

Similarly:

$$\begin{aligned} 736 : 3 &= 245 \text{ remainder } 1 \\ 245 : 3 &= 81 \text{ remainder } \ 2 \\ 81 : 3 &= 27 \text{ remainder } \ 0 \\ 27 : 3 &= 9 \text{ remainder } \ 0 \\ 9 : 3 &= 3 \text{ remainder } \ 0 \\ 3 : 3 &= 1 \text{ remainder } \ 0 \\ 1 : 3 &= 0 \text{ remainder } \ 1\,, \end{aligned}$$

and $736_{(10)} = 1000021_{(3)}$. To check this we write $1000021_{(3)} = 1\cdot 3^6 + 2\cdot 3 + 1 = 736_{(10)}$. We also have

$$\begin{aligned} 736 : 16 &= 46 \text{ remainder } 0 \\ 46 : 16 &= 2 \text{ remainder } \ 14 \\ 2 : 16 &= 0 \text{ remainder } \ 2\,, \end{aligned}$$

and $736_{(10)} = 2E0_{(3)}$. To check this we write $2E0_{(16)} = 2\cdot 16^2 + 14\cdot 16 = 736_{(10)}$.

Now, we will try to represent a usual fraction $\frac{3}{7}$ as a "decimal" fraction in different bases. First, in base 10:

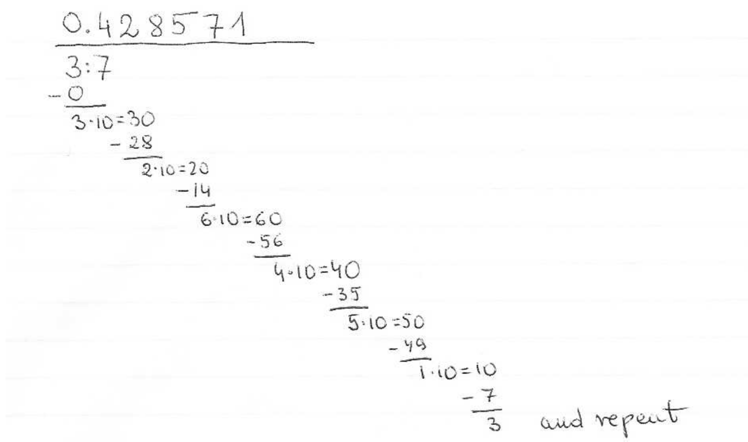

FIGURE 1. Long division 3:7 in base 10

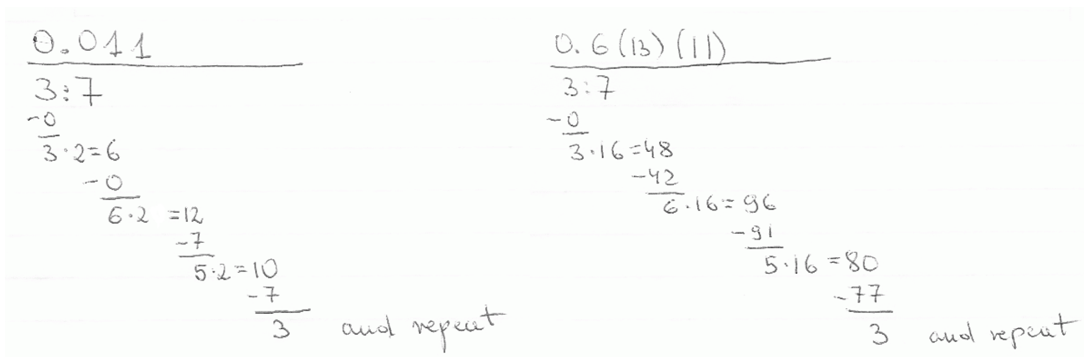

FIGURE 2. Division 3:7 in base 2 and base 16

The important fact we used is that each unit makes 10 units of the lower order. To check correctness of our result we calculate

$$0.428571428571\cdots = \frac{428571}{10^6}\left(\frac{1}{1-1/10^6}\right) = \frac{3}{7}\,.$$

We used the formula for the sum of geometric series

$$1+q+q^2+q^3+\cdots = \frac{1}{1-\frac{1}{q}} \text{ for } |q|<1\,.$$

Similarly, we do this for base 2 and base 16, remembering that for these cases the units make 2 and 16 units of lower order, correspondingly.

To check correctness we calculate:

$$0.011011011\ldots_{(2)} = \left(\frac{1}{4}+\frac{1}{8}\right)\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{3}{7}\,,$$

and

$$0.6(13)(11)6(13)(11)6(13)(11)\ldots_{(16)} = \left(\frac{6}{16}+\frac{13}{16^2}+\frac{11}{16^2}\right)\left(\frac{1}{1-\frac{1}{16^3}}\right) = \frac{3}{7}\,.$$

Of course, we should have used D instead of (13) and B instead of (11).

An easy corollary of the presented algorithm is the theorem:

Theorem: A rational number (a usual fraction) has eventually periodic digital representation in any base. An eventually periodic digital fraction represents a rational number.

Proof of the first claim uses the fact that in base n there is only n possible remainders so the digital representation has to repeat after at most n steps. The second follows from the calculation.

The last question in this section is changing a decimal fraction into digital fraction in other base. First we will do a short geometric introduction into representation of fractions in general. Let us consider base $n = 4$. We will use 4 digits $0, 1, 2, 3$. Let us divide interval $[0, 1)$ into 4 equal subintervals:

$$[0,1) = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{2}{4}\right) \cup \left[\frac{2}{4}, \frac{3}{4}\right) \cup \left[\frac{3}{4}, 1\right) ,$$

see Figure 3 a). The points in the first interval have first digit 0 (since they are between 0 and $1/4$), the points in the second interval have first digit 1 (since they are between $1/4$ and $2/4$), the points in the third interval have first digit 2 (since they are between $2/4$ and $3/4$) and the points in the fourth interval have first digit 3 (since they are between $3/4$ and $4/4$). To assign the second digit we perform the same operation on each of the intervals of the first partition.

In Figure 3 b) we zoomed up the first interval corresponding to the the first digit 0. We divided it into 4 equal subintervals and the digits of the points in these subintervals are:

00 (since they are between 0 and $1/16$),

01 (since they are between $1/16$ and $2/16$),

02 (since they are between $2/16$ and $3/16$),

03 (since they are between $3/16$ and $4/16$).

In Figure 3 c) we zoomed up the third interval of the second generation corresponding to the the first digits 02. We divided it into 4 equal subintervals and the digits of the points in these subintervals are:

020 (since they are between $0/4 + 2/16 + 0/16^2$ and $0/4 + 2/16 + 1/16^2$),

021 (since they are between $0/4 + 2/16 + 1/16^2$ and $0/4 + 2/16 + 2/16^2$),

022 (since they are between $0/4 + 2/16 + 2/16^2$ and $0/4 + 2/16 + 3/16^2$),

023 (since they are between $0/4 + 2/16 + 3/16^2$ and $0/4 + 2/16 + 4/16^2$).

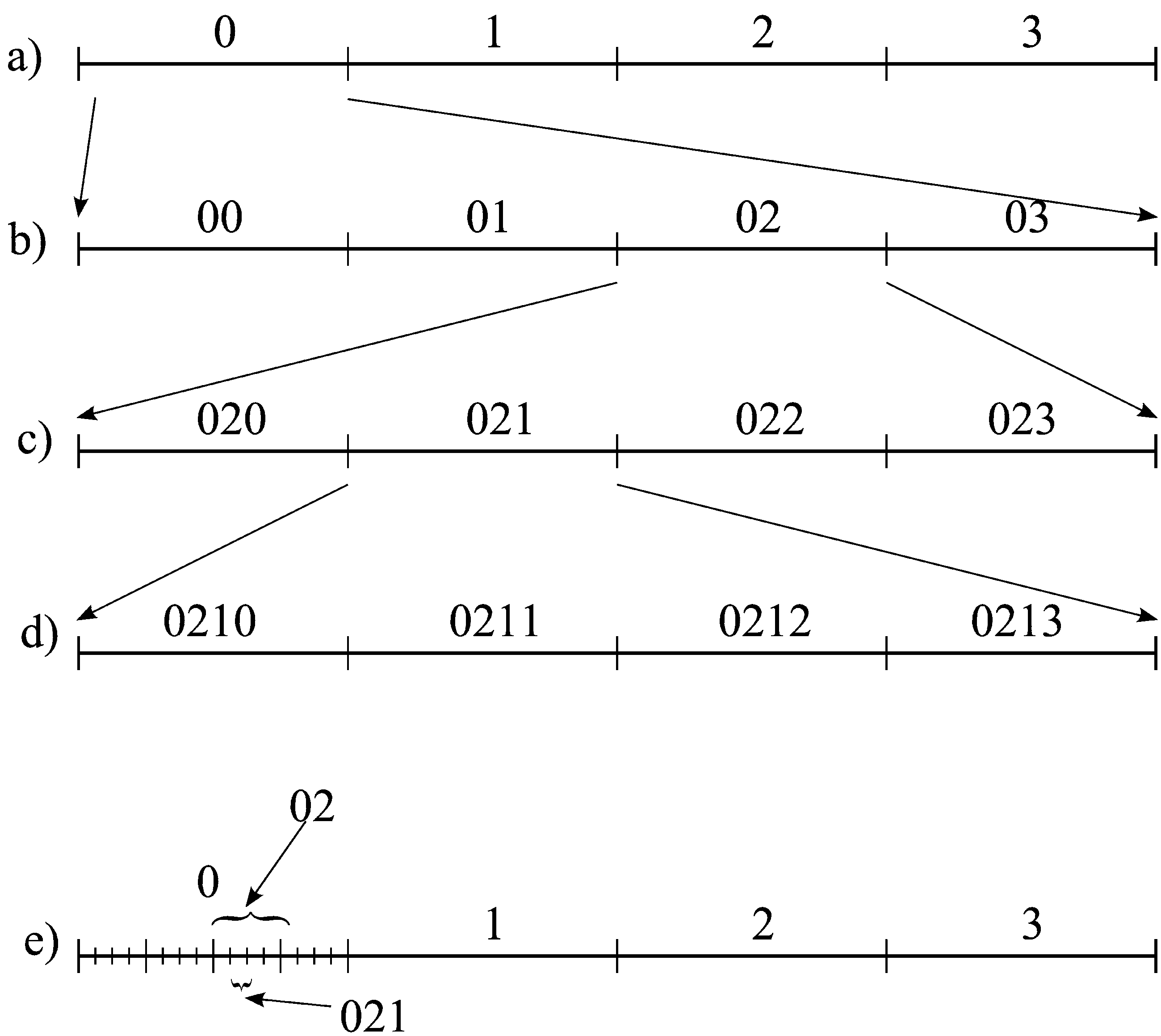

FIGURE 3

In Figure 3 d) we zoomed up the second interval of the the third generation corresponding to the the first digits 021. We divided it into 4 equal subintervals and the digits of the points in these subintervals are:

0210 (since they are between $0/4 + 2/16 + 1/16^2 + 0/16^3$ and $0/4 + 2/16 + 1/16^2 + 1/16^3$),

0211 (since they are between $0/4 + 2/16 + 1/16^2 + 1/16^3$ and $0/4 + 2/16 + 1/16^2 + 2/16^3$),

0212 (since they are between $0/4 + 2/16 + 1/16^2 + 2/16^3$ and $0/4 + 2/16 + 1/16^2 + 3/16^3$), 0213 (since they are between $0/4+2/16+1/16^2+3/16^3$ and $0/4+2/16+1/16^2+4/16^3$).

These operations are repeated infinitely and every point receives it expansion in base 4.

In Figure 3 e) we showed intervals corresponding to digits 02 and 021 without zooming.

Now we will represent a decimal fraction $x = 0.123457$ as a fraction in base 4. We want to represent x in the form $\frac{d_1}{4}+\frac{d_2}{4^2}+\frac{d_3}{4^3}+\frac{d_4}{4^4}+\frac{d_5}{4^5}+\ldots$. How to get d_1: it is integer part of $4\cdot x$ as $4\cdot x = d_1+\frac{d_2}{4^1}+\frac{d_3}{4^2}+\frac{d_4}{4^3}+\frac{d_5}{4^4}+\ldots$. We have $4\cdot 0.123457 = 0.493828$ so $d_1 = 0$. This shows that x is in the first ("0") interval of the first generation in geometric construction above.

To find d_2 we remove d_1, i.e., consider $x_1 = 4x - E(4x) = \frac{d_2}{4^1}+\frac{d_3}{4^2}+\frac{d_4}{4^3}+\frac{d_5}{4^4}+\ldots$, where $E(t)$ denotes the integer part of t. Now, $d_2 = E(4x_1)$. We have $4\cdot 0.493828 = 1.975312$ so $d_2 = 1$. This locates x more precisely in $[0,1)$. x is in the interval corresponding to 01 in the partition of second generation. And so on:

$x_2 = 4x_1 - E(4x_1) = 0.975312$ and $4x_2 = 3.901248$ so $d_3 = 3$ and

$x_3 = 4x_2 - E(4x_2) = 0.901248$ and $4x_3 = 3.604992$ so $d_4 = 3$ and

$x_4 = 4x_3 - E(4x_3) = 0.604992$ and $4x_4 = 2.419968$ so $d_5 = 2$ and so on. We have

$$0.123457_{(10)} = 0.01332\ldots_{(4)}\ .$$

To check we write

$$0.01332\ldots_{(4)} = 0/4+1/4^2+3/4^3+3/4^4+2/4^5 = 0.1230468750\,,$$

which agrees with what we expected on the first three places. Precision in base 4 in much worse than in base 10, i.e., one needs much more digits to represent a number to the same precision.