CHAPTER 4

The Frobenius–Perron Operator

The hero of this book is the Frobenius–Perron operator. With this powerful tool we shall study absolutely continuous invariant measures, their existence and properties. This operator was first introduced by [Kuzmin, 1928ab] and describes the effect of the transformation τ on a probability density function.

4.1 Motivation

Let $\mathcal{X}$ be a random variable on the space $I = [a, b]$ having the probability density function f. Then, for any measurable set $A \subset I$,

$$\text{Prob}\{\mathcal{X} \in A\} = \int_A f d\lambda,$$

where λ is the normalized Lebesgue measure on I. Let $\tau : I \to I$ be a transformation. Then $\tau(\mathcal{X})$ is also a random variable and it is reasonable to ask: *What is the probability density function of* $\tau(\mathcal{X})$? We write

$$\begin{aligned}\text{Prob}\{\tau(\mathcal{X}) \in A\} &= \text{Prob}\{\mathcal{X} \in \tau^{-1}(A)\} \\ &= \int_{\tau^{-1}A} f d\lambda.\end{aligned}$$

To obtain a probability density function for $\tau(\mathcal{X})$, we have to write this last integral as

$$\int_A \phi d\lambda,$$

for some function ϕ. Obviously, if such a ϕ exists, it will depend both on f and on the transformation τ.

Let us assume that τ is non-singular and define

$$\mu(A) = \int_{\tau^{-1}A} f d\lambda,$$

where $f \in \mathfrak{L}^1$ and A is an arbitrary measurable set. Since τ is nonsingular, $\lambda(A) = 0$ implies $\lambda(\tau^{-1}A) = 0$, which in turn implies that $\mu(A) = 0$.

Hence $\mu << \lambda$. Then, by the Radon–Nikodym Theorem, there exists a $\phi \in \mathfrak{L}^1$ such that for all measurable sets A,

$$\mu(A) = \int_A \phi d\lambda.$$

ϕ is unique a.e., and depends on τ and f. Set $P_\tau f = \phi$. Thus, the probability density function f has been transformed to a new probability density function $P_\tau f$. P_τ obviously depends on the transformation τ (hence the subscript) and is an operator from the space of probability density functions on I into itself. P_τ is referred to as the *Frobenius–Perron operator associated with* τ. It is the major tool used in this book and will make it possible to prove the existence of absolutely continuous invariant measures and to establish many useful properties of these measures and their densities.

Since $f \in \mathfrak{L}^1, P_\tau f \in \mathfrak{L}^1$. Hence $P_\tau : \mathfrak{L}^1 \to \mathfrak{L}^1$ is a well-defined operator. If we let $A = [a, x] \subset I$, we have

$$\int_a^x P_\tau f d\lambda = \int_{\tau^{-1}[a,x]} f d\lambda.$$

On differentiating both sides with respect to x, we obtain

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}[a,x]} f d\lambda.$$

Example 4.1.1. Let $I = [0, 1]$ and let $\tau : I \to I$ be defined by

$$\tau(x) = \sin(\pi x), \qquad 0 \leq x \leq 1.$$

It is easy to see that

$$\tau^{-1}([0, x]) = [0, \frac{1}{\pi} \sin^{-1} x] \cup [1 - \frac{1}{\pi} \sin^{-1} x, 1], \quad 0 \leq x \leq 1.$$

Hence, for any $f \in \mathfrak{L}^1$,

$$\int_{\tau^{-1}[0,x]} f d\lambda = \int_0^{\frac{1}{\pi} \sin^{-1} x} f d\lambda + \int_{1-\frac{1}{\pi} \sin^{-1} x}^1 f d\lambda, \quad 0 \leq x \leq 1.$$

Using Leibniz's rule, we obtain

$$\begin{aligned} P_\tau f &= \frac{d}{dx} \int_{\tau^{-1}[0,x]} f d\lambda \\ &= \frac{1}{\pi\sqrt{1-x^2}} [f(\frac{1}{\pi} \sin^{-1} x) + f(1 - \frac{1}{\pi} \sin^{-1} x)]. \end{aligned}$$

Example 4.1.2. Let $I = [0,1]$ and let $\tau : I \to I$ be defined by

$$\tau(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ -\frac{4}{3}x + \frac{5}{3}, & \frac{1}{2} < x \le 1, \end{cases}$$

as shown in Figure 4.1.1.

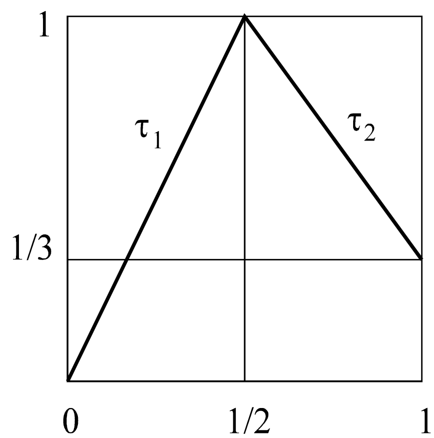


FIGURE 4.1.1

It is easy to see that

$$\tau^{-1}([0,x]) = [0, \frac{1}{2}x], \qquad \text{if } x < \frac{1}{3}$$

and

$$\tau^{-1}([0,x]) = [0, \frac{1}{2}x] \cup [\frac{5}{4} - \frac{3}{4}x, 1], \qquad \text{if } x \ge \frac{1}{3}.$$

Thus, we have

$$\tau^{-1}([0,x]) = [0, \frac{1}{2}x] \cup \{[\frac{5}{4} - \frac{3}{4}x, 1] \cap B\}, \qquad 0 \le x \le 1,$$

where $B = [\frac{1}{2}, 1]$. Hence, for any $f \in \mathfrak{L}^1$,

$$\int_{\tau^{-1}[0,x]} f d\lambda = \int_0^{\frac{x}{2}} f d\lambda + \int_{\frac{5}{4} - \frac{3}{4}x}^{1} f \chi_B d\lambda, \qquad 0 \le x \le 1.$$

Using Leibniz's rule, we obtain

$$P_\tau f(x) = \frac{1}{2} f(\frac{x}{2}) + \frac{3}{4} f(\frac{5}{4} - \frac{3}{4}x)\chi_J(x), \tag{4.1.1}$$

where $J = \tau(B) = [\frac{1}{3}, 1]$. In Section 4.3, we shall derive a formula for $P_\tau f$ which generalizes (4.1.1).

In Chapter 5 we shall use the Frobenius–Perron operator to establish the existence of absolutely continuous invariant measures for a large and important class of transformations. Now we present some properties of the operator.

4.2 Properties of the Frobenius–Perron Operator

In this section we define and present the basic properties of the Frobenius–Perron operator. We do it formally on an interval $I = [a, b]$, but all the ensuing results can be easily extended to a general measure space case.

Definition 4.2.1. Let $I = [a, b]$, $\mathfrak{B}$ be the Borel σ-algebra of subsets of I and let λ denote the normalized Lebesgue measure on I. Let $\tau : I \to I$ be a nonsingular transformation. We define the Frobenius–Perron operator $P_\tau : \mathfrak{L}^1 \to \mathfrak{L}^1$ as follows: For any $f \in \mathfrak{L}^1$, $P_\tau f$ is the unique (up to a.e. equivalence) function in $\mathfrak{L}^1$ such that

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda$$

for any $A \in \mathfrak{B}$.

The validity of this definition, i.e., the existence and the uniqueness of $P_\tau f$, follows by the Radon–Nikodym Theorem.

Proposition 4.2.1. *(Linearity)* $P_\tau : \mathfrak{L}^1 \to \mathfrak{L}^1$ *is a linear operator.*

Proof. Let $A \subset I$ be measurable and let α, β be constants. Then, if $f, g \in \mathfrak{L}^1$,

$$\begin{aligned}
\int_A P_\tau(\alpha f + \beta g) d\lambda &= \int_{\tau^{-1}A} (\alpha f + \beta g) d\lambda = \alpha \int_{\tau^{-1}A} f d\lambda + \beta \int_{\tau^{-1}A} g d\lambda \\
&= \alpha \int_A P_\tau f d\lambda + \beta \int_A P_\tau g d\lambda \\
&= \int_A (\alpha P_\tau f + \beta P_\tau g) d\lambda.
\end{aligned}$$

Since this is true for any measurable set A,

$$P_\tau(\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g \quad \text{a.e.}$$

$\square$

Proposition 4.2.2. *(Positivity)* *Let $f \in \mathfrak{L}^1$ and assume $f \geq 0$. Then $P_\tau f \geq 0$.*

Proof. For $A \in \mathfrak{B}$,

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}A} f d\lambda \geq 0.$$

Since $A \in \mathfrak{B}$ is arbitrary, $P_\tau f \geq 0$. $\square$

Proposition 4.2.3. *(Preservation of Integrals)*

$$\int_I P_\tau f d\lambda = \int_I f d\lambda.$$

Proof. Since

$$\int_I P_\tau f d\lambda = \int_{\tau^{-1}(I)} f d\lambda = \int_I f d\lambda,$$

the result follows. $\square$

Proposition 4.2.4. *(Contraction Property)*
$P_\tau : \mathfrak{L}^1 \to \mathfrak{L}^1$ is a contraction, i.e., $\|P_\tau f\|_1 \leq \|f\|_1$ for any $f \in \mathfrak{L}^1$.

Proof. Let $f \in \mathfrak{L}^1$. Let $f^+ = \max(f, 0)$ and $f^- = -\min(0, f)$. Then $f^+, f^- \in \mathfrak{L}^1$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. By the linearity of $P_\tau f$, we have

$$P_\tau f = P_\tau(f^+ - f^-) = P_\tau f^+ - P_\tau f^-.$$

Hence,

$$|P_\tau f| \leq |P_\tau f^+| + |P_\tau f^-| = P_\tau f^+ + P_\tau f^- = P_\tau |f|,$$

and

$$\|P_\tau f\|_1 = \int_I |P_\tau f| d\lambda \leq \int_I P_\tau |f| d\lambda = \int_I |f| d\lambda = \|f\|_1,$$

where we have used Proposition 4.2.3. $\square$

It follows from this result that $P_\tau : \mathfrak{L}^1 \to \mathfrak{L}^1$ is continuous with respect to the norm topology since

$$\|P_\tau f - P_\tau g\|_1 \leq \|f - g\|_1.$$

Proposition 4.2.5. *(Composition Property) Let $\tau : I \to I$ and $\sigma : I \to I$ be nonsingular. Then $P_{\tau o \sigma} f = P_\tau \circ P_\sigma f$. In particular, $P_{\tau^n} f = P_\tau^n f$.*

Proof. Since τ and σ are nonsingular their composition $\tau \circ \sigma$ is also nonsingular. Let $f \in \mathfrak{L}^1$ and $A \in \mathfrak{B}$:

$$\int_A P_{\tau o \sigma} f d\lambda = \int_{(\tau \circ \sigma)^{-1} A} f d\lambda = \int_{\sigma^{-1}(\tau^{-1} A)} f d\lambda$$

and

$$\int_A P_\tau (P_\sigma f) d\lambda = \int_{\tau^{-1} A} P_\sigma f d\lambda = \int_{\sigma^{-1}(\tau^{-1} A)} f d\lambda.$$

Hence $P_{\tau o \sigma} f = P_\tau P_\sigma f$ a.e. By induction, it follows that $P_{\tau^n} f = P_\tau^n f$ a.e. □

Recall that the *Koopman operator* $U_\tau : \mathfrak{L}^\infty \to \mathfrak{L}^\infty$ is defined by $U_\tau g = g \circ \tau$ and that for $f \in \mathfrak{L}^1, g \in \mathfrak{L}^\infty$, we denote $\int_I f g d\lambda$ by $\langle f, g \rangle$.

Proposition 4.2.6. *(Adjoint Property)*
If $f \in \mathfrak{L}^1$ and $g \in \mathfrak{L}^\infty$, then $\langle P_\tau f, g \rangle = \langle f, U_\tau g \rangle$, i.e.,

$$\int_I (P_\tau f) \cdot g d\lambda = \int_I f \cdot U_\tau g d\lambda. \tag{4.2.1}$$

Proof. Let A be a measurable subset of I and let $g = \chi_A$. Then the left hand side of (4.2.1) is

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1} A} f d\lambda$$

and the right hand side is

$$\int_I f \cdot (\chi_A \circ \tau) d\lambda = \int_I f \cdot \chi_{\tau^{-1} A} d\lambda = \int_{\tau^{-1} A} f d\lambda.$$

Hence (4.2.1) is verified for characteristic functions. Since the linear combinations of characteristic functions are dense in $\mathfrak{L}^\infty$, (4.2.1) holds for all $f \in \mathfrak{L}^1$. □

The following proposition says that a density f^* is a fixed point of P_τ if and only if it is the density of a τ-invariant measure μ, absolutely continuous with respect to a measure λ.

Proposition 4.2.7. *Let $\tau : I \to I$ be nonsingular. Then $P_\tau f^* = f^*$ a.e., if and only if the measure $\mu = f^* \cdot \lambda$, defined by $\mu(A) = \int_A f^* d\lambda$, is τ-invariant, i.e., if and only if $\mu(\tau^{-1}A) = \mu(A)$ for all measurable sets A, where $f^* \geq 0$, $f^* \in \mathfrak{L}^1$ and $\|f^*\|_1 = 1$.*

Proof. Assume $\mu(\tau^{-1}A) = \mu(A)$ for any measurable set A. Then

$$\int_{\tau^{-1}A} f^* d\lambda = \int_A f^* d\lambda$$

and therefore

$$\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda.$$

Since $A \in \mathfrak{B}$ is arbitrary, $P_\tau f^* = f^*$ a.e.

Assume $P_\tau f^* = f^*$ a.e. Then

$$\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda = \mu(A)$$

By definition,

$$\int_A P_\tau f^* d\lambda = \int_{\tau^{-1}A} f^* d\lambda = \mu(\tau^{-1}A)$$

and so $\mu(\tau^{-1}A) = \mu(A)$. □

Let $\mathfrak{D}(X, \mathfrak{B}, \mu)$ denote the probability density functions on the measure space $(X, \mathfrak{B}, \mu)$. When we wish to emphasize the underlying measure in the Frobenius–Perron operator, we shall write $P_{\tau,\mu}$. $P_{\tau,\mu}$ acts on $\mathfrak{D}(X, \mathfrak{B}, \mu)$, while $P_{\tau,\nu}$ acts on $\mathfrak{D}(X, \mathfrak{B}, \nu)$.

Suppose $\mu << \nu$ and $\nu << \mu$, i.e., μ is equivalent to ν. Then

$$\tau_* \mu << \tau_* \nu << \nu << \mu.$$

The following result presents a relation between $P_{\tau,\mu}$ and $P_{\tau,\nu}$.

Proposition 4.2.8. *Let μ be equivalent to ν. Then $\mu = f\nu$, where $f \in \mathfrak{D}(X, \mathfrak{B}, \nu)$, and for any $g \in \mathfrak{L}^1(X, \mathfrak{B}, \mu)$,*

$$P_{\tau,\mu} g = \frac{P_{\tau,\nu}(f \cdot g)}{f}. \tag{4.2.2}$$

Proof. For any $A \in \mathfrak{B}$,

$$\int_A (P_{\tau,\mu} g) d\mu = \int_{\tau^{-1}A} g d\mu,$$

and

$$\int_A \frac{P_{\tau,\nu}(fg)}{f} d\mu = \int_A P_{\tau,\nu}(fg) d\nu = \int_{\tau^{-1}A} fg d\nu = \int_{\tau^{-1}A} g d\mu.$$

Since $A \in \mathfrak{B}$ is arbitrary, the result is proved. □

Now we will prove some properties of $P_\tau = P_{\tau,\mu}$, where μ is a τ-invariant measure.

Proposition 4.2.9. *Let $\tau : I \to I$ and let μ be a τ-invariant measure. Then*

$$(P_\tau f) \circ \tau = E(f|\tau^{-1}(\mathfrak{B})) \text{ a.e.}$$

Proof. Since $(P_\tau f) \circ \tau$ is obviously $\tau^{-1}\mathfrak{B}$-measurable, it is enough to prove that $(P_\tau f) \circ \tau$ satisfies the condition of Theorem 2.4.1. Let $A = \tau^{-1}(B), B \in \mathfrak{B}$. Then

$$\begin{aligned}\int_A (P_\tau f) \circ \tau d\mu &= \int_{\tau^{-1}B} (P_\tau f) \circ \tau d\mu \\ &= \int_B P_\tau f d\mu = \int_{\tau^{-1}B} f d\mu = \int_A f d\mu.\end{aligned}$$

□

Corollary 4.2.1. *If $\tau : I \to I$ and μ is a τ-invariant measure, then P_τ is a contraction on any space $\mathfrak{L}^p$, $1 \le p \le +\infty$.*

Proof. Let $1 \le p < +\infty$. Then

$$\begin{aligned}(\|P_\tau f\|_p)^p &= \int_X |P_\tau f|^p d\mu = \int_X |(P_\tau f) \circ \tau|^p d\mu \\ &= \int_X |E(f|\tau^{-1}\mathfrak{B})|^p d\mu \\ &\le \int_X E(|f|^p|\tau^{-1}\mathfrak{B}) d\mu = \int_X |f|^p d\mu = (\|f\|_p)^p\end{aligned}$$

and $\|P_\tau f\|_p \le \|f\|_p$. Let $p = +\infty$. Then

$$\begin{aligned}\|P_\tau f\|_\infty &= \text{ess sup}\,|P_\tau f| = \text{ess sup}\,|(P_\tau f) \circ \tau| \\ &= \text{ess sup}\,|E(f|\tau^{-1}(\mathfrak{B}))| \le \text{ess sup}\,|f|.\end{aligned}$$

□

Proposition 4.2.10. *Let $\tau : I \to I$ and μ be τ-invariant measure. Let $\mathbf{1}$ denote the constant function equal to 1 everywhere. Then,*
(a) τ is ergodic $\Longleftrightarrow$ for any $f \in \mathfrak{D}(X, \mathfrak{B}, \mu)$

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k_{\tau,\mu} f \to \mathbf{1},$$

weakly in $\mathfrak{L}^1$ as $n \to +\infty$.
(b) τ is weakly mixing $\Longleftrightarrow$ for any $f \in \mathfrak{D}(X, \mathfrak{B}, \mu)$

$$\frac{1}{n}\sum_{k=0}^{n-1} |P^k_{\tau,\mu} f - \mathbf{1}| \to 0,$$

weakly in $\mathfrak{L}^1$ as $n \to +\infty$.
(c) τ is mixing $\Longleftrightarrow$ for any $f \in \mathfrak{D}(X, \mathfrak{B}, \mu)$

$$P^n_{\tau,\mu} f \to \mathbf{1},$$

weakly in $\mathfrak{L}^1$ as $n \to +\infty$.

Proof. All statements are direct consequences of properties (a), (b), and (c) of Theorem 3.4.2 and the Adjoint Property (Proposition 4.2.6). □

Proposition 4.2.11. *Let $\tau : I \to I$ and μ be a τ-invariant measure. Then τ is exact $\Longleftrightarrow$ for any $f \in \mathfrak{D}(X, \mathfrak{B}, \mu)$,*

$$P^n_{\tau,\mu} f \to \mathbf{1},$$

as $n \to \infty$ in the $\mathfrak{L}^1$-norm.

Proof. Assume τ is exact. The σ-algebras $\tau^{-n}(\mathfrak{B})$ form a decreasing sequence of σ-algebras. Since τ is exact, the σ-algebra $\mathfrak{B}^T = \bigcap_{n\geq 1} \tau^{-n}(\mathfrak{B})$ consists of sets of measure 0 or 1. By Proposition 4.2.5 and Proposition 4.2.9,

$$(P^n_\tau f \circ \tau^n) = (P_{\tau^n} f) \circ \tau^n = E(f|\tau^{-n}(\mathfrak{B})) \longrightarrow E(f|\mathfrak{B}^T)$$

in $\mathfrak{L}^1(X, \mathfrak{B}, \mu)$ as $n \to \infty$. Since $\mathfrak{B}^T$ consists of sets of μ-measure 0 or 1, $E(f|\mathfrak{B}^T) = \int_X f d\mu = 1$. Thus, we have

$$\int_X |(P^n_\tau f) \circ \tau^n - \mathbf{1}| d\mu \to 0 \quad \text{as} \quad n \to +\infty.$$

But

$$\int_X |(P_\tau^n f) \circ \tau^n - \mathbf{1}| d\mu = \int_X |(P_\tau^n f) \circ \tau^n - \mathbf{1} \circ \tau^n| d\mu$$
$$= \int_X |P_\tau^n f - \mathbf{1}| d\mu.$$

Thus $P_\tau^n f \to \mathbf{1}$ in $\mathfrak{L}^1$ as $n \to \infty$.

Now let $P_\tau^n f \to \mathbf{1}$ as $n \to \infty$ in $\mathfrak{L}^1$. Let $A \in \mathfrak{B}$ and assume $\mu(A) > 0$. We will show that as $n \to \infty$

$$\mu(\tau^n A) \to 1.$$

Let $f_A = \frac{1}{\mu(A)} \chi_A$. Then $\int_X f_A d\mu = 1$ and

$$v_n \equiv \|P_\tau^n f_A - \mathbf{1}\|_1 \to 0$$

as $n \to \infty$. We have

$$\begin{aligned}
\mu(\tau^n(A)) &= \int_{\tau^n(A)} \mathbf{1} d\mu \\
&= \int_{\tau^n(A)} P_\tau^n f_A d\mu - \int_{\tau^n(A)} (P_\tau^n f_A - \mathbf{1}) d\mu \\
&\geq \int_{\tau^n(A)} P_\tau^n f_A d\mu - v_n = \int_{\tau^{-n}(\tau^n A)} f_A d\mu - v_n \\
&\geq 1 - v_n.
\end{aligned}$$

Since $v_n \to 0$ as $n \to \infty$, we have the result. □

Recall that the transformation $\tau_* : \mathfrak{M}(I) \to \mathfrak{M}(I)$ is defined by $\tau_* \nu(A) = \nu(\tau^{-1} A)$. Let $\nu = f \cdot \mu$. Then

$$\begin{aligned}
\tau_*^n \nu(A) = \nu(\tau^{-n} A) &= \int_{\tau^{-n} A} f d\mu = \int_A P_{\tau^n} f d\mu \\
&= \int_I (P_{\tau^n} f) \chi_A d\mu = \int_I f \chi_A(\tau^n) d\mu. \qquad (4.2.3)
\end{aligned}$$

Let $\mathfrak{M}_1(I) \subset \mathfrak{M}(I)$ denote the space of probability measures.

Proposition 4.2.12. *Let $\tau : I \to I$ be strongly mixing on the normalized measure space $(I, \mathfrak{B}, \mu)$. Let $\nu \in \mathfrak{M}_1(I)$ be absolutely continuous with respect to μ. Then on any set $A \in \mathfrak{B}, \tau_*^n \nu \to \mu$ as $n \to +\infty$.*

Proof. Since $\nu << \mu$, there exists $f \in \mathfrak{D}(\mu)$ such that

$$\nu(A) = \int_I f d\mu.$$

Then by (4.2.3),

$$\tau_*^n \nu(A) = \int_I f \cdot \chi_A(\tau^n) d\mu.$$

Since τ is strongly mixing, we have

$$\int_I f \cdot \chi_A(\tau^n) d\mu \to \int_I f d\mu \int_I \chi_A d\mu = \mu(A)$$

as $n \to \infty$. □

We already know that P_τ is continuous with respect to the norm topology on $\mathfrak{L}^1(I, \mathfrak{B}, \lambda)$. The final property of P_τ establishes the fact that P_τ is also continuous in the weak topology of $\mathfrak{L}^1(I, \mathfrak{B}, \lambda)$.

Proposition 4.2.13. *Let $(I, \mathfrak{B}, \mu)$ be a normalized measure space and let $\tau : I \to I$ be nonsingular. Then $P_\tau : \mathfrak{L}^1 \to \mathfrak{L}^1$ is continuous in the weak topology on $\mathfrak{L}^1$.*

Proof. Let $f_n \to f$ weakly in $\mathfrak{L}^1$ as $n \to \infty$. We want to prove that $P_\tau f_n \to P_\tau f$ weakly in $\mathfrak{L}^1$ as $n \to \infty$, i.e., for all $g \in \mathfrak{L}^\infty$,

$$\int_I (P_\tau f_n) g d\mu \to \int_I (P_\tau f) g d\mu.$$

Now, by Proposition 4.2.6,

$$\int_I (P_\tau f_n) g d\mu = \int_I f_n (g \circ \tau) d\mu.$$

Since $g \circ \tau \in \mathfrak{L}^\infty$ and $f_n \to f$ weakly, we have

$$\int_I f_n (g \circ \tau) d\mu \to \int_I f (g \circ \tau) d\mu = \int_I (P_\tau f) g d\mu.$$

Thus,

$$\int_I (P_\tau f_n) g d\mu \to \int_I (P_\tau f) g d\mu$$

as $n \to \infty$, i.e., $P_\tau f_n \to P_\tau f$ weakly in $\mathfrak{L}^1$. □

4.3 Representation of the Frobenius–Perron Operator

In this section we derive an extremely useful representation for the Frobenius–Perron operator for a large class of one-dimensional transformations. These transformations, which are piecewise monotonic functions on an interval into itself, contain many of the transformations of interest in one-dimensional dynamical modeling and analysis.

Definition 4.3.1. Let $I = [a, b]$. The transformation $\tau : I \to I$ is called *piecewise monotonic* if there exists a partition of I, $a = a_0 < a_1 < \dots < a_q = b$, and a number $r \geq 1$ such that
(1) $\tau_{|(a_{i-1}, a_i)}$ is a C^r function, $i = 1, \dots, q$ which can be extended to a C^r function on $[a_{i-1}, a_i]$, $i = 1, \dots, q$, and
(2) $|\tau'(x)| > 0$ on (a_{i-1}, a_i), $i = 1, \dots, q$.

If, in addition to (2), $|\tau'(x)| \geq \alpha > 1$ wherever the derivative exists, then τ is called *piecewise monotonic* and *expanding*. Note that (2) implies that τ is monotonic on each (a_{i-1}, a_i). An example of such a transformation is shown in Figure 4.3.1.

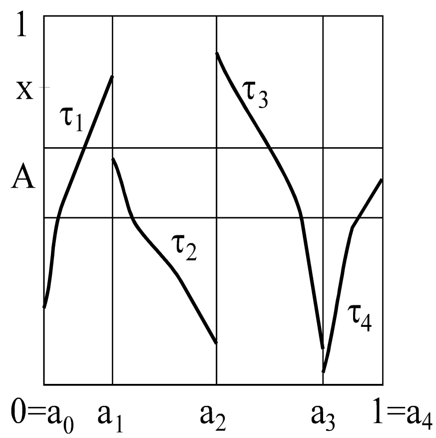


FIGURE 4.3.1

We now proceed to find P_τ for τ piecewise monotonic. By the definition of P_τ, we have

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}A} f d\lambda, \tag{4.3.1}$$

for any Borel set A in I.

Since τ is monotonic on each $(a_{i-1}, a_i), i = 1, \dots, q$, we can define an inverse function for each $\tau_{|(a_{i-1}, a_i)}$. Let

$$\phi_i = \tau^{-1}|_{B_i},$$

where $B_i = \tau([a_{i-1}, a_i])$. Then $\phi_i : B_i \to [a_{i-1}, a_i]$ and

$$\tau^{-1}(A) = \cup_{i=1}^q \phi_i(B_i \cap A), \tag{4.3.2}$$

where the sets $\{\phi_i(B_i \bigcap A)\}_{i=1}^q$ are mutually disjoint. See Figure 4.3.1. Note also that, depending on $A, \phi_i(B_i \bigcap A)$ may be empty. On substituting (4.3.2) into (4.3.1), we obtain

$$\begin{aligned}\int_A P_\tau f d\lambda &= \sum_{i=1}^q \int_{\phi_i(B_i \cap A)} f d\lambda \\ &= \sum_{i=1}^q \int_{B_i \cap A} f(\phi_i(x)) \,|\, {\phi_i}'(x) \,|\, d\lambda\,,\end{aligned}$$

where we have used the change of variable formula for each i. Now

$$\begin{aligned}\int_A P_\tau f d\lambda &= \sum_{i=1}^q \int_A f(\phi_i(x))|{\phi_i}'(x)|\chi_{B_i}(x) d\lambda \\ &= \int_A \sum_{i=1}^q \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}\chi_{\tau(a_{i-1},\, a_i)}(x) d\lambda.\end{aligned}$$

Since A is arbitrary,

$$P_\tau f(x) = \sum_{i=1}^q \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}\chi_{\tau(a_{i-1},\, a_i)}(x) \tag{4.3.3}$$

for any $f \in \mathfrak{L}^1$. There is a more compact way of writing (4.3.3):

$$P_\tau f(x) = \sum_{z \in \{\tau^{-1}(x)\}} \frac{f(z)}{|\tau'(z)|}. \tag{4.3.4}$$

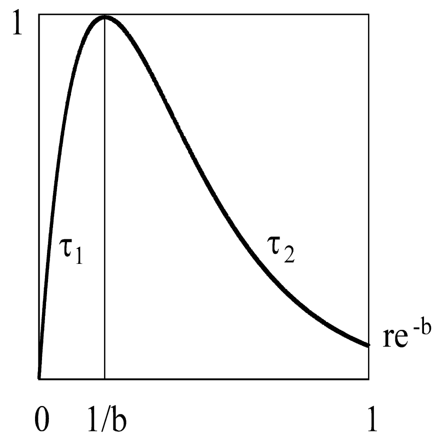
For any x, the set $\{\tau^{-1}(x)\}$ consists of at most q points; if z is one of these points, i.e., $z \in (a_{i-1}, a_i)$ for some i, the corresponding term $\dfrac{f(z)}{|\tau'(z)|}$ will appear on the right hand side of (4.3.4).

Remark 4.3.1. The operator P_τ is not 1-to-1. To see this, let us consider τ, the symmetric triangle transformation on $[0, 1]$. Let $f = 1$ on $[0, \frac{1}{2})$ and -1 on $[\frac{1}{2}, 1]$. Then, $P_\tau f = 0$ a.e. Thus P_τ is *not* a 1-to-1 operator.

Example 4.3.1. Let $\tau : [0, 1] \to [0, 1]$ be defined by $\tau(x) = rxe^{-bx}$, where r and b are such that τ is well-defined (i.e., $b > 1$ and $r \leq be$). The graph of such a τ is shown in Figure 4.3.2. Then,

$$P_\tau f(x) = \frac{f(\tau_1^{-1}(x))}{|\tau_1'(\tau_1^{-1}(x))|} + \frac{f(\tau_2^{-1}(x))}{|\tau_2'(\tau_2^{-1}(x))|}\chi_{[re^{-b},\, 1]}(x),$$

where τ_1 and τ_2 are the two monotonic components of τ.



b = 5 , r = 5e

FIGURE 4.3.2

Problems for Chapter 4

Problem 4.2.1. Let $\tau : [0,1] \to [0,1]$ be defined by $\tau(x) = rx(1-x)$, where $0 < r \leq 4$. Find $P_\tau f$.

Problem 4.2.2. Let $I = [0,1]$, $\mathfrak{B}$ = Borel σ-algebra on I and let λ be Lebesgue measure on I. Let $\tau : I \to I$ be defined by $\tau(x) = px$ mod 1, where p is a positive integer greater than or equal to 2. Find $P_\tau f$. Then, prove that τ is exact.

Problem 4.2.3. For τ and f as in Figures 4.4.1 and 4.4.2 respectively, find $P_\tau f$.

Problem 4.2.4. For

$$\tau(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ -x + \frac{3}{2}, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

find P_τ. Let

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ g(x), & \frac{1}{2} \leq x \leq 1, \end{cases}$$

where g is symmetric with respect to the line $x = \frac{3}{4}$. Show that $P_\tau f = f$.

Problem 4.2.5. Let I be an interval of the real line. Let τ_1 and τ_2 be measurable, nonsingular transformations from $I \to I$. Show that $P_{\tau_1 \circ \tau_2} = P_{\tau_1} \circ P_{\tau_2}$.

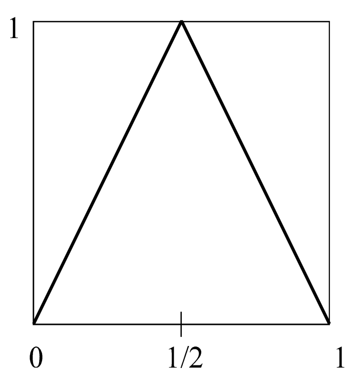


FIGURE 4.4.1

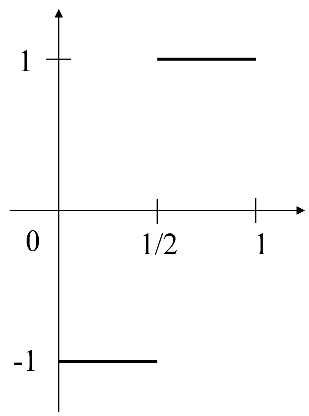


FIGURE 4.4.2

Problem 4.2.6. Let I be an interval of the real line and $\tau : I \to I$ be a measurable and nonsingular transformation. Show that

$$P_{\tau^n} = P_\tau^n,$$

where $\tau^n = \tau \circ \tau \circ \ldots \tau$.

Problem 4.2.7. Let $\tau_n \to \tau$ uniformly and let f_n be the invariant density associated with τ_n, i.e., $P_{\tau_n} f_n = f_n$. If $f_n \to f$ weakly in $\mathfrak{L}^1$, show that $P_\tau f = f$.

Problem 4.2.8. Let $f \in \mathfrak{L}^1$, $g \in \mathfrak{L}^\infty$ or the other way around. Prove that

$$P_\tau((f \circ \tau) \cdot g) = f \cdot P_\tau(g), a.e.$$

Problem 4.3.1. Let $\tau(x) = 4x(1-x)$ on $[0,1]$. Show that $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ is a fixed point of $P_\tau f$.

Problem 4.3.2. Let

$$\tau(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ -\frac{4}{3}x + \frac{5}{3}, & \frac{1}{2} \le x \le 1. \end{cases}$$

Find $V_0^1 P_\tau f$, where
(a) $f(x) = x^2$
(b) $f(x) = \sin x$.

Problem 4.3.3. Let

$$\tau(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} < x \le 1. \end{cases}$$

Let S consist of all functions f of the form $f = \alpha\chi_{[0,\frac{1}{2}]} + \beta\chi_{(\frac{1}{2},1]}$ where $\alpha, \beta \in [0,1]$. Let $f = (f_1, f_2)$ where $f_1 = \alpha\chi_{[0,\frac{1}{2}]}$ and $f_2 = \beta\chi_{(\frac{1}{2},1]}$. Show that $P_\tau f = (f_1, f_2)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Problem 4.3.4. Let $\tau : [0,1] \to [0,1]$ be defined by

$$\tau(x) = \tau_1(x)\chi_{[0,\frac{1}{4})}(x) + \tau_2(x)\chi_{[\frac{1}{4},\frac{1}{2})}(x) + \tau_3(x)\chi_{[\frac{1}{2},1]}(x),$$

where $\tau_1(x) = 4x$, $\tau_2(x) = \frac{3}{2} - 2x$, $\tau_3(x) = 2x - 1$. Let S denote the class of all functions $f : [0,1] \to [0,1]$, where $f = \alpha_1\chi_{[0,\frac{1}{4})} + \alpha_2\chi_{[\frac{1}{4},\frac{1}{2})} + \alpha_3\chi_{[\frac{1}{2},1]}$ and $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$. For any $f : [0,1] \to [0,1]$, let $f \equiv (f_1, f_2, f_3)$ where $f_1 = f\chi_{[0,\frac{1}{4})}$, $f_2 = f\chi_{[\frac{1}{4},\frac{1}{2})}$, and $f_3 = f\chi_{[\frac{1}{2},1]}$. Show that for $f \in S$,

$$P_\tau f = (f_1, f_2, f_3)\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Find a fixed point of P_τ.

Problem 4.3.5. Let $\tau : [0,1] \to [0,1]$ be defined by

$$\tau(x) = \begin{cases} -2x + 1, & x \in [0, \frac{1}{2}], \\ 2x - 1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Find P_τ.

Problem 4.3.6. $\tau_3 : [0,1] \to [0,1]$ be defined by $\tau_3 = \tau_2 \circ \tau_1$, where

$$\tau_1(x) = rx(1-x), \quad 0 \le r \le 4$$

and

$$\tau_2(x) = \begin{cases} -2x + 1, & x \in [0, \frac{1}{2}], \\ 2x - 1, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Find P_{τ^3}.

Problem 4.3.7. Let $\tau : [0,1] \to [0,1]$ be defined by $\tau(x) = rxe^{-bx}$, where $r > 0$, $b > 0$. Find P_τ.

Problem 4.3.8. Show that
(a) $f(x) = 1$ is an invariant density for

$$\tau(x) = \begin{cases} \dfrac{x}{\alpha}, & 0 \le x \le \alpha \\ \dfrac{1-x}{1-\alpha}, & \alpha \le x \le 1; \end{cases}$$

(b)

$$f(x) = \begin{cases} 0, & 0 \le x < \dfrac{1}{4} \\ 1, & \dfrac{1}{4} \le x < \dfrac{1}{2} \\ \dfrac{3}{2}, & \dfrac{1}{2} \le x \le 1 \end{cases}$$

is an invariant density for

$$\tau(x) = \begin{cases} 2x, & 0 \le x \le \dfrac{1}{2} \\ 1 + \dfrac{3}{2}(\dfrac{1}{2} - x), & \dfrac{1}{2} \le x \le 1; \end{cases}$$

(c) $f(x) = \frac{4}{\pi} \cdot \frac{1}{1+x^2}$ is an invariant density for

$$\tau(x) = \begin{cases} \dfrac{2x}{1-x^2}, & 0 \le x \le \sqrt{2} - 1 \\ \dfrac{1-x^2}{2x}, & \sqrt{2} - 1 \le x \le 1; \end{cases}$$

(d) $f(x) = 12(x - \frac{1}{2})^2$ is an invariant density for $\tau : [0, 1] \to [0, 1]$ given by

$$\tau(x) = (\frac{1}{8} - 2|x - \frac{1}{2}|^3)^{\frac{1}{3}} + \frac{1}{2};$$

(e) $f(x) = \frac{2x}{(1+x)^2}$ is an invariant density for

$$\tau(x) = \begin{cases} \dfrac{2x}{1-x}, & 0 \le x \le \dfrac{1}{3} \\ \dfrac{1-x}{2x}, & \dfrac{1}{3} \le x \le 1; \end{cases}$$

(f) $f(x) = px^{p-1}$ is an invariant density for

$$\tau(x) = \begin{cases} \dfrac{x}{(\frac{1}{2})^{\frac{1}{p}}}, & 0 \le x \le (\dfrac{1}{2})^{\frac{1}{p}} \\ \dfrac{(1-x^p)^{\frac{1}{p}}}{(\frac{1}{2})^{\frac{1}{p}}}, & (\dfrac{1}{2})^{\frac{1}{p}} \le x \le 1; \end{cases}$$

(g) $f(x) = \frac{1}{1-\cos x}$ is invariant under

$$\tau(x) = 2\arctan(\frac{1}{2}\tan x)\ ,\ -\pi \le x \le \pi.$$

Problem 4.3.9. (Difficult) Let $\tau : [0,1] \to [0,1]$ be given by

$$\tau(x) = \begin{cases} \frac{\alpha x}{\alpha p + (\alpha - p)x}, & 0 \le x \le \alpha \\ \frac{q(1-\alpha) - q(1-\alpha)x}{q - q\alpha - \alpha + (1 - q + q\alpha)x}, & \alpha \le x \le 1, \end{cases}$$

where $0 < p \le 1$ and $q > 0$ are real numbers. Let

$$S_1(x) = \frac{\alpha - p + \alpha x}{\alpha p}$$

and

$$S_2(x) = \frac{1 - q + q\alpha - q(1-\alpha)x}{q - q\alpha - \alpha + q(1-\alpha)x}.$$

Let β, γ, δ be defined by the equations $S_1(\delta) = \delta$, $S_2(\gamma) = \delta$, $S_1(\beta) = \gamma$, respectively. Then verify that an invariant density of τ is given as follows:

Case (a): $\gamma \neq \delta$.

$$f(x) = \left| \frac{1}{x + \frac{1}{\gamma}} - \frac{1}{x + \frac{1}{\delta}} \right|;$$

Case (b): $\beta = \gamma = \delta \neq 0$.

$$f(x) = \frac{1}{(x + \frac{1}{\beta})^2};$$

Case (c): $\beta = \gamma = \delta = 0$.

$$f(x) \equiv 1.$$

Problem 4.3.10. Consider a family of transformations

$$\tau_a(x) = \begin{cases} \frac{ax}{1-(2-a)x}, & \text{for } 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \text{for } \frac{1}{2} < x \le 1, \end{cases}$$

$a \ge 1$. Show that the τ_a's satisfy the assumptions of Problem 4.3.9 ($\alpha = \frac{1}{2}$, $p = \frac{1}{a}$, $q = 2$). Find P_{τ_a}-invariant functions f_a for $a > 1$ and show that $f_1 = \lim_{a \to 1^+} f_a$ (pointwise) is a P_{τ_1}-invariant nonintegrable function.

Problem 4.3.11. Let $\tau : [0,1] \to [0,1]$ be nonsingular and let $h : [0,1] \to [0,1]$ be a diffeomorphism. Prove
(a) $P_\tau f = f$ implies $P_\sigma g = g$, where $\sigma = h \circ \tau \circ h^{-1}$ and

$$g = (f \circ h^{-1}) \cdot |(h^{-1})'|;$$

(b) if f is a τ-invariant density, then g is a σ-invariant density.

Problem 4.3.12. Let $\sigma : [0,1] \to [0,1]$ be defined by $\sigma = h \circ \tau \circ h^{-1}$, where $h(x) = \sqrt{x}$ and τ is as in Problem 4.3.4. Find the σ-invariant density g.

Problem 4.3.13. Let $\tau : [0,1] \to [0,1]$ be piecewise monotonic with respect to the partition $\mathcal{P}(\tau) \equiv \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, where $\tau(x) = \sin \pi x$. Find the partition $\mathcal{P}(\tau^3)$ with respect to which τ^3 is piecewise monotonic.

Problem 4.3.14. Let $\tau : [0,1] \to [0,1]$ be defined by

$$\tau(x) = 4x(1-x).$$

Suppose μ is a Borel measure on $[0,1]$ defined by $d\mu = \frac{d\lambda}{2\sqrt{x(1-x)}}$, where λ is Lebesgue measure on $[0,1]$. Find $P_{\tau,\mu}$.

Problem 4.3.15. Let τ be the tent transformation (defined in Problem 4.3.3 and shown in Figure 4.4.1). Let $\sigma(y) = 4 \cdot y \cdot (1-y)$, $y \in [0,1]$. Use the diffeomorphism $h(x) = \sin^2(\frac{\pi}{2} \cdot x)$ and Problem 4.3.11 to show that the density $g(y) = \frac{1}{\pi\sqrt{y(1-y)}}$ is σ-invariant.

Problem 4.3.16. Prove that linear homeomorphism $h(x) = a \cdot x + b$ conjugates $\tau(x) = \alpha x^2 + \beta x + \gamma$ to $\sigma(x) = x^2 + c$, $h(\tau) = \sigma(h)$, if

$$a = \alpha, \qquad b = \frac{\beta}{2}, \qquad c = -\alpha\gamma + \frac{\beta}{2}(1 - \frac{\beta}{2}).$$

In particular, $\tau(x) = 4x(1-x)$ on $[0,1]$ is conjugated to $\sigma(x) = x^2 - 2$ on $[-2, 2]$. Find the density of the σ-invariant absolutely continuous invariant measure.

Problem 4.3.17. a) The transformation $\tau : \mathbb{R} \to \mathbb{R}$, given by $\tau(x) = x - \frac{1}{x}$, is called Boole's transformation. Show that τ preserves Lebesgue measure.

b) A generalized Boole transformation $\tau : \mathbb{R} \to \mathbb{R}$ is given by

$$\tau(x) = \pm(x + a_0 + \sum_{i=1}^{n} \frac{b_i}{x - a_i}),$$

where $n \in \mathbb{N} \cup \{0\}$, $a_i \in \mathbb{R}$, for $i = 0, 1, \dots, n$ and $b_i < 0$, for $i = 1, \dots, n$. Show that τ preserves Lebesgue measure.

Problem 4.3.18. a) Let $\tau : \mathbb{R} \to \mathbb{R}$ be given by

$$\tau(x) = a \tan(x), \quad x \neq k\frac{\pi}{2},$$

where $k = \pm 1, \pm 3, \dots$ and $a > 1$. Show that the density function $f(x) = \frac{p/\pi}{p^2+x^2}$ is τ-invariant for $p > 0$ satisfying equation $a \cdot \tanh(p) = p$.

b) More generally, let

$$\tau(x) = a(\tan(bx) + c), \quad x \neq \frac{k}{b} \cdot \frac{\pi}{2},$$

$k = \pm 1, \pm 3, \dots$. Find the ranges of a, b, c for which we can find $p, q \in \mathbb{R}$ such that the density $f(x) = \frac{p/\pi}{p^2+(x-q)^2}$ is τ-invariant.

c) Let

$$\tau(x) = \tan x, \quad x \neq k\frac{\pi}{2}, \quad k = \pm 1, \pm 3, \dots.$$

Show that the density $f(x) = \frac{1}{x^2}$ is τ-invariant. Note that $\tan(0) = 0$ and $\tan'(0) = 1$. Thus, 0 is a fixed point of τ at which the slope of τ is equal to 1. Such a fixed point is called indifferent.

τ is an example of a transformation with an indifferent fixed point. Such transformations have infinite σ-finite invariant measures ([Thaler, 1980]). The transformation of case c) has σ-finite absolutely continuous invariant measure $\mu = \dfrac{1}{x^2}\lambda$.

Hint: Use the formula

$$\sum_{k=-\infty}^{+\infty} \frac{1}{s^2 + (t + k\pi)^2} = \frac{\tanh(s)(1 + \tan^2(t))}{s(\tan^2(t) + \tanh^2(s))},$$

$s, t \in \mathbb{R}$, or its special form ($s = 0$)

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(t + k\pi)^2} = \frac{1}{\sin^2(t)} = \frac{1 + \tan^2(t)}{\tan^2(t)}.$$

Problem 4.3.19. Let

$$\tau(x) = \begin{cases} \frac{x}{1-x}, & 0 \le x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

Show that $f(x) = \frac{1}{x}$ is a fixed point of P_τ.

Problem 4.3.20. Let τ be the Gauss transformation defined in Problem 3.1.7.

a) Show that $f(x) = \frac{1}{1+x}$ is a fixed point of P_τ.

b) Let $n \geq 2$ and define

$$\tau^{(n)}(x) = \begin{cases} n \cdot x, & \text{for } 0 \leq x \leq \frac{1}{n} \\ \tau(x), & \text{for } \frac{1}{n} < x \leq 1. \end{cases}$$

Show that $\tau^{(n)}$ preserves the same density f.

Problem 4.3.21. Show that the transformation τ with countably many branches, defined by $\tau(x) = \{\frac{x}{1-x}\}$, where $\{t\}$ denotes the fractional part of t, preserves the density $f(x) = \frac{1}{x}$. The transformation τ is called the "backward continued fraction transformation" and has interesting connections with geodesic flow on hyperbolic plane (see [Adler and Flatto, 1984]).

Problem 4.3.22. Let $d \geq 2$ be an integer or $+\infty$ and let $f : [0, d) \to [0, 1)$ be an increasing function with $\lim_{x \to d} f(x) = 1$. We define $\tau(x) = f^{-1}(x) \pmod 1$. τ is a piecewise monotonic transformation with d branches.

a) Show that the equation for the τ-invariant density h is given by

$$h(x) = \sum_{k=0}^{d-1} h(f(x+k)) f'(x+k). \tag{1}$$

b) Let $f_1(x) = \frac{x}{1+x}$, $x > 0$. Show that $h_1(x) = \frac{1}{x}$.

c) Let $\alpha > 0$, $\alpha \neq 1$ and $f_\alpha(x) = (x^{1-\alpha} - (1+x)^{1-\alpha} + 1)^{\frac{1}{1-\alpha}}$, $x > 0$. Show that the τ_α-invariant density is $h_\alpha = \frac{1}{x^\alpha}$.

Problem 4.3.23. (See [Lasota and Yorke, 1982].) Let τ be a μ-nonsingular transformation and let P_τ be the Frobenius–Perron operator induced by τ. A function $h \in \mathfrak{L}^1(X, \mathfrak{B}, \mu)$ is called a lower function for P_τ if $h \geq 0$, $\int_X h d\mu > 0$, and

$$\lim_{n \to +\infty} \|(h - P_\tau^n f)^+\|_1 = 0$$

for all $f \in \mathfrak{D}(X, \mathfrak{B}, \mu)$.

a) Prove that if h is a lower function, then $P_\tau h$ is also a lower function.

b) Prove that if h_1, h_2 are lower functions, then $\max(h_1, h_2)$ is also a lower function.

c) Prove that if $h_n, \; n = 1, 2, \ldots$ are lower functions and

$$h_n \longrightarrow h, \text{ in } \mathfrak{L}^1(X, \mathfrak{B}, \mu)$$

as $\to +\infty$, then h is also a lower function.

d) Prove that if P_τ has a lower function, then P_τ has a lower function h satisfying $P_\tau h = h$.

e) Prove that if h is a lower function and $P_\tau h = h$, then $h_1 = (2 - \|h\|_1)h$ is also a lower function.

f) Prove that if P_τ has a lower function, then τ has an invariant density g such that

$$P_\tau^n f \xrightarrow{\mathfrak{L}^1} g,$$

for any $f \in \mathfrak{D}(X, \mathfrak{B}, \mu)$, i.e., the dynamical system $(X, \mathfrak{B}, \tau, g \cdot \mu)$ is exact.