Subsequential Limits

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Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers: we define its **set of subsequential limits**

$$E = E[x_n] := \{L \in \mathbb{R}^\sharp : \text{ there is a subsequence } x_{n_k} \to L\} \quad (1)$$

The set $E[x_n]$ is always nonempty: if x_n is bounded then by B.W. there is a convergent subsequence and hence E contains at least its limit. If x_n is unbounded above, then x_n has a subsequence that goes to $+\infty$. If it is unbounded below then we can find a subsequence that goes to $-\infty$.

Proposition

The number $\alpha := \sup E[x_n] := \limsup_n x_n$ *belongs to* $E[x_n]$.

Proof.

If $\alpha = -\infty$ it means that $E = \{-\infty\}$ and there is nothing to prove.
If $\alpha = +\infty$ then the sequence x_n must be unbounded above. If it were bounded above then

$$\exists M \in \mathbb{R} : \quad \forall n \in \mathbb{N} \; x_n < M \tag{2}$$

For any $L \in E$ let x_{n_k} be a subsequence that converges to L; since we have $x_n < M$ this must hold also for the given subsequence. In particular $L \leq M$. Thus M is an upper bound for E and $\sup E \leq M < +\infty$, a contradiction.
Since x_n is unbounded above it admits an (increasing) subsequence that tends to $+\infty$ and hence $\alpha \in E$. Such a sequence is constructed by induction as follows. $n_1 = 1$, n_2 is chosen so that $n_2 \geq n_1 + 1$ and $x_{n_2} > 2$ and n_k so that $n_k \geq n_{k-1} + 1$ and $x_{n_k} > k$. At each step the existence of such n_k is guaranteed by the fact that x_n is unbounded above. The constructed subsequence satisfies $x_{n_k} > k$. **[...continues...]** □

...cont'd.

It remains the case where $\alpha = \sup E \in \mathbb{R}$ is a *finite* number. If α is **not an accumulation point for** E then it must belong to E and there is nothing else to prove. It only remains to prove that **if** $\alpha = \sup E$ **is an accumulation point for** E **then** $\alpha \in E$. Since this is true in general (whether α is the sup or not) we prove it separately below. □

Theorem (Closure of E)

Let α be an accumulation point for the set E of all subsequential limits of the sequence $(x_n)_{n\in\mathbb{N}}$. Then $\alpha \in E$ (i.e. it is itself a subsequential limit).

Proof.

By definition of accumulation point, we can find a sequence $y_j \in E$ such that $|y_j - \alpha| < \frac{1}{j}$. We now construct a subsequence x_{n_j} as follows (by induction): n_1 is chosen such that $|x_{n_1} - y_1| < 1$. Since y_2 is also a subsequential limit there is a subsequence $x_{\widetilde{n}_k} \to y_2$. Thus there must be a $\widetilde{n}_{k_0} > n_1$ such that $|x_{\widetilde{n}_{k_0}} - y_2| < \frac{1}{2}$. We take this $\widetilde{n}_{k_0}$ and call it n_2, so that $n_2 > n_1$ and $|x_{n_2} - y_2| < \frac{1}{2}$. Proceeding by induction after $j-1$ steps we have n_{j-1} such that $|x_{n_{j-1}} - y_{j-1}| < \frac{1}{j-1}$. Consider y_j: there is a subsequence $x_{\hat{n}_k} \to y_j$: we can find a $\widehat{n}_{k_0} > n_{j-1}$ such that $|x_{\widehat{n}_{k_0}} - y_j| < \frac{1}{j}$. We call this $\widehat{n}_{k_0}$ by n_j and so on.
We have thus constructed a subsequence x_{n_j} such that $|x_{n_j} - y_j| < \frac{1}{j}$.
I claim that $x_{n_j} \to \alpha$. Indeed

$$|x_{n_j} - \alpha| = |x_{n_j} - y_j + y_j - \alpha| \leq |x_{n_j} - y_j| + |y_j - \alpha| < \frac{2}{j} \xrightarrow[j\to\infty]{} 0 \tag{3}$$

□

Alternative definition of $\limsup$ and $\liminf$

Consider the sequence

$$S_n := \sup\{x_k : \ n \geq k\} \tag{4}$$

This sequence is **decreasing (exercise)**: $L_{n+1} \leq L_n$. It thus has a limit in $\mathbb{R}^\sharp$ (equal to its inf). Then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} S_n = \inf\{S_n : \ n \in \mathbb{N}\} = \lim_{n\to\infty} \sup\{x_k : \ n \geq k\} \tag{5}$$

Similarly

$$L_n := \inf\{x_k : \ n \geq k\} \tag{6}$$

is **increasing** and hence has limit (equal to its sup). Then

$$\liminf_{n} x_n = \lim_{n\to\infty} L_n = \sup\{L_n : \ n \in \mathbb{N}\} = \lim_{n\to\infty} \inf\{x_k : \ k \geq n\} \tag{7}$$

Yet another characterization of $\limsup$

$\alpha = \limsup x_n$ if and only if the following **two conditions** apply:

- for any $M > \alpha$ the sequence is **eventually** less than M (this condition is void if $\alpha = \infty$);
- for any $L < \alpha$ and any $N \in \mathbb{N}$ there is a $n_0 \geq N$ such that $x_{n_0} > L$.

Exercise

Prove that the the three definitions of $\limsup$ *are equivalent.*