Proof of a theorem presented in class

hopefully better explained now.

At the end: second proof.

Theorem: For a sequence (x_n) there exist subsequences (x_{n_k}) and (x_{n_ℓ}) such that

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n \quad , \quad \lim_{\ell\to\infty} x_{n_\ell} = \liminf_{n\to\infty} x_n \, .$$

Proof. First, a remark about notation: Let (x_n) be a sequence : $(x_1, x_2, x_3, x_4, x_5, \dots)$. We will be choosing from it subsequences $\left(x^{(k)}_{n(k)}\right)$ for natural numbers $k = 1, 2, 3 \dots$ The upper (k) is just the number of the subsequence (not a power) and probably could be removed without any loss. $n(k)$ is a sequence of indices, a subsequence of natural numbers. For example, if

$$\left(x^{(6)}_{n(6)}\right) = (x_8, x_{12}, x_{13}, x_{27}, x_{34}, x_{35}, x_{64}, x_{91}, \dots) \, ,$$

then $n(6) = (8, 12, 13, 27, 34, 35, 64, 91, \dots)$. In particular, $n(6)_3 = 13$ and $x^{(6)}_{n(6)_3} = x_{13}$. Similarly, $n(6)_8 = 91$ and $x^{(6)}_{n(6)_8} = x_{91}$.

Now, we can start roofing. Let A be the set of all partial limits of (x_n), i.e., $a \in A$ if and only if there is a convergent subsequence (x_{n_k}) of (x_n) with $a = \lim_{k\to\infty} x_{n_k}$. The set A is not empty as every sequence either goes to infinity or contains a bounded subsequence. And a bounded subsequence contains a convergent one by Bolzano-Weierstrass theorem. Let $\alpha = \sup A$, possibly infinity.

If $\alpha = -\infty$, this means that $x_n \to -\infty$ and there is nothing to prove.

If $\alpha = +\infty$, the sequence (x_n) is unbounded above and we know that unbounded above sequence contains a subsequence convergent to $+\infty$. Again, nothing to proof.

We assume that α is a finite number.

Then, by the definition of the supremum, for any $k = 1, 2, 3, \dots$ there is an element $a_k \in A$ such that

$$\alpha - \frac{1}{k} < a_k \le \alpha \, .$$

In turn, for any $k = 1, 2, 3, \dots$ there is a subsequence of (x_n), we call it $\left(x^{(k)}_{n(k)}\right)$, such that

$$x^{(k)}_{n(k)} \to a_k \text{ as } n(k) \to \infty \, .$$

We start the construction of the subsequence of (x_n) converging to α.

$k = 1$: The subsequence $x^{(1)}_{n(1)} \to a_1$ so for large indices $n(1)_i$ in $n(1)$ the elements $x^{(1)}_{n(1)_i}$ are close (say with error < 1) to a_1. We can find $x^{(1)}_{n(1)_{s(1)}}$ such that

$$\alpha - 1 - 1 < a_1 - 1 < x^{(1)}_{n(1)_{s(1)}} < a_1 + 1 < \alpha + 1\,.$$

$k = 2$: The subsequence $x^{(2)}_{n(2)} \to a_2$ so for arbitrarily large indices $n(2)_i$ in $n(2)$ the elements $x^{(2)}_{n(2)_i}$ are close (say with error $< 1/2$) to a_2. We can find $x^{(2)}_{n(2)_{s(2)}}$ with index $n(2)_{s(2)} > n(1)_{s(1)}$ such that

$$\alpha - \frac{1}{2} - \frac{1}{2} < a_2 - \frac{1}{2} < x^{(2)}_{n(2)_{s(2)}} < a_2 + \frac{1}{2} < \alpha + \frac{1}{2}\,.$$

We make requirement $n(2)_{s(2)} > n(1)_{s(1)}$ because the indices in a subsequence must be strictly increasing.

We proceed by induction: We assume that we have already chosen the elements $x^{(k)}_{n(k)_{s(k)}}$, $k = 1, 2, 3, \ldots, m-1$ satisfying $n(k+1)_{s(k+1)} > n(k)_{s(k)}$ and

$$\alpha - \frac{1}{k} - \frac{1}{k} < a_k - \frac{1}{k} < x^{(k)}_{n(k)_{s(k)}} < a_k + \frac{1}{k} < \alpha + \frac{1}{k}\,.$$

We will choose the next mth element of (x_n).

$k = m$: The subsequence $x^{(m)}_{n(m)} \to a_m$ so for arbitrarily large indices $n(m)_i$ in $n(m)$ the elements $x^{(m)}_{n(m)_i}$ are close (say with error $< 1/m$) to a_m. Thus, we can find $x^{(m)}_{n(m)_{s(m)}}$ with index $n(m)_{s(m)} > n(m-1)_{s(m-1)}$ such that

$$\alpha - \frac{1}{m} - \frac{1}{m} < a_m - \frac{1}{m} < x^{(m)}_{n(m)_{s(m)}} < a_m + \frac{1}{m} < \alpha + \frac{1}{m}\,.$$

By induction, we constructed a subsequence $\left(x^{(k)}_{n(k)_{s(k)}}\right)$ of (x_n) (the indices are strictly increasing) such that for each $k = 1, 2, 3\ldots$ we have

$$\alpha - \frac{2}{k} < x^{(k)}_{n(k)_{s(k)}} < \alpha + \frac{1}{k}\,.$$

By "squeeze" theorem

$$x^{(k)}_{n(k)_{s(k)}} \to \alpha \text{ as } k \to \infty\,.$$

This actually shows that $\alpha \in A$.

The lim sup part of the theorem has been proved. The lim inf part is proven similarly.

Another proof:

Let $\alpha = \limsup_{n\to\infty} x_n$. We want to show that there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = \alpha$. We will consider finite and infinite cases separately.

(1) α is finite. We want to show that

$$\forall_{\ \varepsilon>0} \ \forall_{\ N\geq 1} \ \exists_{\ n\geq N} \ |x_n - \alpha| < \varepsilon.$$

Let us assume that this does not hold, i.e.,

$$(*) \quad \exists_{\ \varepsilon>0} \ \exists_{\ N\geq 1} \ \forall_{\ n\geq N} \ |x_n - \alpha| \geq \varepsilon.$$

Let us fix such an $\varepsilon_0 > 0$ and the corresponding $N_0 \geq 1$. Since $\alpha = \sup A$, where A is the set of all partial limits of (x_n), we can find a partial limit a such that $\alpha - \varepsilon_0/3 < a \leq \alpha$. There is a subsequence $\{x_{n_\ell}\}$ converging to a, i.e., satisfying

$$\forall_{\ \varepsilon>0} \ \exists_{\ M\geq 1} \ \exists_{\ \ell\geq M} \ |x_{n_\ell} - a| < \varepsilon/3.$$

In particular, there exists an element x_{n_ℓ} with $n_\ell > N_0$ satisfying $|x_{n_\ell} - a| < \varepsilon_0/3$. Then,

$$|x_{n_\ell} - \alpha| < |x_{n_\ell} - a| + |a - \alpha| < \varepsilon_0/3 + \varepsilon_0/3 < \varepsilon_0,$$

which contradicts $(*)$.

(1) $\alpha = +\infty$. The idea of the proof is exactly the same but formally it looks different.

We want to show that

$$\forall_{\ K>0} \ \forall_{\ N\geq 1} \ \exists_{\ n\geq N} \ x_n > K.$$

Let us assume that this does not hold, i.e.,

$$(*) \quad \exists_{\ K>0} \ \exists_{\ N\geq 1} \ \forall_{\ n\geq N} \ x_n \leq K.$$

Let us fix such an $K_0 > 0$ and the corresponding $N_0 \geq 1$. Since $+\infty = \sup A$, where A is the set of all partial limits of (x_n), we can find a partial limit a (assume that it is finite, if not there is nothing to prove) such that $a > 3K_0$. There is a subsequence $\{x_{n_\ell}\}$ converging to a, i.e., satisfying

$$\forall_{\ \varepsilon>0} \ \exists_{\ M\geq 1} \ \exists_{\ \ell\geq M} \ |x_{n_\ell} - a| < \varepsilon.$$

In particular, for $\varepsilon = K_0$, there exists an element x_{n_ℓ} with $n_\ell > N_0$ satisfying $a - K_0 < x_{n_\ell}$. Then,

$$x_{n_\ell} > a - K_0 > 3K_0 - K_0 > K_0,$$

which contradicts $(*)$.

$\square$