Generic analysis of Borel homomorphisms for the finite Friedman-Stanley jumps

Assaf Shani

Concordia University

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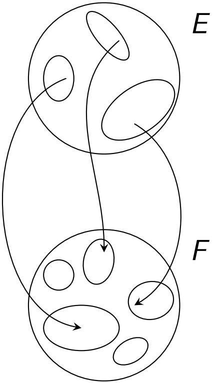
Borel homomorphisms and reductions

An equivalence relation E on a Polish space X is **analytic (Borel)** if $E \subseteq X \times X$ is analytic (Borel).



Definition

Let E and F be equivalence relations on Polish spaces X and Y respectively. $f \colon X \to Y$ a Borel map.

- f is a **Borel homomorphism**, $f \colon E \to_B F$, if $x \, E \, x' \implies f(x) \, F \, f(x')$.
- f is a **Borel reduction** of E to F if $x \, E \, x' \iff f(x) \, F \, f(x')$.
- **E is Borel reducible to F**, denoted $E \leq_B F$, if there is a Borel reduction of E to F.
- E, F are **Borel bireducible** ($E \sim_B F$) if $E \leq_B F$ & $F \leq_B E$.

Some motivations:

- "Borel definable" cardinality for definable quotient spaces.
- Possible complete invariants for classification problems.

Friedman-Stanley jump / countable powerset operation

Definition

Let E be an equivalence relation on a Polish space X.
Define E^+ on the Polish space $X^{\mathbb{N}}$ by

$$x \mathrel{E^+} y \iff \forall n \exists m (x(n) \mathrel{E} y(m)) \;\&\; \forall n \exists m (y(n) \mathrel{E} x(m)),$$

that is, $\{[x(n)]_E ; n \in \mathbb{N}\} = \{[y(n)]_E ; n \in \mathbb{N}\}$.

- The countable powerset operation $\mathcal{P}_{\aleph_0}(-)$, for the quotient X/E, coded on a Polish space.
- Classifiability using hereditarily countable invariants.
  - E is **concretely classifiable** if $E \leq_B =_{\mathbb{R}}$, equality ER on $\mathbb{R}$. (Numerical invariants.)
  - E is classifiable using countable sets of reals as invariants if $E \leq_B =_{\mathbb{R}}^{+}$.
  - Countable sets of countable sets of reals as invariants: $E \leq_B =_{\mathbb{R}}^{++}$.
  - ...

Classification by countable structures

Definition

E is **classifiable by countable structures** if it is Borel reducible to the isomorphism relation for some class of countable objects. E.g.: countable graphs, countable groups ...

- Equivalently: if E is Borel reducible to an orbit equivalence relation induced by S_∞ (or a closed subgroup of S_∞: non-Archimedean groups).

Fact

E a Borel equivalence relation. The following are equivalent.

- *E is classifiable by countable structures;*
- *E is Borel reducible to $=_{\mathbb{R}}^{+\alpha}$ for a countable ordinal α.*

Motivation

Very general goal:

Given equivalence relation E and F, is $E \leq_B F$?

Today's goal:

For $n \leq \omega$, develop methods to prove that $=_{\mathbb{R}}^{+n} \leq_B E$ for some E.

Remark:

For $=_{\mathbb{R}}^{+}$, the situation is well understood. Some examples:

- Foreman - Louveau 1995: $=_{\mathbb{R}}^{+}$ is Borel bireducible with the classification problem of ergodic discrete spectrum measure preserving transformations.
- Marker 2007: Let T be a complete first order theory whose space of types is uncountable. Then $=_{\mathbb{R}}^{+} \leq_B \cong_T$.

Generic dichotomy for Borel homomorphisms

Theorem (Kanovei-Sabok-Zapletal 2013)

Let E be an analytic equivalence relation. Then either

- $=^+_{\mathbb{R}}$ is Borel reducible to E, or
- any Borel homomorphism from $=^+_{\mathbb{R}}$ to E maps a comeager subset of $\mathbb{R}^{\mathbb{N}}$ into a single E-class.

Theorem (Marker 2007)

T first order theory, uncountable type space. Then $=^+_{\mathbb{R}} \leq_B \cong_T$.

- Fix a perfect set of types C, identified with $\mathbb{R}$.
- Naive idea: map a countable set of reals $A \subseteq C$ to a model M satisfying "precisely" A.
- Can be done if A is a Scott set: sufficiently closed under some countably many operations.
- Improved idea: $A \mapsto \mathrm{closure}(A) \mapsto M$.
- This gives a Borel homomorphism, not trivial on comeager sets. Therefore $=^+_{\mathbb{R}} \leq_B \simeq_T$.

Some difficulties in generalizing for $n \geq 2$

Kanovei-Sabok-Zapletal 2013: E analytic ER. Then either

- $=^+_{\mathbb{R}}$ is Borel reducible to E, or
- any $f : =^+_{\mathbb{R}} \to_B E$ maps a comeager set into a single E-class.

Already for $=^{++}_{\mathbb{R}}$:

- On a comeager subset $C \subseteq (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$, $(=^{++}_{\mathbb{R}} \restriction C) \leq_B =^+_{\mathbb{R}}$.
  $C =$ all $x \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ s.t. $(n, m) \neq (l, k) \implies x(n, m) \neq x(l, k)$.
- There **is** an "interesting" Borel homomorphism $=^{++}_{\mathbb{R}} \to_B =^+_{\mathbb{R}}$:
  $$(x_{i,j} \mid i, j \in \mathbb{N}) \mapsto (x_{<i,j>} \mid i, j \in \mathbb{N}).$$

More generally:

- For $n \geq 2$, need a different presentation / topology.
- Need to consider the homomorphisms $=^{+n}_{\mathbb{R}} \to_B =^{+k}_{\mathbb{R}}$, $k < n$, essentially taking a hereditarily countable set of rank n to the set of its rank k elements.

Main result

Theorem (S.)

There are equivalence relations F_n on Polish spaces X_n, s.t.

1. $F_n \sim_B =_{\mathbb{R}}^{+n}$, $n = 1, 2, 3, \ldots, \omega$, and

there are Borel homomorphism $u_k^n \colon F_n \to_B F_k$, $k < n \leq \omega$, s.t.

2. Let E be classifiable by countable structures. Then either
   - F_n is Borel reducible to E, or
   - every Borel homomorphism $f \colon F_n \to_B E$ factors through u_k^n on a comeager set, for $k < n$. (That is, there is a homomorphism $h \colon F_k \to_B E$ so that $(h \circ u) \mathrel{E} f$ on a comeager set.)

To prove $=_{\mathbb{R}}^{+n} \leq_B E$, enough to find a “non-trivial” homomorphism.

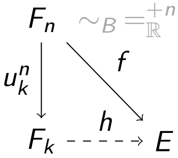


Figure: $(\forall f \colon F_n \to_B E)(\exists k < n \, \exists h \colon F_k \to_B E)$

Definition of F_n and u^n_m

- $X_n =\subseteq ((2^{\mathbb{N}})^{\mathbb{N}})^n$, for $n = 1, 2, 3, \dots, \omega$. Fix $x \in X_n$.
- $A^x_1 = \{x(0)(k);\ k \in \mathbb{N}\} \subseteq 2^{\mathbb{N}}$.

$$a^{x,l}_1 = \{x(0)(k);\ x(1)(l)(k) = 1\} \subseteq A^x_1$$

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
* & 1 & 0 & 1 & \dots \\
* & 1 & 1 & 0 & \dots \\
* & 0 & 1 & 1 & \dots \\
* & 0 & 1 & 0 & \dots \\
x(0) & x(1)(0) & x(1)(1) & x(1)(2) & \\
(2^{\mathbb{N}})^{\mathbb{N}} & 2^{\mathbb{N}} & 2^{\mathbb{N}} & 2^{\mathbb{N}} &
\end{array}
\quad \mapsto \quad
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
* & - & * & \dots \\
* & * & - & \dots \\
- & * & * & \dots \\
- & * & - & \dots \\
a^{x,0}_1 & a^{x,1}_1 & a^{x,2}_1 &
\end{array}$$

- $A^x_2 = \left\{a^{x,l}_1;\ l \in \mathbb{N}\right\};\ a^{x,l}_2 = \left\{a^{x,k}_1;\ x(2)(l)(k) = 1\right\} \subseteq A^x_2 \dots$

$$\mathbf{x\ F_n\ y} \iff \mathbf{A^x_n = A^y_n}$$

- $u^n_m \colon X_n \to X_m$, for $m < n$, projection.

An application to a question of Clemens

The following answers positively a question of Clemens.

Corollary (S.)

Suppose $E <_B =_{\mathbb{R}}^{+\omega}$. Then for any Borel homomorphism $f\colon =_{\mathbb{R}}^{+\omega} \to_B E$, $=_{\mathbb{R}}^{+\omega}$ retains its complexity on a fiber, that is, there is y in the domain of E so that $=_{\mathbb{R}}^{+\omega} \sim_B =_{\mathbb{R}}^{+\omega} \restriction \{x; f(x) \mathrel{E} y\}$.
(That is, $=^{+\omega}$ is **regular**.)

- Can replace $=^{+\omega}$ with F_ω.
- By the main theorem, any $f\colon F_\omega \to_B E$ factors through u_k^ω for some k, on a comeager set.
- From the definitions, F_ω is equivalent to its restriction to any fiber of u_k^ω.
- It remains to show that F_ω retains its complexity on comeager sets: $F_\omega \leq_B F_\omega \restriction C$ for any comeager C.

Spectrum of the meager ideal

Corollary (S.)

For any $n \leq \omega$, F_n retains its complexity on comeager sets: $F_n \leq_B F_n \restriction C$ for any comeager set C.

In particular, $=_{\mathbb{R}}^{+n}$ is in the **spectrum of the meager ideal**. This was proved by Kanovei, Sabok, and Zapletal for $n = 1$. *For $n > 1$, the different presentation F_n is necessary.*

- Fix a comeager set C (assume it is F_n-invariant). Fix $f \colon F_n \to_B F_n \restriction C$ which is the identity on C.
- From the definitions, u_k^n is not a reduction on any comeager set, for $k < n$.
- So f does not factor through u_k^n, for $k < n$.
- By the main theorem, $F_n \leq_B F_n \restriction C$.

What else is good about F_n? Group action

$S_\infty = \mathrm{Sym}(\mathbb{N})$, $S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \rightsquigarrow =^+_{\mathbb{R}}$ (on a large set).
Consider the action $S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}}$.

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
 & * & 1 & 0 & 1 & \dots \\
 & * & 1 & 1 & 0 & \dots \\
 & * & 0 & 1 & 1 & \dots \\
 & * & 0 & 1 & 0 & \dots \\
S_\infty & \curvearrowright & \curvearrowright & \curvearrowright & \curvearrowright &
\end{array}$$

F_2 is induced (on a large set) by the action

$$S_\infty \times S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$$

Similarly: F_n is induced by a natural action of $(S_\infty)^n$ on $((2^{\mathbb{N}})^{\mathbb{N}})^n$.
In contrast, $=^{++}_{\mathbb{R}}$ is naturally induced by an action of

$$S_\infty \ltimes (S_\infty)^{\mathbb{N}} \text{ on } (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$$

What else is good about F_n? Borel complexity

Note: $=^{+}$ is $\mathbf{\Pi}^0_3$; $=^{++}$ is $\mathbf{\Pi}^0_5$; $=^{+++}$ is $\mathbf{\Pi}^0_7$.

Theorem (Hjorth-Kechris-Louveau 1998)

$=^{+n}$ is *potentially* $\mathbf{\Pi}^0_{2+n}$: it is Borel reducible to a $\mathbf{\Pi}^0_{2+n}$ ER.
In fact it is maximal potentially $\mathbf{\Pi}^0_{2+n}$ for S_∞-actions.

Note:

F_n is $\mathbf{\Pi}^0_{2+n}$.
e.g., F_2 is $\mathbf{\Pi}^0_4$. Main point: given x, y, we want

$$\forall n \exists m (\forall i, j \, [\, x(0)(i) = y(0)(j) \rightarrow x(1)(n)(i) = y(1)(m)(j) \,])$$

*	1	1	0	*	0	1	1
*	0	1	1	*	0	0	0
*	0	1	0	*	1	1	0
$x(0)$	$x(1)(0)$	$x(1)(1)$	$x(1)(2)$	$y(0)$	$y(1)(0)$	$y(1)(1)$	$y(1)(2)$

New ideas going into the proof

1. To build a reduction from F_n to some equivalence relation, need a construction which is invariant under the action

$$S_\infty \times S_\infty \curvearrowright (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$$

2. For this action, need to understand Vaught transforms:

$$\{x; \text{ for a comeager set of } g \in S_\infty \times S_\infty,\ g \cdot x \in C\},$$

for comeager sets C.

3. Lemma: given a Borel homomorphism f from F_n to E, E classifiable by countable structures, there are:

$$\begin{array}{ccccccccc} F_1 & \leftarrow & F_2 & \leftarrow & F_3 & \leftarrow & \dots & \leftarrow & F_n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ E_1 & \leftarrow & E_2 & \leftarrow & E_3 & \leftarrow & \dots & \leftarrow & E \end{array}$$

which commute on comeager sets.

Some questions

- Find a model-theoretic condition on a first order theory T so that $=_{\mathbb{R}}^{+n} \leq_B \simeq_T$. (Done by Marker 2007 for $=_{\mathbb{R}}^{+}$.)
- Extend to transfinite jumps, $=_{\mathbb{R}}^{+\alpha}$ for a countable ordinal α.
- Find an equivalent jump operation which also gives a good topology.
- Prove generic dichotomies for homomorphisms for other equivalence relations.
- Prove a measure theoretic version of the dichotomy for $=_{\mathbb{R}}^{+n}$. (Done by Kanovei-Sabok-Zapletal 2013 for $=_{\mathbb{R}}^{+}$.)
- Find a set theoretic condition for $=_{\mathbb{R}}^{+n} \leq_B E$, e.g. based on properties of E in $L(\mathbb{R})$ or the Solovay model. (Done by Larson-Zapletal 2020 for $=_{\mathbb{R}}^{+}$.)