145B SET THEORY II: AN INTRODUCTION TO FORCING

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1. Introduction

We assume the axioms ZF, and we call the universe of all sets V. (The axiom of foundation allows us to present V as the union of V_α for all ordinals α.) We can then define Godel's model L of all constructible sets, and prove that L satisfies the axiom of choice and the continuum hypothesis.

In conclusion, the consistency of the axioms ZF implies the consistency of the axioms ZFC+CH. In other words, using ZF, we *cannot* refute the axiom of choice, nor the continuum hypothesis!

But can we prove them? The answer is no. In the 60's, Cohen introduced the method of forcing and used it to show that the axiom of choice *cannot* be proven using ZF, and the continuum hypothesis cannot be proven using ZFC.

Combining Godel's and Cohen's results, we say that the axiom of choice is *independent* of ZF, and that CH is independent of ZFC. Proving this will be one of our main goals.

1.1. Inner models.

How can we prove this? At first, we started with an arbitrary model V of ZF and constructed an *inner model* L which satisfies ZFC+CH. Can we repeat this idea? Perhaps start with an arbitrary model V of ZFC, and construct some inner model M in which the continuum hypothesis fails.

This approach cannot work. What if the model we started with satisfies V=L. Then any class inner model of it will be everything. For example, maybe we can prove, just using ZF, that V=L, that is, that every set is constructible. Using this inner model approach, we cannot even rule out this option.

1.2. Outer models?

Let us take this "worst-case scenario" that V=L. Assume further that there is a countable ordinal α such that L_α is itself a model of ZFC (this is not unreasonable to ask for). L_α is a small, countable, set, and there are many sets outside of it. We hope, perhaps, to create a different model of set theory by *adding* to L_α some sets. For example, L_α contains only countably many real numbers, so there are real numbers outside of L_α that we may try to add.

More specifically, let $\omega_2^{L_\alpha}$ be whatever ordinal L_α thinks is ω_2. (We are assuming L_α is a model of ZFC.) Like everything inside L_α, $\omega_2^{L_\alpha}$ is countable., so we may actually find "$\omega_2^{L_\alpha}$-many" reals and add them to L_α. Perhaps, by adding these reals to L_α, we can find a model of ZFC having ω_2 many reals, and therefore failing to satisfy the continuum hypothesis!

In some sense, this is precisely what we will do. However, the approach above is extremely naive, and is pretty far from reality. (Indeed the Continuum Hypothesis,

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#1 in Hilbert's famous list of problems, was open for a long time.) Roughly speaking, Cohen's method of forcing allows us to add reals, and other sets, to some given model of set theory. It turns out however that we need to choose these sets very carefully for two reasons. One is to ensure that we can construct another model of set theory, containing the original one and the additional sets. The other is to be able to analyze the resulting model, for example, to determine whether or not it satisfies the continuum hypothesis.

2. Axiomatic set theory review

We begin by reviewing some basic set theory. We will take a logic-based and axiomatic approach, emphasizing questions of definability in set theory. It should make sense that paying attention to axiomatic issues will be helpful towards proving independence results. However, the entanglement between logical methods and set theory is much deeper. Often times a careful analysis of definability and axiomatics sheds more light on seemingly unrelated questions.

2.1. Formulas.

Given a language $\mathcal{L}$ (some relation symbols, functions symbols, and constant symbols,) **terms in the language** $\mathcal{L}$ are defined recursively as follows. Any variable is a term (e.g. x, y, x_1, x_2,...); given a function symbol $f(x_1,...,x_n)$ from $\mathcal{L}$ and terms $t_1,...,t_n$, then $f(t_1,...,t_n)$ is a term as well.

For example, in the language of pure set theory, we only have the relation symbol $\in$ (and equality $=$). There are no function symbols, so the terms are only variables.

A **formula in the language** $\mathcal{L}$ is defined recursively as follows. An **atomic formula** is of the form $R(x_1,...,x_n)$, where R is an n-ary relation symbol from $\mathcal{L}$ and $x_1,...,x_n$ are terms. We further construct formulas by logical connectives, for example, given formulas ϕ and ψ, then $\phi \vee \psi$ and $\neg\phi$ are formulas, and quantifiers, if ϕ is a formula then $\forall x\phi$ and $\exists x\phi$ are formulas.

For example, in set theory the atomic formulas are of the form $x = y$ or $x \in y$. $\forall x\exists y\exists z(y \neq z \wedge x \in y \wedge y \in z)$ is a more complex example of a formula (without free variable). $\forall x(x \in y \iff x \in z)$ is a formula with two free variables, y and z, which, assuming the axiom of extensionality, is equivalent to the formula $y = z$.

Remark 2.1. Given a formula ϕ, in which the variable x does not occur, we may still consider x as a free variable, a.k.a., a "dummy variable". That is, we consider it as a variable, but ϕ says nothing about it.

Given a language $\mathcal{L}$, **a model** $\mathcal{M}$ **for** $\mathcal{L}$ is a set M together with interpretations of the symbols in the language. Each constant symbol is interpreted as an element in M, an n-ary relation symbol is interpreted as a subset of M^n, and an n-ary function symbol is interpreted as a function $M^n \to M$. Sometimes M will be a proper (definable) class rather than a set, and the interpretations will be definable subsets and functions with class domain.

For example, working with ZF, we construct V_α, and may view it as a model for the language of set theory by interpreting $x \in y$, for x, y in V_α, according to the true value of "$x \in y$". In this case we may say that $(V_\alpha, \in)$ is a model in the language of set theory, but we will often drop the $\in$ when it is clear. There could be also models in the language of set theory in which the symbol $\in$ is interpreted in a very different way. For example when taking ultrapowers (we will talk more about it).

Given a model $\mathcal{M}$, a formula $\phi(x_1, ..., x_n)$ with free variables $x_1, ..., x_n$, and members of $a_1, ..., a_n$ in M, the **satisfaction relation** $\mathcal{M} \models \phi$ is defined inductively on the construction of formulas, in the logical way. For example, $\mathcal{M}$ satisfies $\exists x \psi(x)$ if there exists $a \in M$ such that $\mathcal{M} \models \psi(a)$.

For example, V_ω satisfies all the axioms of ZFC apart for the axiom of infinity.

2.2. Bounded quantification and Δ_0 formulas.

Definition 2.2 (Bounded quantifiers)**.** Given a formula ϕ and variables x, X (possibly free variables of ϕ), the following are formulas

$$(\forall x \in X)\phi(x, X, ...),\ (\exists x \in X)\phi(x, X, ...)$$

which formally stand for

$$\forall x(x \in X \implies \phi(x, X, ...)),\ \exists x(x \in X \wedge \phi(x, X, ...)).$$

These are called **bounded quantifiers**.

Definition 2.3 (Δ_0 formulas)**.** A formula in the language of set theory is Δ_0 if all the quantifiers appearing in it are bounded. (Formally, we define these recursively, stating that all the atomic formulas are Δ_0, applying any connectives to Δ_0 formulas gives Δ_0 formulas, and applying bounded quantifiers to Δ_0 formulas give Δ_0 formulas.)

Example 2.4.
- $(\exists x \in X)(\forall y \in X)(x \in y)$ is a Δ_0 formula with one free variable X;
- $(\forall x \in X)(x \cup \{x\} \in X)$ is a Δ_0 formula with one free variable X. How to write it formally?

$$(\forall x \in X)(\exists y \in X)[x \in y \wedge (\forall z \in y)(z = x \vee z \in x)]$$

Remark 2.5. We care about formulas up to equivalence. For example, the formula $\forall x(x = x)$ is technically not Δ_0, but it is logically equivalent to a tautology, and we consider it Δ_0.

2.3. Transitive models.

Definition 2.6. Recall that a set X is **transitive** if for any $x \in X$ if $y \in x$ then $y \in X$.

Remark 2.7. The formula $\phi(X)$ defined by $(\forall x \in X)(\forall y \in x)(y \in X)$ is a Δ_0 formula such that X is transitive if and only if $\phi(X)$ holds.

Theorem 2.8. *Suppose X is a transitive set. Consider $(X, \in)$ as a model in the language of set theory. Let $\phi(x_1, ..., x_n)$ be a Δ_0 formula and $a_1, ..., a_n$ members of X. Then*

$$(*)\ \ \phi(a_1, ..., a_n) \text{ is true if and only if } (X, \in) \models \phi(a_1, ..., a_n).$$

Slogan: "Δ_0 statements are absolute between transitive models of set theory".

Proof. The proof is by induction, along the construction sequence of a formula. First consider atomic formulas. If ϕ is of the form $x \in y$, then $(*)$ is true by definition (we defined the relation $\in$ in X to be precisely the real relation $\in$). Similarly, if ϕ is of the form $x = y$, $(*)$ holds.

To deal with logical connectives, suppose $(*)$ holds for ψ and ϕ, then show that $(*)$ holds for $\neg\psi$ and for $\psi \vee \phi$. This is left as an exercise.

Finally, we need to show that if $(*)$ is true for $\phi(x, y, x_1, ..., x_n)$ them $(*)$ is true after applying a bounded quantifier. Let $\psi(y, x_1, ..., x_n)$ be the formula $(\exists x \in y)\phi(x, y, x_1, ..., x_n)$, and fix $b, a_1, ..., a_n$ in X. Suppose first that $(X, \in) \models \psi(b, a_1, ..., a_n)$. Then, working in $(X, \in)$, there is some $a \in b$ such that $\phi(a, b, a_1, ..., a_n)$ holds in $(X, \in)$. By induction hypothesis (by $(*)$ for ϕ), it follows that $\phi(a, b, a_1, ..., a_n)$ is true (in V), and therefore $\psi(b, a_1, ..., a_n)$ is true.

Assume now that $\psi(b, a_1, ..., a_n)$ is true. By the definition of ψ, there is some $a \in b$ such that $\phi(a, b, a_1, ..., a_n)$ holds. *Since X is transitive*, then $a \in X$. Therefore in $(X, \in)$ we conclude that $\psi(b, a_1, ..., a_n)$ is true. □

Corollary 2.9. If X is a transitive set, then $(X, \in)$ satisfies the axiom of extensionality. That is, given $A, B \in X$, if A and B have the same members, then $A = B$.

Proof. Given $A, B \in X$, the axiom of extensionality for A, B can be written as a Δ_0 formula

$$((\forall x \in A)x \in B) \wedge ((\forall x \in B)x \in A).$$

□

2.4. **Relativization.** When studying different models of set theory, we want to know when a model M satisfies a sentence ϕ. There is a simple way to express the statement $M \models \phi$ with a formula, as follows.

Definition 2.10. Let M be a set, or a definable class. For a formula $\phi(x_1, ..., x_n)$ we define the *relativization* of ϕ to M, denoted $\phi^M(x_1, ..., x_n)$, by induction on the construction of formulas.

- $(x \in y)^M$ is defined to be $x \in y$;
- $(x = y)^M$ is defined to be $x = y$;
- given $(\phi \vee \psi)^M$ is defined to be $\phi^M \vee \psi^M$;
- $(\neg\phi)^M$ is defined to be $\neg(\phi^M)$;
- $(\exists x\phi)^M$ is defined to be $(\exists x \in M)\phi^M$.

Remark 2.11. M is a parameter in the formula ψ^M. If M is a definable class, defined using the formula $\chi(x)$, we replace $x \in M$ with $\chi(x)$.

Example 2.12. Let ϕ be the sentence $\forall x\forall y\exists z(x \in z \wedge y \in z)$. Then ϕ^{V_ω} is the formula $(\forall x \in V_\omega)(\forall y \in V_\omega)(\exists z \in V_\omega)(x \in z \wedge y \in z)$.

Exercise 2.13. Show that for any set M, considered as a model $(M, \in)$ in the language of set theory, for any formula $\phi(x_1, ..., x_n)$ and any $a_1, ..., a_n \in M$,

$$\phi^M(a_1, ..., a_n) \text{ is true if and only if } (M, \in) \models \phi(a_1, ..., a_n).$$

Exercise 2.14.
(1) Show that for any set M and formula ϕ, ϕ^M is a Δ_0 formula.
(2) Suppose ϕ is a Δ_0 formula and M is any set, show that ϕ^M is ϕ.

2.5. **The Levy Hierarchy.**

Definition 2.15. Say that a formula is Σ_0, or Π_0, if it is Δ_0. Define recursively:

- a formula is Σ_{n+1} if it is of the form $\exists x\phi$ for some Π_n formula ϕ;
- a formula is Π_{n+1} if it is of the form $\forall x\phi$ for some Σ_n formula ϕ.

Say that a property P (that is, some definable relation) is Σ_n (resp. Π_n) if there is a Σ_n (resp. Π_n) formula ϕ which defines P. Say that a property P is Δ_n if it is both Σ_n and Π_n.

Up to logical equivalence, Σ_n formulas are precisely the negations of Π_n formulas, and Π_n formulas are the negations of Σ_n formulas. So a property P is Δ_n if and only if both P and $\neg P$ are Σ_n (equivalently, if both P and $\neg P$ are Π_n).

The proof that a specific formula "correctly represents the property P" will often use some axioms of ZF (and will not be a simple logical equivalence). When saying that "P is Σ_n" we usually mean that this is proved in ZF. In some contexts we will allow ourselves to use more, or restrict ourselves to less, than ZF.

Example 2.16. The property $P(R, X)$ saying that R is a (strict) well ordering on X, is Π_1. It can be described as follows: R is a well ordering on X if

(1) for any $x, y \in X$, if $x \neq y$ then either $x \; R \; y$ or $y \; R \; x$, and not both;
(2) for any $x, y, z \in X$, if $x \; R \; y$ and $y \; R \; z$ then $x \; R \; z$;
(3) For any set Y, if $Y \subseteq X$ then there is some $y \in Y$ such that for any other $z \in Y$, either $y \; R \; z$ or $y = z$.

Properties (1) and (2) are Δ_0 in fact. (3) is of the form $\forall Y$ followed by a Δ_0 statement (note that saying $Y \subseteq X$ is Δ_0, $(\forall y \in Y) y \in X$, and therefore is Π_1.

Example 2.17. The property $P(x)$ saying that "x is an ordinal", is Δ_0. It can be expressed as follows: x is an ordinal if x is transitive and $\in$ is a linear ordering of x. We saw above that being a linear order can be expressed by a Δ_0 formula.

Note that we used ZF to argue that the above Δ_0 formula captures the concept of an ordinal. Specifically, the axiom of foundation tells us that if $\in$ is a linear order on x then it must be a well order. Without the axiom of foundation, we would need to stipulate this extra assumption, which would give a Π_1 instead of Δ_0.

Example 2.18. The property $P(x, y)$ saying that "x is the powerset of y", is Π_1. This can be expressed by the Π_1 formula $\phi(x, y)$:

$$\forall a (a \in x \iff a \subseteq y)$$

Exercise 2.19. The property $P(x, y)$ saying that "x is the union of y, $x = \bigcup y$", is Δ_0.

Exercise 2.20. (1) Suppose that M is a transitive set, $\psi(x_1, ..., x_n)$ is a Π_1 formula and $a_1, ..., a_n$ are in M. Show that if $\psi(a_1, ..., a_n)$ is true, then $(M, \in) \models \psi(a_1, ..., a_n)$.

(2) Suppose that M is a transitive set, $\psi(x_1, ..., x_n)$ is a Σ_1 formula and $a_1, ..., a_n$ are in M. Show that if $(M, \in) \models \psi(a_1, ..., a_n)$ then $\psi(a_1, ..., a_n)$ is true.
(3) Conclude that if M is a transitive set, $P(x_1, ..., x_n)$ is a Δ_1 property, then for any $a_1, ..., a_n$ in M, $P(a_1, ..., a_n)$ holds if and only if $(M, \in) \models P(a_1, ..., a_n)$. That is, Δ_1 *statements are absolute between transitive models of set theory.*
(4) Show that part (1) can fail for Σ_1 formulas. That is, find a Σ_1 formula $\phi(x_1, ..., x_n)$ (for some n), a transitive set M and $a_1, ..., a_n$ in M such that $\phi(a_1, ..., a_n)$ is true, yet $(M, \in) \models \neg\phi(a_1, ..., a_n)$.

Closure properties. While the syntactic requirement for being Σ_n or Π_n are rather strict, the collection of Σ_n properties and Π_n properties are closed under a variety of operations.

Theorem 2.21. For $n \geq 1$.

(1) If P and Q are Σ_n (respectively Π_n) properties, then $P \vee Q$, $P \wedge Q$, $(\exists x \in X)P$, and $(\forall x \in X)P$, are Σ_n (respectively Π_n) properties.
(2) If P is a Σ_n property then $\exists x P$ is a Σ_n property.
(3) If P is a Π_n property then $\forall x P$ is a Π_n property.

Proof. The proof goes by induction on n. Let us consider the case $n = 1$, as the inductive step is similar.

We show that if P is Σ_1 then $\exists x P$ is Σ_1 as well. Let $\phi(x, x_1, ..., x_n)$ be a Σ_1 formula such that $P(x, x_1, ..., x_n)$ holds if and only if $\phi(x, x_1, ..., x_n)$. By definition, ϕ is of the form $\exists y \psi(y, x, x_1, ..., x_n)$, where ψ is a Σ_0 formula. Now P is equivalent to the formula $\exists x \exists y \psi(y, x, x_1, ..., x_n)$, which is equivalent to the Σ_1 formula

$$\exists X(\exists x, y \in X)\psi(y, x, x_1, ..., x_n).$$

Next we show that if P is Σ_1 then $(\forall x \in X)P$ is Σ_1. Let $\phi(X, x, x_1, ..., x_n)$ be a Σ_1 formula such that $P(X, x, x_1, ..., x_n)$ holds if and only if $\phi(X, x, x_1, ..., x_n)$. By definition, ϕ is of the form $\exists y \psi(y, X, x, x_1, ..., x_n)$, where ψ is a Σ_0 formula. Now P is equivalent to the statement $(\forall x \in X)\exists y \psi(y, X, x, x_1, ..., x_n)$, which is equivalent to the Σ_1 property

$$\exists Y(\forall x \in X)(\exists y \in Y)\psi(y, X, x, x_1, ..., x_n).$$

Note that for the latter equivalence we used the axiom of replacement as well as the axiom of foundation.

The remaining cases are easier, and left as an exercise. For example, we show that if P and Q are Σ_1 then $P \wedge Q$ is Σ_1. For notational simplicity assume $P(x)$ and $Q(x)$ are properties of a single variable. By assumption, there are Σ_1 formulas $\phi(x)$ and $\psi(x)$ such that which define P and Q respectively. There are Σ_0 formulas $\psi_0(x, y)$ and $\phi_0(x, y)$ such that ϕ is $\exists y \phi_0$ and ψ is $\exists y \psi_0$. Now $P \wedge Q$ can be written as $\exists y \psi_0(x, y) \wedge \exists y \phi_0(x, y)$, which is equivalent to the Σ_1 formula $\exists y(\psi_0(x, y) \wedge \phi_0(x, y))$. $\square$

Exercise 2.22. Complete the proof of Theorem 2.21

2.6. **The V_α hierarchy.** Define recursively along the ordinals

- $V_\emptyset = \emptyset$;
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, the powerset of V_α; and
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$, if α is a limit ordinal.

The axiom of foundation is equivalent to the statement that the whole universe is $V = \bigcup_\alpha V_\alpha$. That is, any set x is in V_α for some ordinal α.

Consider the formula $\phi(y, F)$ saying that y is an ordinal and F codes the construction of V_α up to y. That is:

- $F(\emptyset) = \emptyset$;
- $(\forall \beta \in y)(F(\beta + 1) = \mathcal{P}(F(\beta)))$;
- $\forall \beta \in y$, if β is limit then $F(\beta) = \bigcup_{\gamma<\beta} F(\gamma)$.

Note that the second bullet is Π_1 and the rest is Δ_0, so ϕ is a Π_1 property.

Corollary 2.23. The property $P(X, \alpha)$ saying that "X is V_α" is Δ_2.

Proof. First, X is equal to V_α if and only if there exists an F such that $\phi(\alpha + 1, F)$ and $X = F(\alpha)$. This is a statement of the form $\exists(\Pi_1)$, and is therefore Σ_2.

Second, X is equal to V_α if *for any* F, $\phi(\alpha+1, F) \implies F(\alpha) = X$. This is a statement of the form $\forall(\Pi_1 \to \Delta_0)$, equivalently, $\forall(\Delta_0 \vee \neg\Pi_1)$, which is $\forall(\Sigma_1)$, which is Π_2. □

Corollary 2.24. The property $P(X)$ saying "X is V_α for some ordinal α" is Σ_2.

Proof. By the previous corollary, we may write this statement as $\exists(\Sigma_2)$, which is Σ_2. □

2.7. **The axioms of ZFC.** Recall the axioms of ZFC:

- Extensionality: for any A, B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.
- Empty set: there exists a set $\emptyset$ with no elements.
- Comprehension (scheme): given a formula $\phi(x, x_1, ..., x_n)$, we include the axiom saying that for any $a_1, ..., a_n$ and any set A there exists a set B whose elements are precisely those $x \in A$ for which $\phi(x, a_1, ..., a_n)$ holds.
- Pairing: for any a, b there exists a set whose elements are precisely a and b.
- Union: for any set X there exists a set Y such that $y \in Y \iff \exists x \in X(y \in x)$.
- Infinity: there exists an inductive set, that is, a set X such that $\emptyset \in X$ and $x \cup \{x\} \in X$ for any $x \in X$.
- Power set: for any set X there exists a set $\mathcal{P}(X)$ whose elements are all subsets of X.
- Foundation: for a non-empty set X there exists $x \in X$ such that $x \cap X = \emptyset$.
- Replacement (scheme): given a formula $\phi(x, y, x_1, ..., x_n)$ we include the axiom saying that for any $a_1, ..., a_n$ and for any set X, if for any $x \in X$ there exists a unique y such that $\phi(x, y, a_1, ..., a_n)$ holds, then *there exists a set* Y such that for any $x \in X$ there is a $y \in Y$ such that $\phi(x, y, a_1, ..., a_n)$.
- The axiom of choice: for any set X, if all members of X are non-empty then there exists a function f with domain X such that $f(x) \in x$ for every $x \in X$.

Remark 2.25. The axioms of Extensionality, and Foundation are Π_1 statement. The axiom of Infinity is a Σ_1 statement. The axioms of Pairing, Union, and Choice are Π_2 statements.

2.8. $H(\kappa)$**.** Let κ be an infinite regular cardinal. Say that a set x is **hereditarily smaller than** κ if $|\text{t.c.}(x)| < \kappa$, where $\text{t.c.}(x)$ is the transitive closure of x. Let $H(\kappa)$ be the collection of all sets x which are hereditarily smaller than κ.

Remark 2.26. For a regular cardinal κ, if x is hereditarily smaller than κ then x is in V_κ. This can be shown by induction on the rank of x. So $H(\kappa) \subseteq V_\kappa$ and $H(\kappa)$ is a set by comprehension.

It follows from the definition that $H(\kappa)$ is a transitive set.

Exercise 2.27. Show that $H(\omega) = V_\omega$. However, $H(\omega_1)$ is strictly contained in V_{ω_1}.

Theorem 2.28 (Assume ZFC)**.** For an uncountable regular cardinal κ, $H(\kappa)$ satisfies all the axioms of ZFC apart from the powerset axiom.

Proof. The main point is showing that the axiom of replacement holds. The others are easier.

For the union axiom, take $x \in H(\kappa)$. Then the transitive closure of $\bigcup x$ is contained in the transitive closure of $\{x\}$. So $\bigcup x \in H(\kappa)$. To be precise, we must also show that the set $z = \bigcup x$ is in fact the union of x as calculated in $H(\kappa)$. That is, that $H(\kappa) \models$ "z is the union of x". This is easy to show directly, but it also follows at once from our observations regarding the complexity of this statement. Recall that the definition of "z is the union of x" is Π_1, and $H(\kappa)$ is transitive, therefore "z is the union of x" also holds in $H(\kappa)$.

Fix a formula $\phi(x, y)$ and a set $X \in H(\kappa)$ such that for any $x \in X$ there exists a unique y in $H(\kappa)$ such that $M \models \phi(x, y)$. Then

$$(\forall x \in X)(\exists! y \in H(\kappa))(\phi^M(x, y)).$$

We may now apply the axiom of replacement for the formula ϕ^M, to find a set Y of all sets $y \in H(\kappa)$ for which there exists some $x \in X$ such that $\phi^M(x, y)$ holds.

Since there is a surjective map from X onto Y, $|Y| < \kappa$. It suffices to prove the following:

Lemma 2.29. If $Y \subseteq H(\kappa)$ and $|Y| < \kappa$ then $Y \in H(\kappa)$.

Proof.

$$\text{t.c.}(Y) = \bigcup_{y \in Y} \text{t.c.}(y).$$

So t.c.(Y) can be written as a union of $< \kappa$ many sets of size $< \kappa$. Since κ is a regular cardinal, we conclude that t.c.(Y) is of size $< \kappa$. □

□

Exercise 2.30. Show that the powerset axiom fails in $H(\aleph_1)$.

It follows that $H(\aleph_1)$ satisfies all the axioms of ZFC but not the powerset axiom, and therefore the powerset axiom is independent from the other axioms of ZFC.

2.9. **The reflection principle.**

Theorem 2.31. For any formula $\phi(x_1, ..., x_n)$ there is an ordinal α such that

for any $a_1, ..., a_n \in V_\alpha$, $\phi^{V_\alpha}(a_1, ..., a_n)$ holds if and only if $\phi(a_1, ..., a_n)$.

That is, V_α "reflects" correctly the truth value of ϕ.

Definition 2.32. Given a function f whose domain and range are ordinals, say that α is a **closure point** of f if for any $\beta < \alpha$, $f(\beta) < \alpha$. We also consider **definable functions** whose domain may be the class of all ordinals. That is, f is defined by some formula $\chi(x, y)$ in the sense that for each ordinal α there is a unique ordinal θ such that $\chi(\alpha, \theta)$ holds.

Exercise 2.33. Given a definable function f from ordinals to ordinals, the collection of all closure points of f is

(1) definable via a formula;
(2) closed, that is, if X is a set of closure points of f and $\alpha = \sup X = \bigcup X$, then α is a closure point of f;
(3) unbounded, that is, for any ordinal β there is an ordinal $\alpha > \beta$ such that α is a closure point of f.

Hint: part (3) makes an essential use of the axiom of replacement.

We will prove the following strong version of Theorem 2.31: for any formula ϕ there is a definable function f all of whose closure points reflect ϕ correctly. That is, there is a **closed unbounded** class of ordinals which reflect ϕ.

Exercise 2.34. Suppose $\chi_1, ..., \chi_n$ are definable functions from ordinals to ordinals. Show that there is a formula χ defining a function from ordinals to ordinals such that any closure point of χ is a closure point for each of $\chi_1, ..., \chi_n$.

Lemma 2.35. Let $\phi(x_1, ..., x_n)$ be a formula, then there is a definable function f from ordinals to ordinals such that for any closure point of f

($\star$) for any $a_1, ..., a_n \in V_\alpha$, $\phi^{V_\alpha}(a_1, ..., a_n)$ holds if and only if $\phi(a_1, ..., a_n)$.

Proof. We prove this inductively on the construction of formulas. If ϕ is an atomic formula then ($\star$) is satisfied for any α. (In fact this is the case for any Δ_0 formula ϕ.)

The main inductive case is the existentional quantifier. Suppose ϕ is of the form $\exists x \psi(x, x_1, ..., x_n)$, where we already know the lemma for ψ. That is, there is a definable function h such that for any closure point α for h, ($\star$) holds for V_α and ψ.

Define a function f recursively as follows. $f(0) = 0$. Suppose we define f on all ordinals below α. Let $\theta = \sup_{\beta<\alpha} f(\beta)$. Let $\zeta > \theta$ be the minimal ordinal such that

$$\forall a_1, ..., a_n \in V_\theta (\exists x \psi(x, a_1, ..., a_n) \implies (\exists x \in V_\zeta)\psi(x, a_1, ..., a_n)).$$

Such ζ exists by the axiom of replacement. Define $f(\alpha) = h(\beta)$ where β is minimal such that $h(\beta) \geq \zeta$. Note that any closure point for f is also a closure point for h. (Also any closure point of f is a limit ordinal.)

Let α be a closure point of f and take $a_1, ..., a_n$ in V_α. Since α is a closure point for h, we have

$$\psi^{V_\alpha}(a, a_1, ..., a_n) \text{ if and only if } \psi(a, a_1, ..., a_n),$$

for any $a \in V_\alpha$. In particular

$$\phi^{V_\alpha}(a_1, ..., a_n) \implies \phi(a_1, ..., a_n).$$

Conversely, suppose $\phi(a_1, ..., a_n)$ is true. Fix $\beta < \alpha$ such that $a_1, ..., a_n$ are all in V_β. Since $\exists x \psi(x, a_1, ..., a_n)$, it follows from the definition of f that there is an a in $V_{f(\beta)}$ for which $\psi(a, a_1, ..., a_n)$ holds, and so $\psi^{V_\alpha}(a_1, ..., a_n)$ holds. As α is a closure point of f, $f(\beta) < \alpha$, and so $a \in V_\alpha$. It follows that $\phi^{V_\alpha}(a_1, ..., a_n)$ holds, as required. □

2.10. **Coding of formulas.** We may code formulas as sets in V_ω as follows. (We fix an infinite list of variable $x_0, x_1, ...$ to be used by our formulas.) For each formula ϕ we assign a code $[\phi] \in V_\omega$ recursively.

- $[x_i = x_j] = \langle 0, i, j \rangle$, $[x_i \in x_j] = \langle 1, i, j \rangle$;
- $[\neg \phi] = \langle 2, [\phi] \rangle$;
- $[\phi \wedge \psi] = \langle 3, [\phi], [\psi] \rangle$;
- $[\exists x_i \phi] = \langle 4, i, [\phi] \rangle$.

Say that x codes a formula, Formula(x), if $x = [\phi]$ for some formula ϕ.

Claim 2.36. Formula(x) is a Δ_1 property (using ω as a parameter).

Proof. x codes a formula if for any set Z, if

$$(\forall i,j\in\omega)\,\langle 0,i,j\rangle\in Z\wedge\langle 1,i,j\rangle\in Z\wedge(\forall y,z\in Z)\,(\langle 2,z\rangle\in Z\wedge\langle 3,z,y\rangle\in Z\wedge\langle 4,z,y\rangle\in Z)$$

then $x\in Z$. That is, any set Z that has the codes for atomic formulas, and is closed under the operations for formula constructions, must contain codes for all formulas and therefore x. This is a Π_1 statement.

Also, x is a formula if and only if there exists a function F whose domain is a natural number and F codes a construction sequence of formulas $F(0),F(1),...$ where x is $F(n)$ for some n in the domain of F. The latter can be written as a Σ_1 statement. □

If x codes a formula we will write ϕ_x for the formula coded by x.

Exercise 2.37. For each natural number n there is a Δ_1 formula $\mathrm{Formula}_n(x)$ such that x is a code for a Σ_n formula if and only if $\mathrm{Formula}_n(x)$ holds.

Exercise 2.38. There is a Δ_1 formula $\mathrm{Free}(x,y)$ which holds if and only if x codes a formula and $y\subseteq\omega$ is the set of all free variables appearing in ϕ_x.

Similarly, the property $P_k(x,y,z_1,...,z_k)$ saying "y codes a formula with k variables and x is the code for the sentence $\phi_y(z_1,...,z_k)$" is Δ_1 (ϕ_x is the sentence obtained from the formula ϕ_y by substituting the free variables with constants).

Fact 2.39. For each natural number $n\geq 1$ there is a Σ_n formula $\mathrm{Truth}_n(x)$ which holds if and only if x is a code of a true Σ_n sentence.

Proof sketch. The proof is by induction. We sketch the Σ_1 case. First we find a Σ_1 formula $\mathrm{Truth}_{\mathrm{qf}}(x,z)$ which holds if and only if $z=\langle z_1,...,z_k\rangle$ is a sequence of length k, for some $k\in\omega$, x is a code for a quantifier free formula with k free variables and $\phi_x(z_1,...,z_k)$ is true.

$\mathrm{Truth}_{\mathrm{qf}}(x,z)$ says that there are sequences F and T coding a construction sequence $F(0),F(1),...,F(n)$ which does not use the "$\exists$" clause at all and $F(n)$ is x, and a sequence of "truth values" $T(0),...,T(n)\in\{0,1\}$ such that for each $i\leq n$

- if $F(i)$ is of the form $\langle 0,l,t\rangle$, then $T(i)=1$ iff $z_l=z_t$;
- if $F(i)$ is of the form $\langle 1,l,t\rangle$ then $T(i)=1$ iff $z_l\in z_t$;
- if $F(i)$ is of the form $\langle 2,F(j)\rangle$, $j<i$ then $T(i)=1$ iff $T(j)=0$;
- if $T(i)$ is of the form $\langle 3,F(j_1),F(j_2)\rangle$ then $T(i)=1$ iff $T(j_1)=1$ and $T(j_2)=1$;

and $T(n)=1$. This is a Σ_1 statement, and it works.

Now we can define a Σ_1 formula $\mathrm{Truth}_1(x,z)$ saying: $z=\langle z_1,...,z_k\rangle$ and there is a code y such that x is the code for the formula $\exists x_0\phi_y(x_0,x_1,...,x_k)$ and *there is a* z_0 such that $\mathrm{Truth}_{qf}(x,\langle z_0,z_1,...,z_n\rangle)$.

In particular if x codes a Σ_1 sentence then ϕ_x is true if and only if $\mathrm{Truth}_1(x,\emptyset)$. □

Corollary 2.40. For any n there is an ordinal α such that for any Σ_n formula $\phi(x_1,...,x_k)$ and for any $a_1,...,a_k\in V_\alpha$

$$\phi^{V_\alpha}(a_1,...,a_k)\iff\phi(a_1,...,a_k)$$

In this case we say that V_α is a Σ_n **elementary substructure** of V, written $V_\alpha\prec_{\Sigma_n}V$.

Proof. Apply the reflection theorem for the formula Truth_n. There is an ordinal α such that for any $x, z \in V_\alpha$

$$(\text{Truth}_n)^{V_\alpha}(x,z) \iff \text{Truth}_n(x,z).$$

Now for any Σ_n formula $\phi(x_1,...,x_k)$, and any $a_1,...,a_k \in V_\alpha$, let $z = \langle a_1,...,a_k\rangle$.

$$\phi^{V_\alpha}(a_1,...,a_k) \iff (\text{Truth}_n)^{V_\alpha}([\phi],z) \iff \text{Truth}_n([\phi],z) \iff \phi(a_1,...,a_k).$$

Remark 2.41. To be precise, in the left most " $\iff$ " above we applied Fact 2.39 *inside* V_α, even though V_α might not satisfy all axiom of ZF. This can be remedied as follows. We proved Fact 2.39 using ZF. The proof, like any proof, used only finitely many of the axioms of ZF. Let Φ be this finite collection. Using the reflection theorem, we can find α such that V_α reflects Truth_n and all sentences in Φ. In particular, Fact 2.39 is true in V_α. (We will use this trick again, in particular when talking about forcing)

□

2.11. **Undefinablity of truth.** [Not covered in class]

Theorem 2.42 (Tarski's undefinability of truth)**.** There is no formula $\text{Truth}(x)$ such that $\text{Truth}(x)$ holds if and only if x is a code of a true sentence.

Proof. Assume otherwise that such formula exists. Let $\theta(y)$ be the formula $\neg\text{Truth}([\phi_y(y)])$. That is, $\theta(y)$ says "y is a code for a formula with one variable, and for the unique $x \in V_\omega$ such that $x = [\phi_y(y)]$, $\neg\text{Truth}(x)$". (This can be done, in a similar way to the exercises above. That is, we can define the relation for "substituting a variable".)

Let $z = [\theta]$. Then $\theta(z)$ holds if and only if $\neg\text{Truth}(\phi_z(z))$ if and only if $\neg\text{Truth}(\theta(z))$ if and only if $\theta(z)$ fails. A contradiction. □

2.12. **Ordinal Definability.**

Definition 2.43. Say that a set A is **ordinal definable** (OD) if there is a formula $\phi(x, x_1,...,x_n)$ and ordinals $\alpha_1,...,\alpha_n$ such that for any set a

$$a \in A \iff \phi(a,\alpha_1,...,\alpha_n)$$

Example 2.44. For any ordinal α, V_α is ordinal definable. For any cardinal κ, $H(\kappa)$ is ordinal definable.

Exercise 2.45. A is ordinal definable if and only if there is a formula $\phi(X, x_1,...,x_n)$ and ordinals $\alpha_1,...,\alpha_n$ such that A is the unique set for whoch $\phi(A,\alpha_1,...,\alpha_n)$ is true.

Definition 2.46. Say that a set A is **hereditarily ordinal definable** if every set in the transitive closure of $\{A\}$ is ordinal definable. That is, A is OD, the members of A are OD, the members of $\bigcup A$ are OD, and so on...

Example 2.47. Every ordinal is hereditarily ordinal definable.

Lemma 2.48. A is ordinal definable if and only if there exists an ordinal α, a formula ϕ, ordinals $\alpha_1,...,\alpha_n < \alpha$ such that

$$a \in A \iff a \in V_\alpha \wedge \phi^{V_\alpha}(a,\alpha_1,...,\alpha_n)$$

In particular, being ordinal definable is a definable property: A is ordinal definable if and only if "there is an ordinal alpha and ordinals $\alpha_1, ..., \alpha_n < \alpha$ and a set X such that $X = V_\alpha$ and $a \in A \iff \phi^X(a, \alpha_1, ..., \alpha_N)$. The reason we can definably quantify over all formulas is because we only quantify over Δ_0 formulas. Furthermore, truth for Δ_0 formulas is definable. (It is Σ_1, and in fact Δ_1.)

Similarly, the property "A is a hereditarily ordinal definable set" is a definable property.

Definition 2.49. Let HOD be the class of all hereditarily ordinal definable sets.

By definition HOD is a transitive class. That is, if A is in HOD and $a \in A$ then a is in HOD. This may not be the case for OD.

Theorem 2.50 (Assuming ZF)**.** The class HOD satisfies ZFC.

Corollary 2.51 (Gödel)**.** If ZF is consistent then ZFC is consistent. In other words, we cannot refute the axiom of choice using ZF set theory.

Proof. We prove for example the union axiom. Suppose A is HOD. In particular A is OD. Let $\phi(x, \alpha_1, ..., \alpha_n)$ be such that $a \in A \iff \phi(a, \alpha_1, ..., \alpha_n)$. Define $\psi(x, \alpha_1, ..., \alpha_n)$ as $\exists y(\phi(y, \alpha_1, ..., \alpha_n) \wedge x \in y)$. This shows that $\bigcup A$ is OD as well. Furthermore, the transitive closure of $\bigcup A$ is contained in the transitive closure of A, so $\bigcup A$ is hereditarily ordinal definable as well.

The rest of the ZF axioms can be verified in a similar way, and the proofs are similar to those we have seen in $H(\kappa)$ and V_α before. This is left for the reader.

The main point is showing that the axiom of choice holds in HOD, even without assuming it in V. It suffices to show the following: given an OD set A there is an OD relation $<$ which is a well ordering of A. (If A is HOD then $<$ will also be.)

Let A be ordinal definable. Fix α so that for any a in A there is a formula ϕ, ordinals $\alpha_1, ..., \alpha_n < \alpha$ such that $a = \left\{x \in V_\alpha : \phi^{V_\alpha}(x, \alpha_1, ..., \alpha_n)\right\}$. We say that a is defined via $\phi, \alpha_1, ..., \alpha_n$. ($\alpha$ is now fixed, and there could be many tuples $\phi, \alpha_1, ..., \alpha_n$ defining the same set.)

Recall that given well orders $<_1, ..., <_m$ on sets $X_1, ..., X_m$, the **lexicographic ordering** on $X_1 \times ... \times X_m$ is defined as follows: $(x_1, ..., x_m) < (y_1, ..., y_m)$ if and only if for the first i such that $x_i \neq y_i$, $x_i <_i y_i$.

We coded formulas as members of V_ω, which is countable. (Fix some well ordering of V_ω, for example, via an enumeration of it.) The ordinal α is well ordered by the membership relation $\in$. Consider the lexicographic ordering on $V_\omega \times \alpha^n$.

We have a map from $V_\omega \times \alpha^{<\omega}$ sending $\langle [\phi], \alpha_1, ..., \alpha_n\rangle$ to $a_{\langle [\phi], \alpha_1, ..., \alpha_n\rangle} = \left\{x \in V_\alpha : \phi^{V_\alpha}(x, \alpha_1, ..., \alpha_n)\right\}$, where the image includes the transitive closure of A. Furthermore this map is definable using the parameters ω and α. For $a \in A$ let $n(a)$ be the minimal n such that $a = a_{\langle \phi, \alpha_1, ..., \alpha_n\rangle}$ for some $\langle \phi, \alpha_1, ..., \alpha_n\rangle \in V_\omega \times \alpha^n$.

Finally, we can use this map to well order A: say that $a < b$ if and only if either $n(a) < n(b)$ or $n(a) = n(b) = n$ and the minimal tuple $\langle \phi, \alpha_1, ..., \alpha_n\rangle$ such that $a = a_{\langle \phi, \alpha_1, ..., \alpha_n\rangle}$ is lexicographically smaller than the minimal tuple $\langle \psi, \beta_1, ..., \beta_n\rangle$ such that $b = b_{\langle \psi, \beta_1, ..., \beta_n\rangle}$. $<$ well orders A and is ordinal definable. □

2.13. **Lowenheim Skolem theorems.** Recall:

Definition 2.52. Given a language $\mathcal{L}$, a structure M and a substructure N of M, we say that N is an **elementary substructure** of M, denoted $N \prec M$, if for any formula $\phi(x_1, ..., x_n)$ and for any $a_1, ..., a_n \in N$,

$$M \models \phi(a_1, ..., a_n) \iff N \models \phi(a_1, ..., a_n).$$

(By definition, N is a substructure of M if $N \subseteq M$ and the above holds for *atomic* formulas. In the language of set theory, this simply means that N is contained in M and they interpret $\in$ the same way.)

Given the notation introduced above, if M is a model of set theory and N is a submodel of M, then $N \prec M$ if and only if $N \prec_{\Sigma_n} M$ for each n.

Lemma 2.53 (Tarski-Vaught test)**.** Suppose N is a substructure of M. Then $N \prec M$ if and only if for any formula $\phi(x, x_1, ..., x_n)$ and any $a_1, ..., a_n \in N$, if there is a b in M such that $M \models \phi(b, a_1, ..., a_n)$ then there exists a $a \in N$ such that $M \models \phi(a, a_1, ..., a_n)$.

Theorem 2.54 (Using ZFC)**.** Suppose M is a model.

(1) For any $X \subseteq M$ there exists an elementary substructure $N \prec M$ such that $X \subseteq N$ and $|N| = |X| + \aleph_0$. (That is, if X is infinite then $|N| = |X|$ and if X is finite then $|N| = \aleph_0$.)
(2) For any cardinal $\kappa \geq |M|$ there is an elementary extension of M, $M \prec N$ such that $|N| = \kappa$.

Part (1) will be more relevant for us, so we outline the proof.

Proof. We define N as $N = \bigcup_{n<\omega} N_n$ where N_n is defined recusively as follows. Fix some choice function f defined on all non-empty subsets of M. Given a formula $\phi(x, x_1, ..., x_k)$ and $a_1, ..., a_k$ in M let $M(\phi, a_1, ..., a_k) = \{a \in M : M \models \phi(a, a_1, ..., a_k)\}$.

Let $N_0 = X$ and

$$N_{n+1} = N_n \cup \bigcup \{f(M(\phi, a_1, ..., a_k)) : \phi \text{ is a formula and } a_1, ..., a_k \in N_n\}.$$

This is a set, again using the fact that "truth in M" can be defined by a formula, and then using the axiom of replacement.

Now N satisfies the Tarski-Vaught test, and therefore $N \prec M$. □

Remark 2.55. There is a subtlety here. This looks, and is, very similar to the proof of the reflection principle. Why could we not guarantee $V_\alpha \prec V$ there? The point is that truth in V is *not* definable, and we had to limit our interest to Σ_n formulas for a bounded n in order to define truth. However, since M is a set, truth in M is always definable (in a Δ_0 way), for all formulas.

Definition 2.56. Two models M and N are said to be **elementary equivalent**, denoted $N \equiv M$, if for any sentence ϕ, $M \models \phi$ if and only if $N \models \phi$.

Definition 2.57. Two models M and N are **isomorphic** if there is a bijective function $f : M \to N$ such that for any formula $\phi(x_1, ..., x_n)$ and any $a_1, ..., a_n$ in M

$$M \models \phi(a_1, ..., a_n) \iff N \models \phi(f(a_1), ..., f(a_n)).$$

Note that if $N \prec M$ then N and M are elementary equivalent. Also if N and M are isomorphic then they are elementary equivalent.

Recall:

Fact 2.58 (a corollary of the Mostowski collapse theorem)**.** Suppose M is a set and $(M, \in)$ satisfies the axiom of extensionality. Then there is a transitive set N and an isomorphism $\pi : M \to N$. Furthermore, if X is transitive and $X \subseteq M$ then $\pi \restriction X = \mathrm{id}$, that is, $\pi(x) = x$ for any $x \in X$.

Proof. Since $\in$ is well founded, have a rank function on $\in$ in M:

- $x \in M$ is of rank 0 if there is no $y \in M$ such that $y \in x$.
- $x \in M$ is of rank α if for all $y \in M$, if $y \in x$ then y has rank $< \alpha$.

Note that by extensionality there is precisely one set x in M of rank 0, and we define $\pi(x) = \emptyset$. π is defined recursively by $\pi(x) = \{\pi(y) : y \in x\}$.

□

Corollary 2.59. For any transitive set M there is a countable transitive model N which is elementary equivalent to N.

Proof. By Lowenheim Skolem we may find $N' \prec M$ which is countable. Applying the mostowski collapse we find a transitive set $(N, \in)$ which is isomorphic to $(N', \in)$. So N is countable and is elementary equivalent to M. □

Exercise 2.60. Show that if $N \prec V_{\omega_2}$ *and* N is transitive then N is not countable.

Corollary 2.61. The property "κ is a cardinal" is Π_1 but *not* Σ_1.

Proof. We saw that this is a Π_1 property. It remains to show that it is not Σ_1. If it were, then it would be absolute between transitive models of set theory.

Suppose first that we may find some transitive model of ZF, M. Then there is a model N which is countable, transitive, and is elementary equivalent to M. In particular N is also a model of ZF. There is an ordinal κ such that $N \models$ "κ is the first uncountable cardinal". Note that $\kappa > \omega$. Since N is countable and transitive, then κ is in fact a countable ordinal. In particular, κ is *not* a cardinal.

Remark 2.62. What if we cannot find a transitive model satisfying ZF? The proof can be remedied by using the finiteness of proofs. That is, if we could prove, using ZF (or ZFC), that "κ is a cardinal" is a Σ_1 property, this proof would following from some finitely many axioms $\Phi \subseteq$ ZFC. By the reflection principle we may find $V_\alpha \models \Phi$. So we may find a countable transitive model satisfying all axioms in Φ, again leading to the same contradiction.

□

The same proof also shows that "$|X| \leq |Y|$" is not Π_1.

Theorem 2.63 (Levy)**.** Let κ be an uncountable cardinal. Then $H(\kappa) \prec_{\Sigma_1} V$. That is, for any Σ_1 formula $\phi(x_1, ..., x_n)$ and any $a_1, ..., a_n \in H(\kappa)$,

$$\phi^{H(\kappa)}(a_1, ..., a_n) \iff \phi(a_1, ..., a_n).$$

(Note that we are not assuming that κ is regular.)

Proof. Since Σ_1 formulas are "upwards absolute" between transitive models, it suffices to show the implication $\Leftarrow$. Let $\phi(x_1, ..., x_n)$ be $\exists y \psi(y, x_1, ..., x_n)$ where ψ is a Δ_0 formula. Assume that $\phi(a_1, ..., a_n)$ holds, for $a_1, ..., a_n \in H(\kappa)$. Fix b such that $\psi(b, a_1, ..., a_n)$.

Fix α large enough so that $a_1, ..., a_n, b \in V_\alpha$. Since ψ is a Δ_0 formula and V_α is transitive we have $\psi^{V_\alpha}(b, a_1, ..., a_n)$. Let X be the transitive closure of $\{a_1, ..., a_n\}$. Then $|X| < \kappa$. By Lowenhein-Skolem, we may find an elementary substructure $M \prec V_\alpha$ such that $X \cup \{b\} \subseteq M$ and $|M| = |X| + \aleph_0 < \kappa$. Applying Mostowski collapse, there is a transitive set N and an isomorphism $\pi\colon M \to N$. Furthermore, $\pi(x) = x$ for any $x \in X$.

Let $c = \pi(b)$. It follows that $N \models \psi(c, a_1, ..., a_n)$. Finally, $N \subseteq H(\kappa)$, and they are both transitive, so they agree on Δ_0 statements. That is, $H(\kappa) \models \psi(c, a_1, ..., a_n)$, and therefore $H(\kappa) \models \phi(a_1, ..., a_n)$.

□

Exercise 2.64. Show that $V_{\omega_1} \not\prec_{\Sigma_1} V$.

3. Forcing

From now on we will assume that there exists a transitive set M such that $(M, \in)$ is a model of ZFC. (We will discuss later how to avoid this additional assumption. We have seen similar arguments.) By applying Lowenheim-Skolem, and a Mostowski collapse, we may assume that there is a countable transitive set M such that $(M, \in)$ is a model of ZFC.

Remark 3.1. In particular, we may calculate L^M (L as calculated by M). By absoluteness of being L, it follows that L^M is L_α for some ordinal α. So in fact we have a countable ordinal α such that L_α satisfies ZFC.

Let α be the minimal ordinal such that L_α is a model of ZFC. Then in L_α there is no transitive model of ZFC at all!

Again, this is evidence that in order to find a model in which, say, the continuum hypothesis fails, we cannot rely on inner models. Instead we must find a way to go outside a given model of set theory.

We start by just adding *something*, for example, to prove the consistency if $V \neq L$. For example, we will want to take some set $a \subseteq \omega$ which is not in M, and find a model M' such that $M \subseteq M'$, $a \in M'$ and M' is again a model of ZFC. Not any set a would work, and part of the work will be to find a's that do work.

Recall that a set $a \subseteq \omega$ can be naturally identified with its *characteristic function* $\chi_a \colon \omega \to \{0, 1\}$, $\chi_a(k) = 1 \iff k \in a$.

Definition 3.2. An **approximation** to a subset of ω is a function p whose domain is a finite subset of ω, and it take values in $\{0, 1\}$.

We say that an approximation p is **stronger** than q if p extends q as a function. That is, $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and $p(k) = q(k)$ for any $k \in \mathrm{dom}(q)$. (p decides more values of the set a we try to approximate.) In this case we write $p \leq q$ (smaller is stronger).

More generally we will consider arbitrary partially ordered sets $(\mathbb{P}, \leq)$.

Definition 3.3. A set $\mathbb{P}$ is partially ordered by $\leq$ if $\leq$ is a transitive symmetric relation on $\mathbb{P}$. We usually assume that $\mathbb{P}$ has a maximal element (sometimes denoted as $1_{\mathbb{P}}$). We call $(\mathbb{P}, \leq)$ a partially ordered set, or poset.

For example, let $\mathbb{P}$ be the set of all finite approximations to a subset of ω, and define $p \leq q$ as above. The maximal element is the empty function $\emptyset$.

Definition 3.4. Let $(\mathbb{P}, \leq)$ be a partially ordered set. $p, q \in \mathbb{P}$ are **compatible** if there exists $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$.

In our example, p and q are compatible if and only if they agree on their common domain $\mathrm{dom}\, p \cap \mathrm{dom}\, q$, which is true if and only if the set $p \cup q$ is a well defined function whose domain is $\mathrm{dom}\, p \cup \mathrm{dom}\, q$, a finite subset of ω.

Suppose we do find, as above, $M \subseteq M'$ with $a \in M'$. Then, in M', we can define the set of all approximates of a

$$G = \text{all approximations } p \text{ such that } \chi_a \text{ extends } p.$$

Note that we can also define a from G, as follows: $k \in a$ if and only if there is some $p \in G$ such that $k \in \operatorname{dom} p$ and $p(k) = 1$.

Some properties of G:

Definition 3.5. Suppose $(\mathbb{P}, \leq)$ is a partially ordered set and $G \subseteq \mathbb{P}$. Say that G is a **filter** if it satisfies the following.

- If $p \in G$ and $p \leq q$ then $q \in G$;
- If $p, q \in G$ then there is $r \in G$ such that $r \leq p$ and $r \leq q$.

We will always assume that the filter G is not empty, in which case it must be the case that $1_{\mathbb{P}} \in G$.

An additional property that we needed to define a from G is that for any $k \in \omega$ there is some $p \in G$ such that $k \in \operatorname{dom}(p)$. We will in fact add the set a by adding a filter G, and define a as above.

Fix a transitive model M, a partially ordered set $(\mathbb{P}, \leq) \in M$ and a filter $G \subseteq \mathbb{P}$ (most likely not in M!). We now describe a model $M[G]$ extending M and containing G.

3.1. Names.

Definition 3.6. Define recursively on the ordinals the **names** for $\mathbb{P}$ as follows.

- $N_0 = \emptyset$;
- $N_{\alpha+1} = \mathcal{P}(\mathbb{P} \times N_\alpha)$;
- $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ if α is a limite ordinal.

Define N to be the union of all N_α where α is an ordial in M. (A more precise notation would be $N^M(\mathbb{P})$, the class of $\mathbb{P}$-names defined in the model M.) We say that τ is a $\mathbb{P}$-name if it is in N.

For example, $N_1 = \{\emptyset\}$. The members of N_2 are subsets of $\mathbb{P} \times \{\emptyset\}$, that is, they are sets of pairs $(p, \emptyset)$ where $p \in \mathbb{P}$. Generally, a member of $N_{\alpha+1}$ is a set of pairs (p, τ) where τ is in N_α.

Remark 3.7. The collection of all names for $\mathbb{P}$, in M, is a definable class in M. In fact the relation "$X = N_\alpha$" is definable in M in a Δ_2 manner.

Definition 3.8. For a name τ say that its **rank** is α if α is the minimal ordinal such that $\tau \subseteq \mathbb{P} \times N_\alpha$.

3.2. Extension.

Let $G \subseteq \mathbb{P}$ be a filter. We define a model $M[G]$ as follows.

Definition 3.9. Given a filter $G \subseteq \mathbb{P}$ and a $\mathbb{P}$-name τ, we define its **realization** according to G, $\tau[G]$, recursively on the rank as follows.

(1) If has rank 0, $\tau \subseteq \mathbb{P} \times \emptyset = \emptyset$. So $\tau = \emptyset$ and we define $\tau[G] = \emptyset$.
(2) Given τ of rank $\alpha + 1$ define
$$\tau[G] = \{\sigma[G] : \exists p \in G((p, \sigma) \in \tau)\}.$$

We define the model $M[G]$ as the collection of all $\tau[G]$ for $\tau \in N$.

Lemma 3.10. *$M[G]$ is transitive.*

Proof. Fix $y \in x \in M[G]$. Since $x \in M[G]$, there is a name τ such that $x = \tau[G]$. By definition of $\tau[G]$, there is a name σ and a condition $p \in G$ such that $(p, \sigma) \in \tau$ and $y = \sigma[G]$. In particular, $y \in M[G]$. □

It follows that $M[G]$ satisfies the axioms of extensionality and foundation.

Lemma 3.11. $M \subseteq M[G]$.

Proof. For each $x \in M$ we define a **canonical name** $\check{x}$ recursively on rank, as follows.

- $\check{\emptyset} = \emptyset$;
- Fix x and suppose $\check{y}$ was defined for all $y \in x$. By replacement, applied in M, there must be some ordinal β such that for all $y \in x$, $\check{y}$ is of rank $\leq \beta$. Now define $\check{x}$ to be $\{(1_{\mathbb{P}}, \check{y}) : y \in x\}$, a name of rank $\beta + 1$.

Finally, we prove inductively that $\check{x}[G] = x$ for all $x \in M$ (independent of G). Indeed, $\check{\emptyset} = \emptyset$ and by induction

$$\check{x}[G] = \{\sigma[G] : \exists p \in G((p, \sigma) \in \check{x})\} = \{\check{y}[G] : y \in x\} = \{y : y \in x\} = x.$$

□

Lemma 3.12. $G \in M[G]$.

Proof. We define a $\mathbb{P}$-name $\dot{G}$ so that for any filter $G \subseteq \mathbb{P}$, $\dot{G}[G] = G$. Let $\dot{G} = \{(p, \check{p}) : p \in \mathbb{P}\}$. □

Exercise 3.13. Show that if $G \in M$ then $M[G] = M$.

3.3. Some axioms in $M[G]$.

Lemma 3.14. The union axiom holds in $M[G]$.

Proof. Suppose $x \in M[G]$, $x = \tau[G]$. We want to find a name $\tau^\star$ such that $\tau^\star[G]$ is $\bigcup x$. That is, $z \in \tau^\star[G] \iff$ there is $y \in x$ such that $z \in y$.

Note that $y \in x$ if and only if $y = \sigma[G]$ where $(p, \sigma) \in \tau$ for some $p \in G$. For such y, $z \in y$ if and only if $z = \rho[G]$ where $(q, \rho) \in \sigma$ for some $q \in G$.

Define $\tau^\star$ to be the set of pairs (r, ρ) for which there are $q, p \in \mathbb{P}$ and a name σ such that $r \leq p$ and $r \leq q$ and $(q, \rho) \in \sigma$ and $(p, \sigma) \in \tau$. [Note that if τ is of rank β then $\tau^\star$ is also of rank β.]

Finally, we claim that $\tau^\star[G] = \bigcup \tau[G]$. Assume first $z \in \bigcup \tau[G]$. Then $z = \rho[G]$ and there is $y = \sigma[G]$ such that $z \in y \in \tau[G]$. That is, there are conditions $p, q \in G$ such that $(p, \sigma) \in \tau$ and $(q, \rho) \in \sigma$. *Since G is a filter*, there is a condition $r \in G$ which extends both p and q. We see now that (r, ρ) is in $\tau^\star$, and since $r \in G$, $z = \rho[G] \in \tau^\star[G]$.

For the other direction, assume that $z \in \tau^\star[G]$. So $z = \rho[G]$ and there is some $r \in G$ such that $(r, \rho) \in \tau^\star$. By the definition of $\tau^\star$, there are $p, q \in \mathbb{P}$ such that $r \leq p, q$ and a name σ such that $(q, \rho) \in \sigma$ and $(r, \sigma) \in \tau$. *Since G is a filter* and $r \in G$, then both p and q are in G. It follows that $z = \rho[G] \in \sigma[G] \in \tau[G]$, as required. □

Assume $\mathbb{P}$ is the poset of finite approximations of a subset of ω, $a \subseteq \omega$ and $G = G_a$ is the set of all $p \in \mathbb{P}$ such that a extends p. It follows from Lemma 3.12 and the above that $a \in M[G]$, as it can be defined as $a = \bigcup G$.

Exercise 3.15. Show that, for any filter $G \subseteq \mathbb{P}$, $M[G]$ satisfies

(1) the pairing axiom;
(2) the axiom of infinity.

Lemma 3.16. M and $M[G]$ have the same ordinals.

Proof. We saw that $M \subseteq M[G]$, so every ordinal of M is an ordinal in $M[G]$. We prove by induction on the rank of names that if τ is a name of rank β and $\tau[G]$ is an ordinal then $\tau[G] \leq \beta$.

Indeed if $\tau[G]$ is an ordinal then for any $(p, \sigma) \in \tau$, if $p \in G$ then $\sigma[G]$ is an ordinal as well. By the inductive assumption we get in this case that $\sigma[G] < \beta$. So $\tau[G]$ is a set of ordinal, each of which is $< \beta$, thus $\tau[G] \leq \beta$. □

3.4. **A "bad" filter.** Recall that M is a countable transitive model of ZFC. Let $\kappa = M \cap \mathrm{Ord}$, all the ordinals in M. Since M is transitive, κ is an ordinal, and it is countable. Fix a bijection $f \colon \omega \to \kappa$.

We can "code" f as a subset of ω as follows. First, f can be coded as a relation $R \subseteq \omega \times \omega$ defined by $R(i,j) \iff f(i) \in f(j)$. Next, we can code a relation $R \subseteq \omega \times \omega$ as a subset of ω via some injective map $\omega \times \omega \to \omega$. For example, define $k \in a \iff k = 2^i \cdot 3^j$ where $(i,j) \in R$. (Alternatively use a "snake enumeration" of $\omega \times \omega$.)

Let $G = G_a$ be the set of all finite approximations of a. Then the axiom of replacement fails in $M[G]$. To see this, recall first that $G \in M[G]$. As the union axiom holds in $M[G]$, $f = \bigcup G$ is in $M[G]$ as well, where $f \colon \omega \to \{0,1\}$.

Assume towards a contradiction, that the axioms of replacement and comprehension hold in $M[G]$. It follows that that $a \in M[G]$, and therefore $R \in M[G]$. Note that for each $i \in \omega$ we may consider $R \restriction \{j \in \omega : (j,i) \in R\}$, which is a well ordering of ω, isomorphic to $f(i)$. Specifically the Mostowski collapse of $R \restriction \{j \in \omega : (j,i) \in R\}$ is $f(i)$.

Now consider the formula $\phi(n, \alpha)$ saying that α is an ordinal which is the mostowski collapse of $R \restriction \{j \in \omega : (j,n) \in R\}$. Then in $M[G]$, ϕ defines a function with domain ω, yet its range is not bounded, as it includes all the ordinals of $M[G]$.

3.5. **Generic filters.** Let $(\mathbb{P}, \leq)$ be a poset. Given $p \in \mathbb{P}$ define $\mathbb{P}_p = \{q \in \mathbb{P} : q \leq p\}$, the set of all conditions that extend p.

Definition 3.17. A set $D \subseteq \mathbb{P}$ is **open** if for any $p \in D$, if $q \leq p$ then $q \in D$.

In other words, if $p \in D$ then $\mathbb{P}_p \subseteq D$. We can view the sets $\mathbb{P}_p$ is "basic open sets". Then any open set is a union of basic open sets.

Definition 3.18. A set $D \subseteq \mathbb{P}$ is **dense** if for any $p \in \mathbb{P}$ there is $q \leq p$ such that $q \in D$.

Example 3.19. Let $\mathbb{P}$ be the poset of all finite approximations for a subset of ω.

(1) The set of all $p \in \mathbb{P}$ such that the domain of p is equal to some $n \in \omega$, is dense.
(2) The set of all $p \in \mathbb{P}$ such that $3, 5 \in \operatorname{dom} p$ and $p(3) = 1 - p(5)$ is open.
(3) The set of all $p \in \mathbb{P}$ such that there is some $n \in \omega$ for which $n, n+1, n+2 \in \operatorname{dom} p$ and $p(n) = p(n+1) = p(n+2) = 1$, is dense and open.
(4) Fix $n \in \omega$. The set of all D_n of all $p \in \mathbb{P}$ such that there is some $m \in \omega$ for which $p(2^m \cdot 3^n) = 1$, is a dense open set. (That is, the binary relation coded by the subset approximation by p has the pair (m, n) in it.)

Definition 3.20. Fix M and $(\mathbb{P}, \leq) \in M$. A filter $G \subseteq \mathbb{P}$ is **generic** over M if $G \cap D \neq \emptyset$ for any dense open set $D \in M$.

Example 3.21. Let $G = G_a \subseteq \mathbb{P}$ be the "bad" filter from Section 3.4. Fix $n \in \omega$ such that $f(n) = 0$. Then there is no $i \in \omega$ such that $(i, n) \in R$, and so there is no $i \in \omega$ such that $2^i \cdot 3^n \in a$. So $G \cap D_n = \emptyset$ for the dense open set D_n from Example 3.19 part (4).

Exercise 3.22. The intersection of finitely many dense open subsets of $\mathbb{P}$ is dense and open.

Exercise 3.23. Let $\mathbb{P}$ be the poset of all finite approximations for a subset of ω. Suppose $G \subseteq \mathbb{P}$ is generic over M.

(1) Prove that $G \notin M$.
(2) Prove that $\bigcup G$ is a well defined function from ω to $\{0, 1\}$.

Claim 3.24. The following are equivalent:

- $G \subseteq \mathbb{P}$ is generic over M;
- $G \cap D \neq \emptyset$ for any dense set $D \in M$.

Proof. Suppose $D \subseteq \mathbb{P}$ is dense. Let D^* be all conditions q which extend some condition in D. Then D^* is dense and open, and contains D.

Suppose that $G \subseteq \mathbb{P}$ is a generic filter. We must show that it intersects all dense subsets in M. Given a dense set $D \in M$, we know that $G \cap D^* \neq \emptyset$. Fix $q \in G \cap D^*$. By definition there is some $p \in D$ such that $q \leq p$. Since G is a filter, $p \in G$, so $p \in G \cap D$, as desired. $\square$

Lemma 3.25. Let M be a countable transitive model and $\mathbb{P}$ a poset in M. Then there is a filter $G \subseteq \mathbb{P}$ which is generic over M.

Proof. Let $D_0, D_1, D_2, ...$ be an enumeration of all sets $D \in M$ such that $D \subseteq \mathbb{P}$ is dense and open. We prove a stronger version of the lemma: that for every condition $p \in \mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $p \in G$ and G is generic over M.

We define a sequence of conditions in $\mathbb{P}$, $p_0, p_1, ...$ such that $p_0 \geq p_1 \geq p_2 \geq ...$, as follows:

- let $p_0 = p$;
- given p_n, let p_{n+1} be a member of D_n which extends p_n.

This is possible by the density of each D_n.

Finally, let G be the set of all $q \in \mathbb{P}$ such that $q \geq p_n$ for some n. Then G is a filter. For any n, $p_{n+1} \in G \cap D_n$. Thus G intersects all dense open subsets of $\mathbb{P}$ which are in M, and is generic. $\square$

Given a filter G which is generic over M, we will call $M[G]$ a **generic extension** of M.

3.6. **The forcing relation.**

Definition 3.26 (The forcing relation)**.** Let $\phi(x_1, ..., x_n)$ be a formula in the language of set theory, p a condition in $\mathbb{P}$, and $\tau_1, ..., \tau_n$ $\mathbb{P}$-names. Say that p **forces** $\phi(\tau_1, ..., \tau_n)$, written $p \Vdash \phi(\tau_1, ..., \tau_n)$, if for any generic filter $G \subseteq \mathbb{P}$ over M, if $p \in G$ then $M[G] \models \phi(\tau_1[G], ..., \tau_n[G])$.

Example 3.27. For any poset $\mathbb{P}$ and any $p \in \mathbb{P}$, we defined a name $\tau = \dot{G}$ and showed that $\tau[G] = G$ for any filter G. So for any filter G, if p is in G then $\check{p}[G] \in \tau[G]$. Thus $p \Vdash \check{p} \in \tau$.

Our goal is to establish the following fundamental theorem about the forcing relation.

Theorem 3.28 (Truth and definability for the forcing relation)**.** Fix a countable transitive model M and a poset $(\mathbb{P}, \leq)$ in M.

- [Truth] If $G \subseteq \mathbb{P}$ is a generic filter over M, $\phi(x_1, ..., x_n)$ is a formula in the language of set theory, and $\tau_1, ..., \tau_n$ are names for $\mathbb{P}$ in M, then
$$M[G] \models \phi(\tau_1[G], ..., \tau_n[G]) \iff (\exists p \in G) p \Vdash \phi(\tau_1, ..., \tau_n).$$
- [Definability] For any formula $\phi(x_1, ..., x_n)$ in the language of set theory there is a formula $\psi_\phi(\mathbb{P}, \leq, p, z_1, ..., z_n)$ such that for any p and any $\tau_1, ..., \tau_n$
$$p \Vdash \phi(\tau_1, ..., \tau_n) \iff M \models \psi_\phi(\mathbb{P}, \leq, p, \tau_1, ..., \tau_n).$$

From the definition of $\Vdash$, it is not clear immediately that there are many instance of conditions forcing statements. The first clause above shows that in fact *anything* that is true in some generic extension is forced by some condition. The second clause is even more remarkable. The definition of $\Vdash$ takes into consideration all possible generic extensions of M. A process one can certainly not define inside M. However, it turns out to be completely definable in M!

Before proving Theorem 3.28 let us see how it is used. We show that for a model M of ZF, any generic extension $M[G]$ satisfies ZF, and if M satisfies the axiom of choice then so does $M[G]$.

3.6.1. *The axiom of replacement.* Let M, $\mathbb{P}$ be as above and $G \subseteq \mathbb{P}$ a generic filter over M. We show that $M[G]$ satisfies the axiom of replacement. Fix $a_1, ..., a_n \in M[G]$, a formula $\phi(x, y, z_1, ..., z_n)$ and a set $X \in M[G]$ such that
$$M[G] \models (\forall x \in X)(\exists! y)\phi(x, y, a_1, ..., a_n).$$
We must find a set $Y \in M[G]$ such that $M[G] \models (\forall x \in X)(\exists y \in Y)\phi(x, y, a_1, ..., a_n)$.

Fix names $\rho_1, ..., \rho_n$ and τ such that $X = \tau[G]$ and $a_i = \rho_i[G]$. Given $p \in \mathbb{P}$ and a $\mathbb{P}$-name μ, define $\alpha(p, \mu)$ to be the minimal ordinal α such that there is some name η of rank α with $p \Vdash \phi(\mu, \eta, \rho_1, ..., \rho_n)$.

Let β be the rank of τ. Since M satisfies replacement, there is an ordinal α^* such that for any name μ of rank $< \beta$ and for any $p \in \mathbb{P}$, $\alpha(p, \mu) < \alpha^*$. Here we use the definability clause of Theorem 3.28 to argue that the map $p, \mu \mapsto \alpha(p, \mu)$ is definable in M.

Finally, let σ be the name of all pairs (p, η) with $\alpha(p, \eta) < \alpha^*$ such that there is a name μ of rank $< \beta$ and there is a $q \in \mathbb{P}$ such that $p \leq q$, $(q, \mu) \in \tau$ and $p \Vdash \phi(\mu, \eta, \rho_1, ..., \rho_n)$. We claim that $\sigma[G]$ is the set of all y for which there is $x \in \tau[G]$ such that $\phi^{M[G]}(x, y, a_1, ..., a_n)$ holds.

First, if $y \in \sigma[G]$, $y = \eta[G]$ where $p \in G$ and $(p, \eta) \in \sigma$. Then there is a $p \leq q$ and $(q, \mu) \in \tau$ such that $p \Vdash \phi(\mu, \eta, \rho_1, ..., \rho_n)$. Then p, q are both in G, and so $x = \mu[G] \in \tau[G]$ and $\phi^{M[G]}(x, y, a_1, ..., a_n)$ holds.

Conversely, fix $x \in \tau[G]$. Then there is a pair $(q, \mu) \in \tau$ with $q \in G$ and $\mu[G] = x$. In particular μ is of rank $< \beta$. By assumption, there is some $y \in M[G]$ such that $M[G] \models \phi(x, y, a_1, ..., a_n)$. Fix η' such that $y = \eta'[G]$. By Theorem 3.28 there is some condition $p' \in G$ forcing that $\phi(\mu, \eta', \rho_1, ..., \rho_n)$. By choice of α^*, we may find η so that $p' \Vdash \phi(\mu, \eta, \rho_1, ..., \rho_n)$ and $\alpha(p', \eta) < \alpha^*$. Since $p', q \in G$, there is a $p \in G$ with $p \leq p'$ and $p \leq p$. Now $(p, \eta) \in \sigma$, and so $y = \eta[G] \in \sigma[G]$, as required.

3.7. **Proof of the forcing theorem.**

Definition 3.29 (Strong forcing)**.** For a condition $p \in \mathbb{P}$, a sentence ϕ in the forcing language, we define the relation $p \Vdash^* \phi$ (read "p strongly forces ϕ"). More specifically, one can think of $p \Vdash^* \phi$ as "p directly witnesses that ϕ must hold".

First, consider atomic sentences and their negations. We define recursively on the ranks of the names:

(1) $p \Vdash^* \sigma \in \tau$ if there is a $q \in \mathbb{P}$ and a name λ such that $p \leq q$ and $(q, \lambda) \in \tau$ and $p \Vdash^* \lambda = \sigma$;
(2) $p \Vdash^* \sigma \notin \tau$ if there is no $q \leq p$ such that $q \Vdash^* \sigma \in \tau$.
(3) $p \Vdash^* \sigma \neq \tau$ if either
   - there is a $q \in \mathbb{P}$ and a name λ such that $p \leq q$ and $(q, \lambda) \in \sigma$ and $p \Vdash^* \lambda \notin \tau$, or
   - there is a $q \in \mathbb{P}$ and a name λ such that $p \leq q$ and $(q, \lambda) \in \tau$ and $p \Vdash^* \lambda \notin \sigma$, (or both);
(4) $p \Vdash^* \sigma = \tau$ if there is no $q \leq p$ such that $q \Vdash^* \sigma \neq \tau$.

(So, p witnesses that $\sigma = \tau$ if there is no way to extend p and witness that some name is going to be in one set but not the other.)

We continue to define the relation $\Vdash^*$ on all formulas, this time by recursion on the construction of formulas:

(5) $p \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$ if there is no $q \leq p$ such that $q \Vdash^* \phi(\tau_1, ..., \tau_n)$;
(6) $p \Vdash^* \phi(\tau_1, ..., \tau_n) \vee \psi(\tau_1, ..., \tau_n)$ if $p \Vdash^* \phi(\tau_1, ..., \tau_n)$ or $p \Vdash^* \psi(\tau_1, ..., \tau_n)$;
(7) $p \Vdash^* \exists x \phi(x, \tau_1, ..., \tau_n)$ if there is a name σ such that $p \Vdash^* \phi(\sigma, \tau_1, ..., \tau_n)$.

Lemma 3.30. If $p \Vdash^* \phi(\tau_1, ..., \tau_n)$ and $r \leq p$ then $r \Vdash^* \phi(\tau_1, ..., \tau_n)$. That is, if p knows that ϕ will hold, and r has more information than p, then r also knows that ϕ must hold.

Lemma 3.31. For any formula $\phi(x_1, ..., x_n)$ and names $\tau_1, ..., \tau_n$, the set D of all conditions $p \in \mathbb{P}$ such that either $p \Vdash^* \phi(\tau_1, ..., \tau_n)$ or $p \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$, is dense and open in $\mathbb{P}$.

Proof. That D is open follow from the preceding lemma. Now for any condition p, either there is some $q \leq p$ for which $q \Vdash^* \phi(\tau_1, ..., \tau_n)$, in which case $q \in D$, or there is no such q, in which case $p \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$ by definiton, and so $p \in D$. $\square$

Theorem 3.32. Let M be a countable transitive model, $(\mathbb{P}, \leq)$ a poset in M and $G \subseteq \mathbb{P}$ a generic filter over M. For any formula $\phi(x_1, ..., x_n)$ and names $\tau_1, ..., \tau_n$,

$$M[G] \models \phi(\tau_1[G], ..., \tau_n[G]) \iff (\exists p \in G) p \Vdash^* \phi(\tau_1, ..., \tau_n).$$

In particular, if $p \Vdash^* \phi$ then ϕ holds in any generic extension $M[G]$ such that $p \in G$, and so $p \Vdash \phi$. In particular, if $M[G] \models \phi(\tau_1[G], ..., \tau_n[G])$, then there is some $p \in G$ such that $p \Vdash^* \phi(\tau_1, ..., \tau_n)$ and so $p \Vdash \phi(\tau_1, ..., \tau_n)$. So the Truth clause of Theorem 3.28 follows.

Proof. We first prove the theorem for atomic formulas and their negations, by induction on the rank of names. Then for all formulas by induction on the construction of formulas.

(1) Assume first $p \Vdash^* \sigma \in \tau$, then there is a $q \in \mathbb{P}$ and a name λ such that $p \leq q$ and $(q, \lambda) \in \tau$ and $p \Vdash^* \lambda = \sigma$. For any generic filter $G \subseteq \mathbb{P}$ over M, if $p \in G$ then also $q \in G$, and so $\lambda[G] \in \tau[G]$ by definition. Since

the rank of λ is smaller than the rank of τ, it follows from the inductive assumption, since $p \in G$ and $p \Vdash^* \sigma = \lambda$, that $\sigma[G] = \lambda[G]$. We conclude that $\sigma[G] \in \tau[G]$, as required.

Assume now that $G \subseteq \mathbb{P}$ is a generic filter over M and $M[G] \models \sigma[G] \in \tau[G]$. By definition, there is some $(q, \lambda) \in \tau$ with $q \in G$ and $\lambda[G] = \sigma[G]$. Since the rank of λ is smaller than the rank of τ, it follows from the inductive hypothesis that there is some $p' \in G$ such that $p' \Vdash^* \lambda = \sigma$. Since G is a filter, we may find $p \in G$ with $p \leq p'$ and $p \leq q$. Now $p \leq q$, $(q, \lambda) \in \tau$ and $p \Vdash^* \lambda = \sigma$, so $p \Vdash^* \sigma \in \tau$ and $p \in G$, as required.

(2) The proof is the same as in (5), we we assume the theorem holds for ϕ and prove it for $\neg\phi$.

(3) Assume first that $p \in G$ and $p \Vdash^* \sigma \neq \tau$. Assume that there is a condition $q \in \mathbb{P}$ and a name λ such that $p \leq q$ and $(q, \lambda) \in \sigma$ and $p \Vdash^* \lambda \notin \tau$. (In the other case the proof is the same, replacing the roles of σ and τ.) Then $q \in G$, so by the inductive assumption $\lambda[G] \notin \tau[G]$. Furthermore, by definition $\lambda[G] \in \sigma[G]$, so $\sigma[G] \neq \tau[G]$.

Assume now that $G \subseteq \mathbb{P}$ is a generic filter over M and $M[G] \models \sigma[G] \notin \tau[G]$. Assume without loss of generality that there is some $x \in \sigma[G] \setminus \tau[G]$. By definition there is some $(q, \lambda) \in \sigma$ with $q \in G$ and $\lambda[G] = x$. By the inductive assumption, there is some $p' \in G$ such that $p' \Vdash^* \lambda \notin \tau$. Take $p \in G$ with $p \leq p', q$, then $p \Vdash^* \sigma \neq \tau$.

(4) The proof is the same as in (5), we we assume the theorem holds for ϕ and prove it for $\neg\phi$.

(5) Assume that $p \in G$ and $p \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$. Then there is no $q \leq p$ such that $q \Vdash^* \phi(\tau_1, ..., \tau_n)$. It follows that $M[G] \models \neg\phi(\tau_1[G], ..., \tau_n[G])$. Otherwise, $M[G] \models \phi(\tau_1[G], ..., \tau_n[G])$, and so by the inductive assumption there is some $q' \in G$ with $q' \Vdash^* \phi(\tau_1, ..., \tau_n)$. Find $q \in G$ with $q \leq q', p$. This contradicts our assumption.

Now assume $M[G] \models \neg\phi(\tau_1[G], ..., \tau_n[G])$. The set D of all conditions $p \in G$ for which either $p \Vdash^* \phi(\tau_1, ..., \tau_n)$ or $p \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$ is open and dense, and therefore intersects G. By the inductive assumption, if $p \in G$ and $p \Vdash^* \phi(\tau_1, ..., \tau_n)$, then $M[G] \models \phi(\tau_1[G], ..., \tau_n[G])$, which cannot happen. Thus there must be some $p \in G$ with $p \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$, as required.

(6) Left as an exercise.

(7) Left as an exercise.

$\square$

Finally, we see that the difference between forcing and strong forcing is a double negation.

Theorem 3.33. $p \Vdash \phi(\tau_1, ..., \tau_n) \iff p \Vdash^* \neg\neg\phi(\tau_1, ..., \tau_n)$.

Since $\Vdash^*$ was defined, in M, the definability of $\Vdash$ follows immediately.

Proof. Assume first that $p \Vdash^* \neg\neg\phi(\tau_1, ..., \tau_n)$. By Theorem 3.32, if G is a generic filter with $p \in G$ then $M[G] \models \neg\neg\phi(\tau_1[G], ..., \tau_n[G])$, and so $M[G] \models \phi(\tau_1[G], ..., \tau_n[G])$. By definition, $p \Vdash \phi(\tau_1, ..., \tau_n)$.

On the other hand, assume $p \Vdash \phi(\tau_1, ..., \tau_n)$, that is, for any generic filter G with $p \in G$, $M[G] \models \phi(\tau_1[G], ..., \tau_n[G])$. Then there is no $q \leq p$ with $q \Vdash^* \neg\phi(\tau_1, ..., \tau_n)$, as we can always find a generic filter which contains q (and p). By definition, $p \Vdash^* \neg\neg\phi(\tau_1, ..., \tau_n)$. $\square$

It follows from Theorem 3.33 that the forcing relation $\Vdash$ is definable in M, concluding the Definability clause of Theorem 3.28, and so concluding the proof of Theorem 3.28.

3.8. **The remaining ZFC axioms in M[G].**

3.8.1. *The powerset axiom.*

Definition 3.34. Let τ be a $\mathbb{P}$-name. Say that μ is a **canonical name for a subset of** τ if for any $(p, \rho) \in \mu$ there is a $q \in \mathbb{P}$ with $p \leq q$ and $(q, \rho) \in \tau$.

Exercise 3.35. If μ is a canonical name for a subset of τ then for *any* filter G, $\mu[G] \subseteq \tau[G]$.

Note that if μ is a canonical name for a subset of τ then the rank of μ is smaller or equal to the rank of τ.

Lemma 3.36. Suppose τ, μ are $\mathbb{P}$-names, $G \subseteq \mathbb{P}$ is a generic filter and $\mu[G] \subseteq \tau[G]$. Then there is a canonical name for a subset of τ, μ^*, such that $\mu^*[G] = \mu[G]$.

Proof. Define μ^* as the set of all pairs (q, ρ) such that there is some $q \leq p$ with $(p, \rho) \in \tau$ and such that $q \Vdash \rho \in \mu$. The definability clause of Theorem 3.28 allows us to define μ^* in M. Furthermore, by definition μ^* is a canonical name for a subset of τ.

It remains to show that $\mu[G] = \mu^*[G]$. Suppose first $x \in \mu^*[G]$. Then $x = \rho[G]$ where $(q, \rho) \in \mu^*$ for some $q \in G$. By definition, there is some $p \in \mathbb{P}$ such that $q \leq p$, $(p, \rho) \in \tau$ and $q \Vdash \rho \in \mu$. We conclude that $x = \rho[G] \in \mu$.

For the converse, assume that $x \in \mu[G]$. By assumption, $x \in \tau[G]$, so there is some name ρ such that $\rho[G] = x$ and $(p, \rho) \in \tau$ for some $p \in G$. Since $\rho[G] \in \mu[G]$, it follows from the truth clause of Theorem 3.28 that there is some $q' \in G$ such that $q' \Vdash \rho \in \mu$. Since G is a filter, we may find $q \in G$ such that $q \leq q'$ and $q \leq p$. Now $(q, \rho) \in \mu^*$ and so $x = \rho[G] \in \mu^*[G]$, as required. $\square$

Lemma 3.37. Suppose M is a countable transitive model of ZF, $(\mathbb{P}, \leq)$ is a poset in M and $G \subseteq \mathbb{P}$ is a generic filter over M. Then $M[G]$ satisfies the powerset axiom.

Proof. Fix a name τ. We must find a name σ such that $M[G] \models \sigma[G] = \mathcal{P}(\tau[G])$. Define σ to be the name of all pairs $(1_{\mathbb{P}}, \mu)$ where μ is a canonical name for a subset of τ. Note that if τ has rank β then σ has rank $\beta + 1$.

Note that $\sigma[G]$ is equal to the set of all $\mu[G]$ where μ is a canonical name for a subset of τ. It follows that and member of $\sigma[G]$ is contained in $\tau[G]$.

Finally, suppose $x \subseteq \tau[G]$ and $x \in M[G]$. Then $x = \mu[G]$ for some name μ. By the lemma, there is a canonical name for a subset of τ, μ^* such that $\mu^*[G] = M[G] = x$. Thus $x \in \sigma[G]$. $\square$

3.8.2. *The axiom of choice.* We note first that in the proofs so far it was not necessary for M to satisfy the axiom of choice.

Lemma 3.38. Suppose M is a countable transitive model of ZF, $(\mathbb{P}, \leq)$ is a poset in M and $G \subseteq \mathbb{P}$ is a generic filter over M. If M also satisfies the axiom of choice, then so does $M[G]$.

Proof. To prove that the axiom of choice holds, it suffices to show that for each ordinal α there is a well ordering of V_α. Let $M[G]_\alpha$ be $V_\alpha^{M[G]}$, that is, V_α as calculated in $M[G]$.

By Lemma 3.36 the map $\tau \mapsto \tau[G]$ from N_α^M (the set of names of rank α in M) to $M[G]_\alpha$ is surjective. Furthermore, this map is in $M[G]$. Since M satisfies the axiom of choice, then there is a well ordering of N_α^M.

Finally, recall that given a set X and a well ordering of X, and a surjective map $g\colon X \to Y$, we may injectively map Y into X by sending $y \in Y$ to the minimal element in $g^{-1}(y)$. Furthermore, this injective map from Y to X induces a well ordering of Y from the well ordering on X. $\square$

Corollary 3.39. Assuming that ZF is consistent, then ZFC+$V \neq L$ is consistent is well. In other words, ZFC does not prove that every set is construcible.

Proof. As we have seen, we may find a countable transitive model M satisfying ZFC and $V = L$. (If we somehow find such M that satisfies $V \neq L$, then we are done.) Let $\mathbb{P}$ be the poset for all finite approximations of a subset of ω, and fix a filter $G \subseteq \mathbb{P}$ which is generic over M.

By Exercise 3.23 the generic filter G is not in M. Furthermore, as M and $M[G]$ have the same ordinals, $L^{M[G]} = L^M = M$. More specifically: for each ordinal α in M, M thinks that "L_α^M is L_α". Since the latter is a Δ_1 statement, and M and $M[G]$ are transitive models, it is true in $M[G]$ as well. So $M[G]$ thinks that "L_α^M is L_α".

So in $M[G]$ there is a set G such that $M[G] \models G \notin L$, so $M[G]$ thinks that $V \neq L$. (Equivalently, G defines a characteristic function $f_G = \bigcup G$, which defines a subset of ω, $a = \{n \in \omega :\ f_G(n) = 1\}$, and $a \notin L$.) $\square$

The extension $M[G]$ above, where $\mathbb{P}$ is the poset of finite approximations to a subset of ω and $G \subseteq \mathbb{P}$ is generic over M, is often called a "generic extension by a single Cohen subset of ω", and the poset $\mathbb{P}$ is called "Cohen forcing". In $M[G]$ we have the set $a = \{n \in \omega :\ (\exists p \in G)p(n) = 1\}$, often called the "generic Cohen set".

We show that this set a, or alternatively the filter G, are *not* ordinal definable in $M[G]$. In fact, the model $\mathrm{HOD}^{M[G]}$ (HOD as calculated in the model $M[G]$), is contained in M. (So if $M \models$"V=L" then $\mathrm{HOD}^{M[G]} = L^{M[G]} = L^M = M$.

Definition 3.40. Let $(\mathbb{P}, \leq^{\mathbb{P}})$, $(\mathbb{Q}, \leq^{\mathbb{Q}})$ be posets. Say that $f\colon \mathbb{P} \to \mathbb{Q}$ is an **isomorphism** if it is a bijection and satisfies

$$p \leq^{\mathbb{P}} q \iff f(p) \leq^{\mathbb{Q}} f(q),$$

for any $p, q \in \mathbb{P}$.

Note that if f is such an isomorphism then $f(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$. Furthermore f^{-1} is an isomorphism from $\mathbb{Q}$ to $\mathbb{P}$.

Lemma 3.41. Suppose $f\colon \mathbb{P} \to \mathbb{Q}$ is an isomorphism, $f, \mathbb{P}, \mathbb{Q} \in M$, and $G \subseteq \mathbb{P}$ is generic over M. Define $H = \{f(p) : p \in G\}$. Then $H \subseteq \mathbb{Q}$ is generic over M and $M[G] = M[H]$.

Proof. Fix a dense subset $D \subseteq \mathbb{Q}$ with $D \in M$. First we show that $f^{-1}(D) \subseteq \mathbb{P}$ is dense. Fix some $p \in \mathbb{P}$. By assumption there is a $q \leq f(p)$ with $q \in D$. Now $p' = f^{-1}(q) \in D$ and $p' \leq p$ since f is an isomorphism.

Since f is in M then $f^{-1}(D)$ is in M as well and so there is some $p \in G \cap f^{-1}(D)$. Now by definition $f(p) \in H \cap D$, as required.

Finally, note that $H \in M[G]$, since it is defined from f and G, and similarly $G \in M[H]$. The following exercise therefore concludes the proof.

Exercise 3.42. Let M be a countable transitive model, $(\mathbb{P}, \leq)$ a poset in M and $G \subseteq \mathbb{P}$ generic over M. Suppose N is a transitive model of ZF such that $M \subseteq N$ and $G \in N$. Then $M[G] \subseteq N$.

$\square$

Lemma 3.43. Suppose $X \in M[G]$, $X \subseteq M$ and X is ordinal definable in $M[G]$. Then $X \in M$.

Proof. By assumption, there are ordinals $\alpha_1, ..., \alpha_n$, a formula $\phi(x, z_1, ..., z_n)$ such that
$$x \in X \iff \phi^{M[G]}(x, \alpha_1, ..., \alpha_n).$$
By assumption also any $x \in X$ is in M.

Claim 3.44. For any formula ψ and $x_1, ..., x_n \in M$, if $p \Vdash \psi(\check{x}_1, ..., \check{x}_n)$ then $\emptyset \Vdash \psi(\check{x}_1, ..., \check{x}_n)$ (where $\emptyset$ is $1_{\mathbb{P}}$ here).

Proof. Assume towards a contradiction that $\emptyset$ does not force $\psi(\check{x}_1, ..., \check{x}_n)$. Then there is some condition $q \in \mathbb{P}$ such that $q \Vdash \neg\psi(\check{x}_1, ..., \check{x}_n)$ (important step!)

Find n large enough such that both $\operatorname{dom} p$ and $\operatorname{dom} q$ are contained in $n = \{0, ..., n-1\}$. Extend p and q to $p' \leq p$ and $q' \leq q$ with $\operatorname{dom} p' = \operatorname{dom} q' = n$. Let $\Delta = \{k < n : q'(k) \neq p'(k)\} \subseteq n$.

Define a map $f : \mathbb{P} \to \mathbb{P}$ as follows. Given $r \in \mathbb{P}$, $\operatorname{dom}(f(r)) = \operatorname{dom} r$ and for any $k \in \operatorname{dom} r$

- if $k \in \Delta$, $f(r)(k) = 1 - r(k)$ and
- if $k \notin \Delta$, $f(r)(k) = r(k)$.

Then f is a bijection (f is its own inverse) and it is an isomorphism from $\mathbb{P}$ to $\mathbb{P}$ (which we call an automorphism of $\mathbb{P}$). Note also that $f(p') = q'$

Now fix a generic filter $G \subseteq M$ such that $p' \in G$. Let $H = \{f(r) : r \in G\}$. Then $q' \in H$ and so $q \in H$. Applying the forcing theorem twice we conclude

- $M[G] \models \psi(x_1, ..., x_n)$, as $p \in G$ and $p \Vdash \psi(\check{x}_1, ..., \check{x}_n)$;
- $M[H] \models \neg\psi(x_1, ..., x_n)$, as $q \in H$ and $q \Vdash \neg\psi(\check{x}_1, ..., \check{x}_n)$.

Finally, this is a contradiction as $M[G] = M[H]$. $\square$

Back to the proof of the lemma: if $x \in X$ then there is some $p \in G$ forcing $\phi(\check{x}, \check{\alpha}_1, ..., \check{\alpha}_n)$ (by the forcing theorem), and so $\emptyset \Vdash \phi(\check{x}, \check{\alpha}_1, ..., \check{\alpha}_n)$. Also, if $x \in M$ and $x \notin X$, then there is some $p \in G$ forcing $\neg\phi(\check{x}, \check{\alpha}_1, ..., \check{\alpha}_n)$ (by the forcing theorem), and so $\emptyset \Vdash \neg\phi(\check{x}, \check{\alpha}_1, ..., \check{\alpha}_n)$.

In conclusion, X can be defined as
$$\{x : \emptyset \Vdash \phi(\check{x}, \check{\alpha}_1, ..., \check{\alpha}_n)\},$$
and this set is definable in M as $\Vdash$ is definable in M.

[Of course, to apply comprehension we need to bound all the x's in a single set in M. As always, consider the map $X \ni x \mapsto$ minimal α such that $x \in V_\alpha^M$. This is definable in $M[G]$, and the domain is a set X, so by replacement there must be a bound. In other words, if $X \subseteq M$ then $X \subseteq M_\alpha = V_\alpha^M$ for some ordinal α. Now define in M the set X as $\{x \in M_\alpha : \emptyset \Vdash \phi(\check{x}, \check{\alpha}_1, ..., \check{\alpha}_n)\}$, using comprehension.]

□

4. Independence of the Continuum Hypothesis

In the generic extension $M[G]$ discussed so far, adding "a Cohen subset $a \subseteq \omega$", we added of course many different new subsets of ω (e.g. $\omega \setminus a$). It turns out however that if M satisfies the continuum hypothesis, then so would $M[G]$. To find a model in which the continuum hypothesis fails we will add *many* subsets of ω.

Definition 4.1. For an ordinal κ let $\mathbb{P}_\kappa$ be the poset of all finite function $p\colon \operatorname{dom} p \to \{0,1\}$ with $\operatorname{dom} p \subseteq \kappa \times \omega$. Say that $p \leq q$ if p extends q as a function. That is, $\operatorname{dom} q \subseteq \operatorname{dom} p$ and $p \restriction \operatorname{dom} q = q$.

The idea is that for each $\alpha < \kappa$, $p(\alpha, \cdot)\colon \omega \to \{0,1\}$ codes a new subset of ω, and we add κ of those this time.

Lemma 4.2. Suppose M is a countable transitive model, $\kappa \in M$ and $G \subseteq \mathbb{P}_\kappa$ is generic over M. Then $f = \bigcup G$ is a well defined function from $\kappa \times \omega$ to $\{0,1\}$. For $\alpha < \kappa$, let $x_\alpha = \{k \in \omega : \ f(\alpha, k) = 1\}$. Then for $\alpha \neq \beta < \kappa$, $x_\alpha \neq x_\beta$.

Proof. This is a "standard density argument". First note that for each $n \in \omega$ and $\alpha < \kappa$ the set of all conditions $p \in \mathbb{P}_\kappa$ for which $(\alpha, n) \in \operatorname{dom} p$ is a dense subset of $\mathbb{P}_\kappa$. So by genericity there is some $p \in G$ with $(\alpha, n) \in \operatorname{dom} p$, so $(\alpha, n) \in \operatorname{dom} f$. Furthermore, f is a well defined function since G is a filter.

Finally, for any $\alpha \neq \beta < \kappa$, the set of all $p \in \mathbb{P}_\kappa$ for which there is some m such that $(\alpha, m) \in \operatorname{dom} p$ and $(\beta, m) \in \operatorname{dom} p$ and $p(\alpha, m) \neq p(\beta, m)$, is dense in $\mathbb{P}_\kappa$. It follows that for each $\alpha \neq \beta$ there is some m and $p \in G$ as above, and therefore $m \in x_\alpha \iff m \notin x_\beta$. □

We conclude that, by forcing with $\mathbb{P}_\kappa$, κ many distinct subsets of ω are added. So, if we let $\kappa = \omega_2^M$ and force with $\mathbb{P}_\kappa$, we will get a model in which there are ω_2 many subsets of ω, so CH fails? Not so fast!

We know that M and $M[G]$ have the same ordinals, but in general they do not necessarily agree on the cardinals. For example, we know that ω_2^M is just a countable ordinal, so maybe we added a countable enumeration of it to $M[G]$. In this case, the new "ω_2-many" subsets of ω that we added, in $M[G]$, would just be countably many subsets of ω...

Example 4.3. Let κ be a cardinal. Let $\mathrm{Col}(\omega, \kappa)$ be all finite functions $p\colon \operatorname{dom} p \to \kappa$ where $\operatorname{dom} p \subseteq \omega$ is finite. We consider $\mathrm{Col}(\omega, \kappa)$ as a poset with the partial order $p \leq q$ if p extends q as a function.

Exercise 4.4. Let M be a countable transitive model, κ a cardinal in M (for example, $\kappa = \omega_7^M$), and suppose $G \subseteq \mathrm{Col}(\omega, \kappa)$ is a generic filter over M. Then $f = \bigcup G$ is a well defined function and $f\colon \omega \to \kappa$ is surjective. It follows that κ is countable in $M[G]$.

For example, if $\kappa = \omega_1^M$, then $M[G] \models$"κ is a countable ordinal, that is, $\kappa < \omega_1$. There is some other ordinal $\lambda = \omega_1^{M[G]}$. Necessarily in M: "λ is a cardinal greater than $\aleph_1$".

4.1. **Chain condition.** The concept of cardinality, and of being a cardinal, is not preserved between generic extensions in general. Nevertheless, Cohen's forcing to add (many) subsets of ω is "mild" enough, and in fact does preserve cardinals. The central notion is that of anti-chains.

Fix a poset $\mathbb{P}$. Recall that two conditions $p, q \in \mathbb{P}$ are **compatible** if there is some $r \in \mathbb{P}$ with $r \leq p$ and $r \leq q$. In this case we write $p \parallel q$. p and q are **incompatible** if there is no $r \in \mathbb{P}$ with $r \leq p$ and $r \leq q$. In this case, there can be no filter G containing both p and q, and we write $p \perp q$.

Definition 4.5. Let $\mathbb{P}$ be a poset and $A \subseteq \mathbb{P}$.

- Say that A is an **antichain** if for any $p, q \in A$, if $p \neq q$ then p and q are not compatible.
- An antichain $A \subseteq \mathbb{P}$ is **maximal** if for any $p \in \mathbb{P}$ there is some $q \in A$ such that p and q are compatible.

Recall that, assuming the axiom of choice, for any antichain $A \subseteq \mathbb{P}$, there is a maximal antichain $A \subseteq A' \subseteq \mathbb{P}$.

In fact, the arguments in this section, involving chain conditions, do rely on the model M to satisfy the axiom of choice.

Definition 4.6. Say that a poset $\mathbb{P}$ satisfies the **countable chain condition** ("$\mathbb{P}$ is c.c.c.") if all of its antichains are at most countable. (That is, there are no uncountable antichains in $\mathbb{P}$.)

Example 4.7. The poset $\mathbb{P} = \mathbb{P}_\omega$ of finite approximation for a subset of ω satisfies the countable chain condition, since $\mathbb{P}$ is countable.

Example 4.8. Let $\mathbb{P}$ be the set of all non-empty intervals (α, β) where $\alpha < \beta \in \mathbb{R}$. Consider $\mathbb{P}$ as a poset with the partial order $p \leq q$ if $p \subseteq q$. Then $\mathbb{P}$ satisfies the countable chain condition.

Theorem 4.9. Suppose M is a countable transitive model of ZFC, $(\mathbb{P}, \leq)$ is a poset in M, and $M \models$"$\mathbb{P}$ is c.c.c.". Then M and $M[G]$ agree on cardinals.

Proof. We need to show that for any ordinal $\kappa \in M$, $M \models$"κ is a cardinal" if and only if $M[G] \models$"κ is a cardinal". The $\Leftarrow$ implication is immediate, as Π_1 statements are downwards-absolute.

Assume now κ is a cardinal in M. Suppose $f \in M[G]$, $f \colon \alpha \to \kappa$ for some ordinal $\alpha < \kappa$. We need to show that f is not surjective. Fix a name τ such that $\tau[G] = f$. For each $\beta < \alpha$, let $R_\beta = \{\zeta < \kappa : (\exists p \in \mathbb{P}) p \Vdash \tau(\beta) = \zeta \wedge \tau \text{ is a function}\}$.

Claim 4.10. R_β is countable

Proof. Using the axiom of choice, in M, choose a sequence $\langle p_\zeta : \zeta \in R_\beta \rangle$ of conditions such that $p_\zeta \Vdash \tau(\beta) = \zeta$. Note that $\{p_\zeta : \zeta \in R_\beta\}$ is an antichain of $\mathbb{P}$, and so must be countable by assumption. $\square$

Define (in M) $R = \bigcup_{\beta<\alpha} R_\beta$. R is the union of α many countable sets, and so its size is at most $|\alpha| \times \aleph_0 = |\alpha|$ (assume α is infinite). Since κ is a cardinal in M, there is some $\zeta \in \kappa \setminus R$.

Note that for any $p \in \mathbb{P}$ and any $\beta < \alpha$, p does not force $\tau(\beta) = \zeta$. In particular, no $p \in G$ forces that, so $f(\beta) \neq \zeta$ for all $\beta < \alpha$. That is, f is not surjective. $\square$

Exercise 4.11 (ZFC)**.** Suppose M is a countable transitive model of ZFC, $(\mathbb{P}, \leq)$ is a poset in M, and $M \models$"$\mathbb{P}$ is c.c.c.". Then M and $M[G]$ agree on cofinalities. That is, if $M[G]$ thinks κ is a cardinal with cofinality λ, then M thinks that the cofinality of κ is λ as well.

Exercise 4.12. Suppose κ is an uncountable cardinal. Find an uncountable antichain $A \subseteq \mathrm{Col}(\omega, \kappa)$.

Theorem 4.13 (ZFC)**.** Fix a cardinal κ. Let $\mathbb{P}_\kappa$ be the poset of all finite function $p\colon \operatorname{dom} p \to \{0,1\}$ with $\operatorname{dom} p \subseteq \kappa \times \omega$. Then $\mathbb{P}_\kappa$ satisfies the countable chain condition.

Proof. Suppose $A \subseteq \mathbb{P}_\kappa$ is an uncountable antichain. Fix a regular cardinal $\theta > \kappa$ for example, $\theta = \kappa^+$). Apply Downwards LS to find a countable elementary submodel $M \prec H(\theta)$ such that $\kappa \in M$, $\mathbb{P}_\kappa \in M$, and $A \in M$. [M is not transitive!]

For any $p \in A$ consider $p \cap M$, which is the function p restricted to the domain $\operatorname{dom} p \cap M$. Note that $p \cap M$ is in M (any finite subset of M is in M). Since M is countable and A uncountable, there is a condition $p \in A \setminus M$. Let $r = p \cap M$. Note that $p \neq r$ and $r \in M$.

Now $H(\theta) \models$ "$\exists t(t \in A \wedge t \leq r)$" (this is a statement about A and r). Since $M \prec H(\theta)$, $A, r \in M$, $M \models$"$\exists t(t \in A \wedge t \leq r)$" as well. Fix $p' \in M$ such that $M \models p' \in A$ and $p' \leq r$. Note that this p' is in fact in A, and extends r as a function.

Since $\operatorname{dom} p'$ is finite and in M, then $\operatorname{dom} p' \subseteq M$. So the only common domain of p and p' is $\operatorname{dom} r$, on which they both agree. So p and p' are compatible, contradicting the assumption that A is an antichain. $\square$

A collection $\mathcal{F}$ of sets is called a Δ-system if there is a set r such that for any $x, y \in \mathcal{F}$, $x \cap y = r$.

Lemma 4.14 (The Δ-system lemma)**.** Suppose $\mathcal{F}$ is a set of finite sets and $\mathcal{F}$ is uncountable. Then there is a $\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathcal{F}'$ is uncountable and $\mathcal{F}'$ forms a Δ-system.

Exercise 4.15.
(1) Prove the Δ-system lemma.
(2) Use the Δ-system lemma to prove that $\mathbb{P}_\kappa$ satisfies is c.c.c.

4.2. **The precise size of the continuum.** We have shown that in a generic extension by $\mathbb{P}_\kappa$, $M[G]$ thinks that there are at least κ-many subsets of ω, and the cardinals in M and in $M[G]$ are the same. In particular, if $\kappa = \omega_2^M$ then $M[G]$ thinks "there are at least $\aleph_2$-many subsets of ω, and so CH fails. Next we show that we did not add "too many" subsets of ω, and in this model in fact $|\mathcal{P}(\omega)| = \aleph_2$.

Lemma 4.16 (ZFC)**.** Let M be a transitive model, $(\mathbb{P}, \leq) \in M$, and $G \subseteq \mathbb{P}$ a filter. The following are equivalent.

- G is generic over M;
- For any maximal antichain $A \subseteq \mathbb{P}$ in M, $G \cap A \neq \emptyset$.

Proof. Given a maximal antichain $A \subseteq \mathbb{P}$, let D_A be all $p \in \mathbb{P}$ with $p \leq a$ for some $a \in A$. Then $D_A \subseteq \mathbb{P}$ is dense: for any $q \in \mathbb{P}$ there is some $a \in A$ with $q \parallel a$. By definition, there is some $p \in \mathbb{P}$ with $p \leq a$ and $p \leq q$. So $p \in D_A$.

Assume now G is generic and $A \in M$ is a maximal antichain. Then $D_A \in M$ is dense, so there is some $p \in G \cap D_A$. By definition, there is some $a \in A$ with $p \leq a$. Therefore $a \in G$, and so $G \cap A \neq \emptyset$.

Assume now that the second clause holds and $D \in M$ is some dense open set. Let $A \subseteq D$ be a maximal antichain. That is, $A \subseteq D$ is an antichain and any $p \in D$ is compatible with some $a \in A$. (Such A exists by Zorn's lemma.) We claim that A is a maximal antichain in $\mathbb{P}$. For any $q \in \mathbb{P}$, there is some $p \in D$ with $p \leq q$. It follows that $p \parallel a$ for some $a \in A$, and therefore $q \parallel a$. Finally, by assumption $A \cap G \neq \emptyset$, so $D \cap G \neq \emptyset$, as $A \subseteq D$. □

Given a formula $\phi(x_1, ..., x_n)$ and $\mathbb{P}$-names $\tau_1, ..., \tau_n$, say that $p \in \mathbb{P}$ **decides** $\phi(\tau_1, ..., \tau_n)$ if either $p \Vdash \phi(\tau_1, ..., \tau_n)$ or $p \Vdash \neg\phi(\tau_1, ..., \tau_n)$. Recall that the set $D_{\phi(\tau_1,...,\tau_n)}$ of all $p \in \mathbb{P}$ such that p decides $\phi(\tau_1, ..., \tau_n)$ is dense and open in $\mathbb{P}$. So if A is a maximal antichain of conditions which decide $\phi(\tau_1, ..., \tau_n)$, then A is maximal in $\mathbb{P}$.

Corollary 4.17. Let $A \in M$ be a maximal antichain among all conditions that force $\phi(\tau_1, ..., \tau_n)$. If G is generic over M then

$$M[G] \models \phi(\tau_1[G], ..., \tau_n[G]) \iff G \cap A \neq \emptyset.$$

Proof. The implication $\Leftarrow$ is clear. For the other direction, note that there is some $p \in G$ which decides $\phi(\tau_1, ..., \tau_n)$. So if $G \cap A = \emptyset$, it must be that $p \Vdash \neg\phi(\tau_1, ..., \tau_n)$, and so $M[G] \models \neg\phi(\tau_1[G], ..., \tau_n[G])$. □

Theorem 4.18 (ZFC)**.** Suppose $\lambda \in M$ is a cardinal. In M: define F to be the set of all functions f with $\operatorname{dom} f = \lambda$ and with the range of f being an antichain in $\mathbb{P}$. Then in $M[G]$

$$2^\lambda \leq |F|.$$

Note that the $|F|$ above is calculated in $M[G]$ and, as ordinals, $|F|^{M[G]} \leq |F|^M$.

Proof. Say that a $\mathbb{P}$-name τ is a **simple** name for a subset of λ if

- $\tau \subseteq \mathbb{P} \times \left\{\check{\beta} : \beta < \lambda\right\}$ and
- for any $\beta < \lambda$ the set A_β of all $p \in \mathbb{P}$ with $(p, \check{\beta}) \in \tau$, is an antichain in $\mathbb{P}$.

Note that if τ is a simple name for a subset of λ then τ is in particular a canonical name for a subset of $\check{\lambda}$, and $\tau[G] \subseteq \lambda$ for any filter G.

Given $f \in F$ it corresponds to a simple name $\tau = \left\{(p, \check{\beta}) : p \in f(\beta)\right\}$. Furthermore, any simple name for a subset of λ is of this form.

Claim 4.19. If $G \subseteq \mathbb{P}$ is generic over M, $x \in M[G]$ is a subset of λ, then there is a simple name τ such that $x = \tau[G]$.

Proof. Let $x = \sigma[G]$. For each $\beta < \lambda$, choose a maximal antichain A_β among all conditions that force $\beta \in \sigma$. Let $\tau = \left\{(p, \check{\beta}) : p \in A_\beta\right\}$.

As we have seen above, $\beta \in x = \sigma[G]$ if and only if $G \cap A_\beta \neq \emptyset$. It follows that $x = \tau[G]$. □

In M there is a bijetion $f \mapsto \tau_f$ between F and simple names for subsets of λ. In $M[G]$ the map $f \mapsto \tau_f[G]$ is onto $\mathcal{P}(\lambda)$ and therefore $2^\lambda \leq |F|$.

□

Let us go back to a generic extension $M[G]$ of M by $\mathbb{P}_\kappa$. Note that $|\mathbb{P}_\kappa| = \kappa$. Furthermore, $\mathbb{P}_\kappa$ satisfies the c.c.c., so there are at most $|\mathbb{P}_\kappa|^{\aleph_0} = \kappa^{\aleph_0}$ many antichains in $\mathbb{P}_\kappa$. Consider F as above with $\lambda = \aleph_0$. We conclude that, in M $|F| \leq (\kappa^{\aleph_0})^{\aleph_0} = \kappa^{\aleph_0 \times \aleph_0} = \kappa^{\aleph_0}$.

Suppose M satisfies CH and $\kappa = \aleph_2^M$. Then in M, $\aleph_2^{\aleph_0} = \aleph_2$. [Recall that assuming CH, $\aleph_1^{\aleph_0} = \aleph_1$. Now since any function from ω to ω_2 is bounded below ω_2, we get $|\omega_2^\omega| \leq |\bigcup_{\gamma<\omega_2} \gamma^\omega| = \sum_{\gamma<\omega_2} |\gamma|^{\aleph_0} = \sum_{\gamma<\omega_2} \aleph_1^{\aleph_0} = \sum_{\gamma<\omega_2} \aleph_1 = \aleph_2$.] Finally, $|F| \leq \aleph_2$ in M, and so $|F| \leq \aleph_2$ in $M[G]$ as well. We conclude that $|2^{\aleph_0}| = \aleph_2$ in $M[G]$.

More generally:

Claim 4.20. Assume GCH. Then for any cardinal λ,

$$\lambda^{\aleph_0} = \begin{cases} \lambda & \mathrm{cf}\lambda > \aleph_0; \\ \lambda^+ & \mathrm{cf}\lambda = \aleph_0. \end{cases}$$

Proof. Note that if $\mathrm{cf}\lambda = \aleph_0$ then $\lambda^+ \leq \lambda^{\aleph_0} \leq (2^\lambda)^{\aleph_0} = 2^\lambda = \lambda^+$, so the claim follows. For λ with $\mathrm{cf}\lambda > \aleph_0$ We prove the claim by induction. If $\mathrm{cf}\lambda > \aleph_0$ then any function from ω to λ is bounded below λ. Therefore

$$\lambda^{\aleph_0} = |\lambda^\omega| = |\bigcup_{\gamma<\lambda} \gamma^\omega| \leq \sum_{\gamma<\lambda} \gamma^{\aleph_0} \leq \sup_{\gamma<\lambda} \gamma^+ \leq \lambda.$$

□

Assume now that M satisfies the GCH (for example, if M satisfies "V=L"). Let κ be a cardinal in M such that $\mathrm{cf}\kappa > \omega$ and G be generic over M for $\mathbb{P}_\kappa$. Let F be the set of all functions with domain ω whose range is an antichain in $\mathbb{P}_\kappa$. Then, in M, $|F| = |(\kappa^\omega)^\omega| = \kappa^{\aleph_0} = \kappa$. Therefore in $M[G]$, $|F| \leq \kappa$. Finally, in $M[G]$ we conclude that $2^{\aleph_0} \geq \kappa$ and also that $2^{\aleph_0} \leq \kappa$, and so $2^{\aleph_0} = \kappa$.

So for the value of the continuum can be any cardinal with cofinality $> \omega$. This is optimal, as $\mathrm{cf}2^{\aleph_0} > \omega$.

5. A change of notation / perspective

As we noted many times, all the "action" in the forcing arguments happen in the countable transitive model M. That is, the names are defined in M and forcing relation is defined in M.

For example, we proved that after forcing with $\mathbb{P} = \mathbb{P}_{\omega_2^M}$ we get a model with the failure of CH. In other words, we proved that $1_\mathbb{P} \Vdash \neg$CH, and the latter holds in M. The only place where we needed M to be a small countable model was to find an actual generic filter. However, even in M, where such filters do not exists, it knows that $1_\mathbb{P} \Vdash \neg$CH, so it knows about the consistency of the negation of CH. In other words, M can "imagine" the world $M[G]$ where ZFC+$\neg$CH holds, even if in M there are no means to construct such a model.

If we take $\mathbb{P} = \mathbb{P}_{\omega_2}$ (in the universe V), then we know that $1_\mathbb{P} \Vdash \neg$CH. Formally, this can only be expressed in terms of the strong forcing relation. This time we cannot find filters which are generic over V. However, we will imagine such filter and consider the statement $1_\mathbb{P} \Vdash \neg$CH as "if $G \subseteq \mathbb{P}$ is a filter which is generic over V, then CH fails in $V[G]$". Similarly, we proved that $1_\mathbb{P} \Vdash$"the universe is not L", which we will now think of as "if G is generic over V, then in $V[G]$ the "Axiom of Constructibility" fails.

In practice, we will even go as far as saying "fix a filter G which is generic over V..." and go on to study the model $V[G]$. If one is ever worried about these issues, at any point we can find a countable transitive model M which is sufficiently similar to V (Σ_{1000}-elementary equivalent) and conclude for $M[G]$ whatever we imagined for $V[G]$.

Sometimes we will force over different (non countable) models. For example, we may say "let G be generic over L, and consider $L[G]$..." So instead of saying "assuming M is a countable transitive model satisfying "V=L" ... then in $M[G]$ "V=L" fails", we can say: in $L[G]$ the axiom "V=L" fails. Also, instead of saying "$L^{M[G]} = L^M$", we simply have $L^{L[G]} = L$, and more generally $L^{V[G]} = L$.

6. Independence of the Axiom of Choice

Next we proceed to construct a model of ZF where the axiom of choice fails, show that the axiom of choice is independent of the axioms of ZF. This result also appeared in Paul Cohen's seminal paper where the method of forcing was introduced.

Recall that if V satisfies ZFC, $\mathbb{P}$ is a poset in V and $G \subseteq \mathbb{P}$ is generic over V, then $V[G]$ satisfies ZFC as well. So, unlike the axiom of constructibility, and the size of the continuum, we cannot "force to change the value of AC". Instead, we will find intermediate models $V \subseteq M \subseteq V[G]$ in which ZF holds yet the axiom of choice fails.

This idea goes back to Fraenkel in the 1920's. Very roughly speaking, Fraenkel's idea to find a model in which the axiom of choice fails was as follows. Fraenkel considered a theory of "set theory with atoms", ZFA. In this theory we assume ZF, yet omit the axiom of extensionality. Moreoever, we stipulate the existence of infinitely many "atoms", $a_0, a_1, \ldots$. These atoms have no members (so in ZF-terms they look like the emptyset), yet are distinct. Finally, consider the unordered set $A = \{a_i : i \in \omega\}$ of these atoms, and let M be a model of all sets "definable from A". For example, we can take something like $L(A)$ or $\mathrm{HOD}(A)$. The point is now that a well ordering of the atoms cannot be recovered from the unordered set A alone, as they all "look the same".

While actual distinct sets can never be atoms, Cohen noticed that his "generic subsets of ω" all look "sufficiently similar" to reproduce Fraenkel's argument within ZF.

6.1. **The basic Cohen model.** Let $\mathbb{P} = \mathbb{P}_\omega$ be the poset of all finite functions $p\colon \omega \times \omega \to \{0,1\}$, where $p \leq q$ if p extends q as a function. Suppose $G \subseteq \mathbb{P}$ is a generic filter (over V). In $V[G]$ we defined $x(n) = \{k \in \omega : (\exists p \in G) p(n,k) = 1\}$. Let $A = \{x(n) : n \in \omega\}$. Let $\dot{x}$ be the name $\{(p,(n,k)) : (n,k) \in \operatorname{dom} p \wedge p(n,k) = 1\}$ (so $\dot{x}[G] = x$), let $\dot{x}(n)$ be the name $\{(p,k) : (n,k) \in \operatorname{dom} p \wedge p(n,k) = 1\}$ (so $\dot{x}(n)[G] = x(n)$, and let $\dot{A}$ be the name $\{(1_{\mathbb{P}}, \dot{x}(n)) : n \in \omega\}$ (so $\dot{A}[G] = A$). Note that $1_{\mathbb{P}}$ forces that $\dot{A}$ is the unordered set of subsets of ω which is enumerated by $\dot{x}$.

The following automorphisms of $\mathbb{P}$ will be useful. Suppose $\pi\colon \omega \to \omega$ is a permutation. We define an automorphism $\Pi\colon \mathbb{P} \to \mathbb{P}$. For $p \in \mathbb{P}$, define $\Pi(p) \in \mathbb{P}$ as follows.

- $(n,k) \in \operatorname{dom} \Pi(p) \iff (\pi^{-1}(n), k) \in \operatorname{dom} p$;
- $\Pi(p)(n,k) = p(\pi^{-1}(n), k)$.

That is, Π permutes the ordering of $x_0, x_1, \ldots$ according to π.

Exercise 6.1. Check that Π is an automorphism.

Note that if $G \subseteq \mathbb{P}$ is generic, its image $G' = \Pi[G] \subseteq \mathbb{P}$ is generic, and $A = \dot{A}[G] = \dot{A}[G']$. Furthermore, for each $m \in \omega$, $\dot{x}(m)[G'] = \dot{x}(\pi^{-1}(m))[G] = x(\pi^{-1}(m))$.

Consider the model $L(A)$, the minimal transitive model of ZF containing all the ordinals and the set A. (We get $L(A)$ by transfinite recursion where $L_{\alpha+1}(A)$ is the set of all subsets of $L_\alpha(A)$ which are definable over $(L_\alpha(A), \in)$. The difference with the usual L-construction is that instead of starting with $L_0 = \emptyset$, we being with $L_0(A) = A$.) We show that the axiom of choice fails in $L(A)$. In fact it fails in a very strong way.

Definition 6.2. Say that a set X is **Dedekind-finite** if there is no injective map from ω to X. (That is, there are no infinite sequences in X.)

Recall that using the axiom of choice, a set is Dedekind-finite if and only if it is finite. (Where a set is finite if and only if there is a bijection between it and some member of ω.)

Theorem 6.3. In $L(A)$, the set A is infinite and Dedekind-finite.

The proof relies on the fact that sets in $L(A)$ are definable from A, and no infinite sequence of members of A can be definable in such a way. It will be useful to consider our generalization of ordinal definability.

Lemma 6.4. Suppose X is in $L(A)$. Then there are finitely many $x_1, ..., x_n \in A$, finitely many ordinals $\alpha_1, ..., \alpha_k$ and a formula ϕ such that in $L(A)$, X is the unique set satisfying $\phi(X, A, a_1, ..., a_n, \alpha_1, ..., \alpha_k)$.

Equivalently, there is a ψ such that in $L(A)$:

$$x \in X \iff \psi(x, A, a_1, ..., a_n, \alpha_1, ..., \alpha_k).$$

Proof. Work in $L(A)$, and consider the model $\mathrm{HOD}^{L(A)}(A)$. Recall that this is a transitive model containing A and all of the ordinals. By minimality, it must be equal to $L(A)$. So any set in $L(A)$ is in $\mathrm{HOD}^{L(A)}$, which is precisely the statement of the lemma. $\square$

Proof of Theorem 6.3. Assume towards a contradiction that there is some $f \in L(A)$ such that $f \colon \omega \to A$ is injective. Fix a formula ϕ, finitely many $a_1, ..., a_n$ from A and some $v \in L$ such that in $L(A)$:

$$f(n) = x \iff \phi(n, x, A, a_1, ..., a_n, v).$$

(v can be a finite sequence of ordinals.) Fix $l_1, ..., l_m \in \omega$ such that $a_{l_i} = x(l_i)$. Since f is injective, there is some k such that $f(k) \neq a_1, ..., a_n$. Fix m such that $f(k) = x(m)$. Finally, fix a condition $p \in G$ forcing that

$$\phi^{L(\dot{A})}(k, \dot{x}(m), \dot{A}, \dot{x}(l_1), ..., \dot{x}(l_n), \check{v}).$$

(Technically we should also write $\check{k}$, but we avoid that.) Given $t \in \omega$, consider the permutation $\pi \colon \omega \to \omega$ swapping m and t. This in turn gives us a permutation Π of $\mathbb{P}$ and a generic $\Pi[G]$.

We want to find such t so that $p \in \Pi[G]$ as well (this is the only condition at the moment which we know forces something interesting). This is possible since G is generic. That is, for any $N \in \omega$, the set of conditions q for which there is some $t \in \omega$ such that for any $k < N$, $q(m, k) = q(t, k)$, is a dense subset of $\mathbb{P}$. Let N be large enough such that for any k, $(m,k) \in \operatorname{dom} p \implies k < N$. We may find q and $t \in \omega$ as above with $q \leq p$ and $q \in G$.

Finally, we have $G' = \Pi[G] \ni p$, as $p \geq \Pi(q)$. Working now in $M[G']$ we conclude that

$$\phi^{L(\dot{A}[G'])}(k, \dot{x}(t)[G'], \dot{A}[G'], \dot{x}(l_1)[G'], ..., \dot{x}(l_n)[G']).$$

Recall that $\dot{A}[G'] = \dot{A}[G] = A$, $\dot{x}(l_i)[G'] = \dot{x}[\pi^{-1}(l_i)][G] = \dot{x}[l_i][G] = a_i$, and $\dot{x}(m)[G'] = \dot{x}(\pi^{-1}(m))[G] = \dot{x}(t)[G] = x(t)$. Thus in $M[G']$ we conclude:

$$\phi^{L(A)}(k, x(t), A, a_1, ..., a_n)$$

and so in $L(A)$, $\phi(k, x(t), A, a_1, ..., a_n)$ holds. However, also $\phi(k, x(m), A, a_1, ..., a_n)$ holds. This contradicts the fact that f, which is defined by $\phi(-, -, A, a_1, ..., a_n)$, is a function, since $x(m) \neq x(t)$. $\square$

Remark 6.5. In the proof above we showed that for any formula ϕ and for any parameter $v \in L$, given distinct $a, a_1, ..., a_n \in A$, if $\phi^{L(A)}(A, v, a_1, ..., a_n, a)$ then there are infinitely many $a' \in A$ such that $\phi^{L(A)}(A, v, a_1, ..., a_n, a')$.

So the set A in $L(A)$ is infinite, yet Dedekind-finite, and so the axiom of choice fails in $L(A)$.

Remark 6.6. $L(A)$ satisfies the linear-ordering principle. That is, for any set X there is some linear ordering of X.

6.2. **Choice for socks.** Recall Russell's famous metaphor for the necessity of the axiom of choice. Given an infinite sequence of pairs of shoes, it is easy to choose one shoe out of each pair in a uniform manner, we simply choose the left shoe of of each pair. However, given the same task with pairs of socks this time, we are left with no way to distinguish between the two socks in each pair, and we need the axiom of choice to find a way to choose one out of each pair.

We show next that the axiom of choice can indeed fail that badly. That is, there could be a countable sequence of sets of size 2 without a choice function.

Let $\mathbb{P}$ be the poset of all finite partial functions $p\colon \operatorname{dom} p \to \{0,1\}$ where $\operatorname{dom} p$ is a finite subset of $\omega \times (\omega \times \{0,1\})$. Let $G \subseteq \mathbb{P}$ be a generic filter over V. For $i \in \{0,1\}$ and $n \in \omega$ let $a^i(n) = \{k \in \omega : (\exists p \in G) p(n,k,i) = 1\}$ (the "generic (n,i) column"). Define $A^i_n = \left\{a^i(n) \Delta X : X \subseteq \omega \text{ finite}\right\}$, where Δ is the symmetric difference. That is, A^i_n is the set of all subsets of ω which differ from $a(n)$ only finitely.

Let $A_n = \{A^0_n, A^1_n\}$, and let $A = \langle A_n : n < \omega \rangle$, the sequence of the sets A_n. Finally, we consider the model $L(A)$, the minimal transitive model of ZF which has the set A as a member.

First note that our proof of Lemma 6.4 works in the following general context.

Lemma 6.7. Let N be some model of ZFC extending V. Let A be a *transitive* set in N. Working in N form the relative constructible universe $L(A)$.

Suppose X is in $L(A)$. Then there are finitely many $x_1, ..., x_n \in A$, finitely many ordinals $\alpha_1, ..., \alpha_k$ and a formula ϕ such that in $L(A)$, X is the unique set satisfying $\phi(X, A, a_1, ..., a_n, \alpha_1, ..., \alpha_k)$.

Equivalently, there is a ψ such that in $L(A)$:

$$x \in X \iff \psi(x, A, a_1, ..., a_n, \alpha_1, ..., \alpha_k).$$

Proof. Work in $L(A)$, and consider the model $\mathrm{HOD}^{L(A)}(A)$. Recall that since A is transitive, this is a transitive model containing A as a set and all of the ordinals.

By the minimality of $L(A)$, it must be equal to $L(A)$. So any set in $L(A)$ is in $\mathrm{HOD}^{L(A)}$, which is precisely the statement of the lemma. $\square$

Theorem 6.8. In $L(A)$, $\prod_n A_n$ is empty. That is, there is no choice sequence f with domain ω such that $f(n) \in A_n$ for each $n \in \omega$.

By applying Lemma 6.7 to the transitive closure of A, we see that the members of $L(A)$ are all definable using A and members of the transitive closure of A. What is in the transitive closure of A? Each set A_n is, each A^i_n, and each $a^i(n)$. Anything else is simply definable from one of those (for example, a finite change of $a^i(n)$, or $\{A_n\}$). Note that A^i_n is definable from $a^i(n)$, each A_n is definable from A, and A^1_n is definable from A and A^0_n (as the only other member of A_n), and vice versa.

As before, the proof of Theorem 6.8 relies on analysing definability questions in $L(A)$.

Proposition 6.9. Fix $n \in \omega$, a formula ϕ, and $v \in V$. In $L(A)$, for any $k > n$:

$$\phi(A, a^0_0, a^1_0, ..., a^0_n, a^1_n, A^0_k, v) \iff \phi(A, A^0_0, ..., A^0_n, A^1_k, v).$$

Proof. We assume that $\phi^{L(A)}(A, a^0_0, a^1_0, ..., a^0_n, a^1_n, A^0_k, v)$ holds and show that $\phi^{L(A)}(A, a^0_0, a^1_0, ..., a^0_n, a^1_n, A^1_k, v)$ holds as well. Fix names $\dot{a}^i_m$, $\dot{A}^i_m$, $\dot{A}_m$, $\dot{A}$ such that $\dot{a}^i_m[G] = a^i_m$, $\dot{A}^i_m[G] = A^i_m$, $\dot{A}_m[G] = A_m$, $\dot{A}[G] = A$. For example, $\dot{a}^i(n) = \left\{(p, \check{k}) : p(n, k, i) = 1\right\}$. Fix a condition $p \in \mathbb{P}$ forcing that $\phi^{L(\dot{A})}(\dot{A}, \dot{a}^0_0, \dot{a}^1_0, ..., \dot{a}^0_n, \dot{a}^1_n, \dot{A}^0_k, \check{v})$.

Let $f = \bigcup G \colon \omega \times \omega \times \{0, 1\} \to \{0, 1\}$. Define $f' \colon \omega \times \omega \times \{0, 1\} \to \{0, 1\}$ by $f'(k, m, 0) = f(k, m, 1)$ and $f'(k, m, 1) = f(k, m, 0)$, and f' agrees with f otherwise.

Exercise 6.10. There is a filter $G' \in V[G]$ such that $G' \subseteq \mathbb{P}$ is generic over V and $f' = \bigcup G'$.

Then

- $\dot{A}^1_k[G'] = \dot{A}^0_k[G] = A^0_k$;
- $\dot{A}^0_k[G'] = \dot{A}^1_k[G] = A^1_k$;
- $\dot{A}^i_m[G'] = \dot{A}^i_m[G] = A^i_m$ for $m \neq k$;
- $\dot{A}_m[G'] = \dot{A}_m[G] = A_m$ for all $m < \omega$;
- $\dot{A}[G'] = \dot{A}[G] = A$.

Define now $\tilde{f}$ by $\tilde{f}(l, m, i) = p(l, m, i)$ whenever $(l, m, i) \in \operatorname{dom} p$, and $\tilde{f}(l, m, i) = f'(l, m, i)$ otherwise.

Exercise 6.11. There is a filter $\tilde{G} \in V[G]$ such that $\tilde{G} \subseteq \mathbb{P}$ is generic over V and $\tilde{f} = \bigcup \tilde{G}$. Furthermore, as $\tilde{f}$ extends p as a function, p is in $\tilde{G}$.

For each l, $\dot{a}^i_l[\tilde{G}]$ and $\dot{a}^i_l[G']$ differ by only a finite amount, so

- $\dot{A}^i_l[\tilde{G}] = \dot{A}^i_l[G']$;
- $\dot{A}[\tilde{G}] = \dot{A}[G'] = A$.

Finally, since p is in the generic $\tilde{G}$, working in $V[\tilde{G}]$ we conclude that

$$\phi^{L(\dot{A}[\tilde{G}])}(\dot{A}[\tilde{G}], \dot{a}^0_0[\tilde{G}], \dot{a}^1_0[\tilde{G}], ..., \dot{a}^0_n[\tilde{G}], \dot{a}^1_n[\tilde{G}], \dot{A}^0_k[\tilde{G}], \check{v}[\tilde{G}])$$

holds, that is,

$$\phi^{L(A)}(A, a^0_0, a^1_0, ..., a^0_n, a^1_n, A^1_k, v)$$

holds, as desired. $\square$

Proof of Theorem 6.8. Suppose towards a contradiction that there is some $f \in L(A)$ with $\operatorname{dom} f = \omega$ and $f(n) \in A_n$ for each n. There is some n, a formula ϕ and $v \in L$ such that f is defined in $L(A)$ by

$$f(k) = A_k^i \iff \phi(k, A_k^i, A, a_0^0, a_0^1, ..., a_n^0, a_n^1, v).$$

Fix $k > n$, then we get from the previous proposition that $f(k) = A_k^0 \iff f(k) = A_k^1$, contradicting that f is a function. $\square$

Remark 6.12. About the names. We wrote above the name $\dot{a}_n^i$ such that for any generic G the name $\dot{a}_n^i[G]$ is precisely the subset of ω defined by the function $\bigcup G(n, -, i)$, and this was used to understand precisely how the names are realized according to the other filters.

Given a finite set $X \subseteq \omega$, you can write explicitly, in the same manner, a name $\dot{a}_n^i(X)$ such that for any G, $\dot{a}_n^i(X)[G] = \dot{a}_n^i[G] \Delta X$. (Exercise!) Now a good name for $\dot{A}_n^i$ would be $\{(1_{\mathbb{P}}, \dot{a}_n^i(X)) : X \subseteq \omega \text{ finite}\}$.

We can also avoid worrying about the names for A_n^i and A_n. Simply, in any statement above replace the set A_n^i by its definition using a_n^i. So any statement $\phi(A_n^i)$ can be translated to a statement $\psi(a_n^i)$, and we can then talk about forcing this statement using the name $\dot{a}_n^i$.

More generally, if we have a name τ and we care about some object definable from τ, we do not really need to worry about finding a name for this object. For example, given a name τ, instead of fixing a different name σ such that σ is always interpreted as the union of τ (we saw how to find such a name), we will simply write $\bigcup \tau$ instead of σ.

Remark 6.13. In $L(A)$, there is no linear ordering of the set $\bigcup_{n<\omega} A_n = \{A_n^i : i \in \{0,1\}, n \in \omega\}$. Otherwise, we could have chosen "$f(n)$ is the smaller among the two elements of A_n".

Exercise 6.14 (ZF)**.** For any ordinal η there is a total linear order on the set $\mathcal{P}(\eta)$.

7. Collapsing

Definition 7.1. Given cardinals $\kappa \leq \lambda$. Let $\operatorname{Col}(\kappa, \lambda)$ be the poset of all functions $p\colon \operatorname{dom} p \to \lambda$ where $\operatorname{dom} p$ is a subset of κ of size $< \kappa$. Define $p \leq q$ if p extends q as a function.

Exercise 7.2. If $G \subseteq \operatorname{Col}(\kappa, \lambda)$ is generic then in $V[G]$, $g \bigcup G$ is a function from κ onto λ.

Definition 7.3. Let κ be a cardinal. A poset $(\mathbb{P}, \leq)$ is called **κ-closed** if any descending sequence of conditions of length less than κ has a lower bound. That is, given $\rho < \kappa$ and a sequence $\langle p_\alpha : \alpha < \rho \rangle$ of conditions $p_\alpha \in \mathbb{P}$ such that for $\alpha < \beta < \rho$, $p_\beta \leq p_\alpha$, then there exists some condition $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for all $\alpha < \rho$.

Say that $(\mathbb{P}, \leq)$ is σ-closed if it is ω_1-closed.

Example 7.4. Suppose κ is a regular cardinal, then $\operatorname{Col}(\kappa, \lambda)$ is κ-closed.

Proof. Given a descending sequence of conditions $\langle p_\alpha : \alpha < \rho \rangle$ for $\rho < \kappa$, let $p = \bigcup_{\alpha<\rho} p_\alpha$. Since the sequence is descending, any two conditions are compatible, and so p is a well defined function from $\operatorname{dom} p$ to λ, where $\operatorname{dom} p = \bigcup_{\alpha<\rho} \operatorname{dom} p_\alpha$. Since

κ is a regular cardinal, $\operatorname{dom} p$ has size $< \kappa$, as a union of $< \kappa$ many sets of size $< \kappa$. $\square$

Lemma 7.5 (ZFC)**.** Suppose $(\mathbb{P}, \leq)$ is ω_1-closed and $G \subseteq \mathbb{P}$ is generic. If $f \in V[G]$ is a function $f \colon \omega \to \theta$, for some ordinal θ, then $f \in V$.

Proof. Suppose τ is a name in V such that $\tau[G] \colon \omega \to \lambda$. We need to show that $\tau[G]$ is a set in V already.

Fix a condition $q \in \mathbb{P}$ and suppose that q forces that τ is a function with domain ω and it takes ordinal values. Define a sequence of conditions p_n, $n < \omega$ such that

- $p_0 \leq q$ and $p_{n+1} \leq p_n$;
- there is an ordinal β such that $p_n \Vdash \tau(n) = \beta$.

Since $\mathbb{P}$ is σ-closed, there is some condition $p \in \mathbb{P}$ such that $p \leq p_n$ for each $n < \omega$. We showed that the set

$$D = \{p \in \mathbb{P} : \forall n \in \omega \exists \beta (p \Vdash \tau(n) = \beta \vee p \Vdash \text{“}\tau \text{ is not a function...”}\}$$

is dense in $\mathbb{P}$. Since G is generic there is some $p \in D \cap G$. Since $\tau[G]$ is a function with domain ω and ordinal values, p must satisfy the first option in the definition of D.

Define in V a function h by $h(n) = \beta$ if and only if $p \Vdash \tau(n) = \beta$. Then $h = \tau[G] = f$ is in V, as required. $\square$

Corollary 7.6 (ZFC)**.** If $\mathbb{P}$ is σ-closed then no new subsets of ω are added when forcing with $\mathbb{P}$.

Corollary 7.7 (ZFC)**.** If $\mathbb{P}$ is σ-closed then ω_1 is not collapsed when forcing by $\mathbb{P}$. That is, $\omega_1^V = \omega_1^{V[G]}$ for any generic filter $G \subseteq \mathbb{P}$.

Similarly, if $\mathbb{P}$ is κ-closed then no cardinal $\leq \kappa$ is collapsed and no new subsets of α are added for any $\alpha < \kappa$.

Using collapsing we can give an alternative proof for the consistency of CH with ZFC. Assume V is some model of ZFC, where CH might fail. Let $\kappa = 2^{\aleph_0}$ and consider $\mathbb{P} = \mathrm{Col}(\omega_1, \kappa)$. Let $G \subseteq \mathbb{P}$ be a filter generic over V.

Since $\mathbb{P}$ is σ-closed, $V[G]$ has no new subsets of ω. So in $V[G]$ we have $|\mathcal{P}(\omega)| = \kappa$ (there is a bijection in V between $\mathcal{P}^V(\omega)$ and κ, and $\mathcal{P}^V(\omega) = \mathcal{P}^{V[G]}(\omega)$). Also, in $V[G]$, $|\kappa| = \aleph_1$, so $2^{\aleph_0} = \aleph_1$.

8. Mutual genericity

Lemma 8.1. Suppose $M \subseteq N$ are transitive models of ZF. Let $\mathbb{P}$ be a poset in M, and suppose $G \subseteq \mathbb{P}$ is generic over N. Then $M[G] \cap N = M$.

Remark 8.2. Note that in this situation $G \subseteq \mathbb{P}$ is generic over M as well. Note however that forcing with $\mathbb{P}$ over M and over N can be very different. For example, if $\mathbb{P}$ is σ-closed in M it is not necesarily so in N, and if $\mathbb{P}$ is c.c.c. in M it is not necessarily so in N.

Proof. First note that for any name $\tau \in M$, and any filter $H \subseteq \mathbb{P}$ generic over N, $\tau[H]$ is the same as calculated over M or over N. Also, for any $x \in M$, the canonical name $\check{x}$ is the same as calculated in M or in N.

Fix a $\mathbb{P}$-name τ in M, and suppose that $\tau[G]$ is in N. Assume first that $\tau[G] \subseteq V_\alpha^M$ for some ordinal α in M. Fix $X \in N$ with $\tau[G] = X$. Note that τ is a $\mathbb{P}$-name in N as well. By the forcing theorem (applied in N), there is some $p \in G$ such that

$$N \models p \Vdash \tau = \check{X}.$$

Define in M

$$X' = \{x \in V_\alpha : p \Vdash \check{x} \in \tau\}.$$

Then X' is in M by comprehension, and the definability of forcing. We claim that $X' = X$.

First note that for $p \in \mathbb{P}$ and $x \in M$,

$$M \models p \Vdash \check{x} \in \tau \iff N \models p \Vdash \check{x} \in \tau.$$

The $\implies$ direction is clear, as any filter generic over N is generic over M as well. For the reverse direction, recall that $p \Vdash \phi$ if and only if there is no $q \leq p$ forcing $\neg\phi$. Now, if $M \models p \nVdash \check{x} \in \tau$, then there is some $q \leq p$ such that $M \models q \Vdash \check{x} \notin \tau$. By the previous argument it follows that $N \models q \Vdash \check{x} \notin \tau$. So $N \models p \nVdash \check{x} \in \tau$ as well.

Finally, since in N, $p \Vdash \tau = \check{X}$, we see that in N, for any x, $p \Vdash \check{x} \in \tau \iff x \in X$. Therefore this is true in M as well, and we conclude that $X' = X$.

To deal with the assumption that $X \subseteq V_\alpha^M$ for some α: the proof is carried by induction on the rank of τ. Assume that for any σ of rank smaller than τ, if $\sigma[G] \in N$ then $\sigma[G] \in M$. Then if $\tau[G] \in N$ is of rank α, it follows from the inductive assumption that $\tau[G] \subseteq V_\alpha^M$, and we are done. $\square$

Corollary 8.3. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are posets in V and $K \subseteq \mathbb{P} \times \mathbb{Q}$ is generic over V. As we have seen in the homework, $K = G \times H$ where H is $\mathbb{Q}$-generic over $V[G]$ and G is $\mathbb{P}$-generic over $V[H]$. Then $V[G] \cap V[H] = V$.

Proof. Apply Lemma 8.1 with $M = V$ and $N = V[G]$. $\square$

9. The Feferman-Levy model

We will now construct a model with the following extreme failure of the axiom of choice: the cofinality of ω_1 is ω, and in fact $\mathcal{P}(\omega)$ can be written as a countable union of countable sets.

Define $\mathbb{P}$ as the poset of all partial functions $p\colon \operatorname{dom} p \to \aleph_\omega$ such that $\operatorname{dom} p \subseteq \omega \times \omega$ is finite and for all $(n, k) \in \operatorname{dom} p$, $p(n, k) \in \aleph_n$.

Let $G \subseteq \mathbb{P}$ be a generic filter over V. Define $f = \bigcup G$, and $f_n\colon \omega \to \aleph_n^V$ by $f_n(k) = f(n, k)$.

Exercise 9.1. Each f_n is onto $\aleph_n^V$.

In $L[G]$, each $\aleph_n^V$, and also $\aleph_\omega^V$, are countable ordinals. We will find a submodel $L(A)$ in which each $\aleph_n^V$ is countable, yet it does not "see" any countable enumeration of $\aleph_\omega^V$. Thus in this model $\aleph_\omega^V$ will be the smallest uncountable cardinal, that is, ω_1.

Let f, g be two functions from ω to κ. Say that f and g are **similar** if there are only finitely many $m \in \omega$ for which $f(m) \neq g(m)$.

For $f\colon \omega \to \kappa$, define

$$[f] = \{g : g\colon \omega \to \kappa \text{ is similar to } f\}.$$

Back to our model $L[G]$, define $A_n = [f_n] = \{g\colon \omega \to \aleph_n^V : g \text{ is similar to } f\}$, and $A = \langle A_n : n < \omega\rangle$. The model of interest will be $L(A)$.

- For $n < \omega$ and a finite function $t\colon m \to \aleph_n$, define $\dot{f}_n^t$ to be all pairs $(p, (\check{k}, \check{\alpha}))$ such that either $k \geq m$ and $p(n,k) = \alpha$ or $k < m$ and $t(k) = \alpha$. Let $\dot{f}_n = \dot{f}_n^t$ where t is the empty function. Then $\dot{f}_n[G] = f_n$ and $\dot{f}_n^t[G]$ is the finite change of f_n according to t.
- $\dot{A}_n = \left\{(1_{\mathbb{P}}, \dot{f}_n^t) : t \in (\aleph_n)^{<\omega}\right\}$. Then $\dot{A}_n[G] = A_n$.
- $\dot{A} = \left\{(1_{\mathbb{P}}, (\check{n}, \dot{A}_n)) : n < \omega\right\}$. Then $\dot{A}[G] = A$.

(Above when we write $(\check{n}, \dot{A}_n)$ we mean a name σ such that $\sigma[H] = (n, \dot{A}_n[H])$ for any filter H.)

The following automorphisms of $\mathbb{P}$ will be crucial for the analysis of $L(A)$. Given two functions $t, s\colon m \to \aleph_n$, define $a = a_n(t,s)\colon \mathbb{P} \to \mathbb{P}$ by sending p to $a(p)$ where

- $\operatorname{dom} a(p) = \operatorname{dom} p$,
- if $p(n,i) = t(i)$ then $a(p)(n,i) = s(i)$, and if $p(n,i) = s(i)$ then $a(p)(n,i) = t(i)$,
- otherwise $a(p)(n,i) = p(n,i)$,
- for $n' \neq n$, $a(p)(n',j) = p(m',j)$.

Exercise 9.2. Show that a is an automorphism of $\mathbb{P}$. Furthermore, if G is generic over L and $G' = a[G]$. Then

- $\dot{f}_m[G']$ is similar to $\dot{f}_n[G] = f_n$.
- $\dot{f}_m[G'] = \dot{f}_m[G] = f_n$ if $m \neq n$;
- $\dot{A}_m[G'] = \dot{A}_m[G] = A_m$ for all m;
- $\dot{A}[G'] = \dot{A}[G] = A$.

We will also apply automorphisms which change finitely many f_n's. This can be done by composing the $a_n(t,s)$'s. Given $t_1, s_1, ..., t_k, s_k$ and $n_1, ..., n_k$, then $a = a_{n_k}(t_k, s_k) \circ ... \circ a_{n_1}(t_1, s_1)$ is an automorphism of $\mathbb{P}$. By applying this a we only change f_n when n is one of $n_1, ..., n_k$ and all A_m's are preserved, as well as A.

For $m < \omega$, let $\mathbb{P}_{<m}$ be the set of all $p \in \mathbb{P}$ whose domain is contained in $m \times \omega = \{0, ..., m-1\} \times \omega$. Let $\mathbb{P}^{\geq m}$ be the set of all $p \in \mathbb{P}$ whose domain is contained in $(\omega \setminus m) \times \omega = \{m, m+1, ...\} \times \omega$. Given $p \in \mathbb{P}$ let $p_{<m}$ be the restriction of p to $m \times \omega$ and let $p^{\geq m}$ be the restriction of p to $(\omega \setminus m) \times \omega$.

Exercise 9.3. The map $\mathbb{P} \to \mathbb{P}_{<m} \times \mathbb{P}^{\geq m}$ sending p to $(p_{<m}, p^{\geq m})$ is an isomorphism.

Given $G \subseteq \mathbb{P}$ generic over L, let $G_{<m} = G \cap \mathbb{P}_{<m}$ and $G^{\geq m} = G \cap \mathbb{P}^{\geq m}$. Then $G_{<m} \subseteq \mathbb{P}_{<m}$ is generic over L and $G^{\geq m} \subseteq \mathbb{P}^{\geq m}$ is generic over $L[G_{<m}]$.

Note that $|\mathbb{P}_{<m}| = \aleph_{m-1}$.

Proposition 9.4. Suppose $g \in L(A)$, $g\colon \lambda \to \theta$ where λ and θ are ordinals. Assume further g is definable as

$$g(\alpha) = \beta \iff \phi^{L(A)}(\alpha, \beta, A, f_0, ..., f_{m-1}, v),$$

where $v \in L$. Then $g \in L[G_{<m}]$.

Before proving the proposition, let us see how to deduce the main result from it.

Theorem 9.5. In the model $L(A)$:

(1) ω_1 has cofinality ω. In fact, ω_1 is $\aleph^L_\omega$;
(2) $\mathcal{P}(\omega)$ is a countable union of countable sets.

Proof. For (1), note first that each of the cardinals $\aleph^L_n$ are in fact countable in $L(A)$, so it suffices to prove that $\aleph^L_\omega$ is not countable in $L(A)$. Indeed, suppose for a contradiction that $g\colon \omega \to \aleph^L_\omega$ were a surjective map in $L(A)$. Then g can be defined in $L(A)$ using A, finitely many $f_0, ..., f_{m-1}$, and ordinals. By Proposition 9.4 we conclude that $g \in L[G_{<m}]$. However, $L[G_{<m}]$ is a $\mathbb{P}_{<m}$-generic extension of L. Working in L: $\mathbb{P}_{<m}$ satisfies the $\aleph_m$-c.c. (as $|\mathbb{P}_{<m}| < \aleph_m$). So $\aleph^L_m, \aleph^L_{m+1}, ..., \aleph^L_\omega$ must remain cardinals in $L[G_{<\omega}]$, and therefore cannot be countable.

Next we prove part (2). As usual, we identify $\mathcal{P}(\omega)$ with 2^ω, all functions from ω to $\{0,1\}$. So by Proposition 9.4 every $x \in \mathcal{P}(\omega)$ is in $L[G_{<m}]$ for some $m < \omega$. Let X_m be the set of all $x \in \mathcal{P}(\omega)$ with $x \in L(G_{<m})$. (Equivalently, the set of all $x \in L(\langle A_i : i < m\rangle) \cap \mathcal{P}(\omega)$.)

Note that the sequence $\langle X_n : n < \omega\rangle$ is in $L(A)$, by an application of comprehension. (We use the fact that we have a single formula defining when $Y = L_\beta(B)$.)

Finally, what is the size of $L[G_{<m}] \cap \mathcal{P}(\omega)$? It is bounded below $\aleph_m^{\aleph_0}$, as calculated in L. (The number of functions from ω to antichains in $\mathbb{P}$.) Since L satisfies the GCH, $\aleph_m^{\aleph_0} = \aleph_m$ (for $m > 0$). So in $L[G_{<m}]$, $|X_m| \le \aleph^L_m$. In $L[G]$ however, $\aleph^L_m$ is countable, so X_m is countable.

Finally, in $L(A)$, $\mathcal{P}(\omega) = \bigcup_{m<\omega} X_m$ is a countable union of countable sets. □

The proof of the proposition relies on the following key lemma.

Lemma 9.6. Let ϕ be a formula, $v \in V$ and $p \in \mathbb{P}$ such that

$$p \Vdash \phi^{L(\dot{A})}(\dot{A}, \check{v}, \dot{f}_0, ...\dot{f}_{m-1}).$$

Then

$$p_{<m} \Vdash \phi^{L(\dot{A})}(\dot{A}, \check{v}, \dot{f}_0, ...\dot{f}_{m-1}). \tag{1}$$

(Note that both forcing relations $\Vdash$ above are referring to the poset $\mathbb{P}$. We consider in this lemma $p_{<m}$ as a condition in $\mathbb{P}$.)

Proof. It suffices to show that for any $q \in \mathbb{P}$, if $q \le p_{<m}$, then q does not force $\neg\phi^{L(\dot{A})}(\dot{A}, \check{v}, \dot{f}_0, ...\dot{f}_{m-1})$. Assume towards a contradiction that q is such a condition.

Assume both q and p have domains contained in $l \times \omega$, $l > m$. For each $m \le n < l$, let $t_n(i) = p(n,i)$ and $s_n(i) = q(n,i)$. Consider the automorphism a of $\mathbb{P}$ which is the composition of $a_n(t_n, s_n)$ for $m \le n < l$.

Now take a generic $H \subseteq \mathbb{P}$ which contains the condition q. The automorphism a was chosen precisely so that p is in the generic filter $a[H]$. Now, working in $V[H]$ we get

$$\neg\phi^{L(A)}(A, v, f_0, ..., f_{m-1}),$$

as it is forced by q. On the other hand, in $M[a[H]]$ we conclude

$$\phi^{L(A)}(A, v, f_0, ..., f_{m-1}),$$

as p forces this. A contradiction! □

Proof of Proposition 9.4. Suppose $g \in L(A)$, $g\colon \lambda \to \theta$ is defined in $L(A)$ by

$$g(\alpha) = \beta \iff \phi^{L(A)}(\alpha, \beta, A, f_0, ..., f_{m-1}, v).$$

By the previous lemma, for any α, β and any $p \in \mathbb{P}$,

$$p \Vdash \phi^{L(\dot{A})}(\check{\alpha}, \check{\beta}, \dot{A}, \dot{f}_0, ..., \dot{f}_{m-1}, \check{v}) \iff p_{<m} \Vdash \phi^{L(\dot{A})}(\check{\alpha}, \check{\beta}, \dot{A}, \dot{f}_0, ..., \dot{f}_{m-1}, \check{v}).$$

In other words, $g(\alpha) = \beta$ if and only if there is $p \in G$ forcing $\phi^{L(\dot{A})}(\check{\alpha}, \check{\beta}, \dot{A}, \dot{f}_0, ..., \dot{f}_{m-1}, \check{v})$, if and only if there is $p \in G$ such that $p_{<m}$ forces $\phi^{L(\dot{A})}(\check{\alpha}, \check{\beta}, \dot{A}, \dot{f}_0, ..., \dot{f}_{m-1}, \check{v})$, if and only if there some $q \in G_{<m}$ forcing $\phi^{L(\dot{A})}(\check{\alpha}, \check{\beta}, \dot{A}, \dot{f}_0, ..., \dot{f}_{m-1}, \check{v})$. Note that the final statement can be defined in $L[G_{<m}]$, where we have $G_{<m}$ and the forcing relation $\Vdash^{\mathbb{P}}$ is definable (in L).

That is, we may define a function g' in $L[G_{<m}]$ by

$$g'(\alpha) = \beta \iff \exists(q \in G_{<m}) q \Vdash \phi^{L(\dot{A})}(\check{\alpha}, \check{\beta}, \dot{A}, \dot{f}_0, ..., \dot{f}_{m-1}, \check{v}),$$

and we see that $g' = g$. □

10. Sacks forcing: a minimal extension

Let $\mathbb{P}$ be the Cohen poset for adding a single subset of ω. Given a generic filter $G \subseteq \mathbb{P}$ over V, let $a \subseteq \omega$ be the subset of ω where $\bigcup G$ is the characteristic function of a. Recall that $V[G]$ is the minimal transitive model of ZF which extends V and contains G as a set. G and a can be simply defined from one another, and we will often denote $V[G]$ as $V[a]$, the minimal transitive extension of V containing a as a set.

Consider the map $f \colon \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ defined as follows.

$$f(p,q)(n) = \begin{cases} p(k) & n = 2k \,\&\, k \in \operatorname{dom} p; \\ q(k) & n = 2k+1 \,\&\, k \in \operatorname{dom} q. \end{cases}$$

(In particular, $n \in \operatorname{dom} f(p,q)$ if and only if either $n = 2k$ and $k \in \operatorname{dom} p$ or $n = 2k+1$ and $k \in \operatorname{dom} q$.

Exercise 10.1. f is an isomorphism between $\mathbb{P} \times \mathbb{P}$ and $\mathbb{P}$.

So if $G \subseteq \mathbb{P}$ is generic over V, then $f^{-1}[G] = K \times H \subseteq \mathbb{P} \times \mathbb{P}$ is generic over V and is in $V[G]$. Let a, b, c be the subsets of ω corresponding to G, K, H. Then $b, c \in V[a]$ yet $c \notin V[b]$ and $b \notin V[c]$ (recall that H is generic over $V[K]$ and K is generic over $V[H]$).

Let us take $V = L$ now. Then the question if whether or not $b \in L[c]$ can be seen as whether b is constructible *relative to* c. Say that b and c have the same *degree of constructibility* if $L[b] = L[c]$. Inside the extension $L[a]$, where a is a Cohen generic subset of ω, there are many different degrees of constructibility.

We will now consider a very different poset, such that after forcing with it we will add a minimal degree $a \subseteq \omega$. That is, if $b \in L[a]$ then either $b \in L$ or $L[a] = L[b]$.

Let $2^{<\omega}$ be the set of all finite binary sequences, that is, all functions $t \colon n \to \{0,1\}$ for some $n \in \omega$. We think of $2^{<\omega}$ as a tree, with root $\emptyset = <>$, which then splits to $< 0 >$ and $< 1 >$, and each $< i >$ splits to $< i0 >$ and $< i1 >$ and so on.. each node $t \in 2^{<\omega}$ splits to $t^\frown 0$ and $t^\frown 1$. (Here $^\frown$ is concatenation of sequences.) For $t \in 2^{<\omega}$ let $l(t) = \operatorname{dom}(t)$ be the length of t. (See Figure 1.)

Say that $T \subseteq 2^{<\omega}$ is a **tree** (a subtree of $2^{<\omega}$) if for any $t \in T$ for any $k < l(t)$, $t \restriction k$ is in T as well (T is "downwards closed"). [We will usually assume that T is not empty, and therefore $\emptyset \in T$.] Say that a node $t \in T$ is **splitting** (in T) if both $t^\frown 0$ and $t^\frown 1$ are in T. Say that a tree T is **perfect** if for any $t \in T$ there is some $u \in T$ such that u is splitting in T. (See Figure 2.)

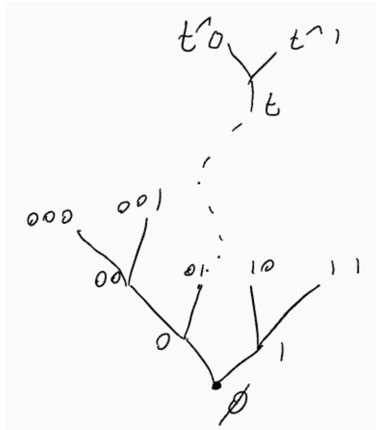

FIGURE 1. $2^{<\omega}$

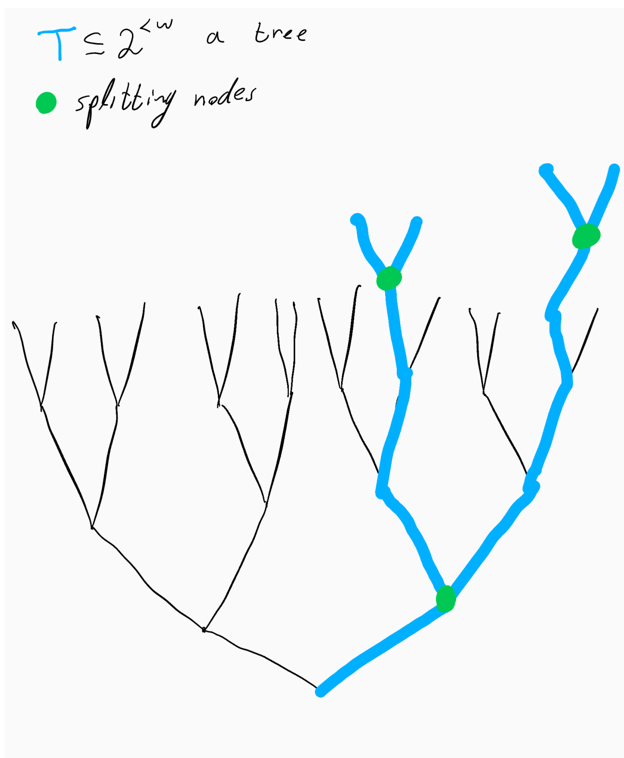

FIGURE 2. A perfect tree with stem $= <1>$

The **stem** of a tree T is the unique $t \in T$ such that t is splitting in t yet $t \restriction k$ is not splitting for any $k < l(t)$. Note that if T_1, T_2 are trees and T_1 is contained in T_2 then the stem of T_1 extends the stem of T_2.

Definition 10.2. Sacks forcing $\mathbb{S}$ is the poset of all non-empty perfect trees in $2^{<\omega}$, where $T_1 \leq T_2$ if $T_1 \subseteq T_2$. (A smaller tree gives more information which gives a stronger condition.)

For $T \in \mathbb{S}$ and $t \in T$ let T_t be the subset of T containing all $u \in T$ such that u either extends t or t extends u. Note that the stem of T_t extends t.

Remark 10.3. We have been working a lot with the set 2^ω of all functions from ω to $\{0,1\}$. We identify this with all subsets of ω. This can also be identified with the "Cantor set" in $\mathbb{R}$. For a tree $T \subseteq 2^{<\omega}$ let $[T]$ be all $f \in 2^\omega$ such that $f \restriction n \in T$ for all $n < \omega$. Then the closed sets in 2^ω are precisely those of the form $[T]$ for some tree T, and the perfect sets are those of the form $[T]$ for a perfect tree T.

Proposition 10.4. Suppose $G \subseteq \mathbb{S}$ is generic over V. Define f as the union of all the stems of the trees in G. Then $f\colon \omega \to \{0,1\}$ is a well defined function, and $f \notin V$.

Proof. First, since G is a filter, f takes at most one value at each $n \in \omega$. Next we show that each $n \in \omega$ is in the domain of f. Consider the set D_n of all $T \in \mathbb{S}$ such that the length of the stem of T is $> n$. Each D_n is dense and therefore there is some T in G whose stem is defined on n. Finally, fix $h\colon \omega \to \{0,1\}$ in V. Let D be the set of all T in $\mathbb{S}$ such that for the stem s of T there is $k < l(s)$ with $s(k) \neq h(k)$. Then D is dense, and so there is some k for which $f(k) \neq s(k)$. $\square$

Let $G \subseteq \mathbb{S}$ be generic over L.

Lemma 10.5. $L[G] = L[f]$, the minimal transitive extension of V containing f.

Proof. We need to show that G is in $V[f]$. In fact, it can be defined as $G = \left\{T \in \mathbb{S}^V : f \in [T]\right\}$. The inclusion $\subseteq$ is immediate. For the other direction, assume that $T \in V$ is such that $f \in [T]$.

Exercise 10.6. Show that for T, U in $\mathbb{S}$, if U is not contained in T then there is a $U' \leq U$ such that $U \cap T$ is finite. (This implies that $[U] \cap [T] = \emptyset$.)

Working now in V, it follows that the set of $U \in \mathbb{S}$ such that either $U \leq T$ or $U \cap T$ is finite is dense in $\mathbb{S}$ (and is in V). So there is such a U in G. Since $f \in [U]$ and $f \in [T]$, it cannot be that $U \cap T$ is finite, therefore $U \leq T$, and so $T \in G$ as well. $\square$

Remark 10.7. Generally speaking, for $T \in V$ the sets of branches $[T]$ as computed in V or $V[G]$ could be very different. For example, of $T = 2^{<\omega}$ is the full binary tree, then $[T]$ is 2^ω, which is different in V and $V[G]$.

However, if $T \cap U$ is finite, then $[T] \cap [U]$ is empty in V and in any generic extension.

Say that $t \in T$ is an n'th splitting node of T if there are precisely n splitting node among $\{t \restriction k : k \leq l(t)\}$.

Lemma 10.8 (Fusion)**.** Suppose T_0, T_1, T_2,... is a sequence of perfect tress such that

- $T_{n+1} \subseteq T_n$
- The $n+1$'th splitting nodes of T_n are $n+1$'th splitting nodes of T_{n+1}.

Then $T = \bigcap T_n$ is a perfect tree.

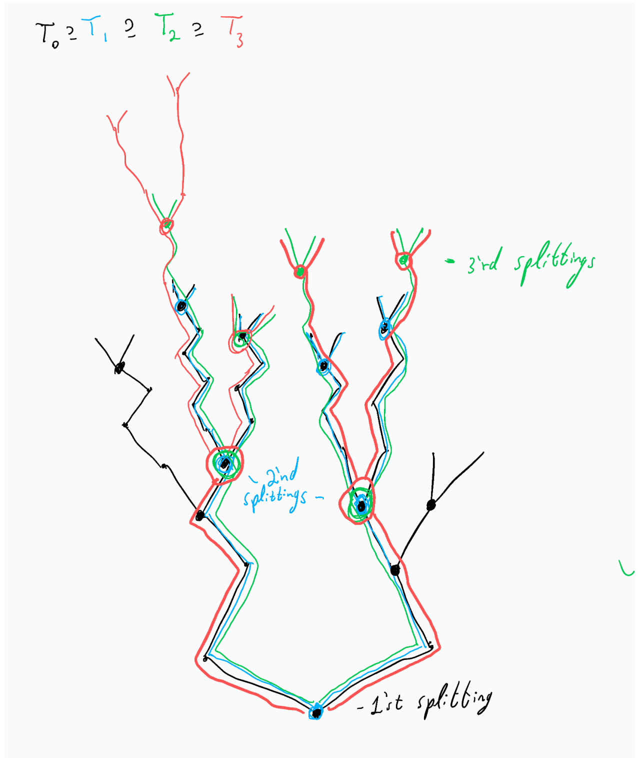

FIGURE 3. Fusion

Proof. Take $t \in T$. Let m be the number of nodes of the form $t \upharpoonright k$ which are splitting nodes of T. Now t is a member of T_m. Since T_m is a perfect tree, we may find t' above t which is splitting, and is an m-splitting node of T_m. By construction, t' is an m-splitting node of each $T_{m'}$, $m' > m$. That is, both $t'^\frown 0$ and $t'^\frown 1$ are in $T_{m'}$ for each m', and so they are in T. Therefore t' is a splitting node of T. $\square$

Theorem 10.9. Suppose $X \in L[G]$, $X \subseteq \eta$ for some ordinal η, then either $X \in L$ or $L[X] = L[G]$.

Proof. Let τ be a name with $\tau[G] = X$. Assume that $X \notin L$ and fix some condition T_0 forcing that τ is not in L (that is, $\tau \notin \check{V_{\eta+1}}$. Note that for any condition $T \leq T_0$, there must be some $\zeta < \eta$ for which T does not force either $\check{\zeta} \in \tau$ nor $\check{\zeta} \notin \tau$. [Otherwise, we could define X in V by $\zeta \in X \iff T \Vdash \check{\zeta} \in \tau$.] In this case we may find extensions of T forcing either statement.

We now work in L. We will find a single tree in L such that going "left or right" will decide different statements about members of $X = \tau[G]$. We will therefore be able to "decode" the function f from X.

Let ζ_0 be the minimal such that T_0 does not decide if $\zeta_0 \in \tau$. Let s be the stem of T_0.

Consider $(T_0)_{s^\frown 0}$ and $(T_0)_{s^\frown 1}$. It cannot be the case that they both force $\zeta_0 \in \tau$. Since the set of conditions $S \in \mathbb{S}$ such that S is either below $(T_0)_{s^\frown 0}$ or $(T_0)_{s^\frown 1}$, is dense in $\mathbb{S}$, below T_0. So this would imply that all extensions of T_0 force the same value for "$\zeta_0 \in \tau$", and therefore T_0 would force this. Similarly, it cannot be the case that they both force $\zeta_0 \notin \tau$.

It follows that there are trees U_0 and U_1 such that $U_i \leq (T_0)_{s^\frown i}$, and such that one of U_0, U_1 forces $\zeta_0 \in \tau$ and the other forces $\zeta_0 \notin \tau$.

Now define T_1 by $T_1 = U_0 \cup U_1$. Note that $T_{s^\frown i} = U_i$. Furthermore, T_1 and T_0 have the same stem s (the same 1-splitting node).

For each U_i, $i = 0, 1$, we apply the same process: let ζ be the minimal such that U_i does not decide $\zeta \in \tau$. Let s_i be the stem of u_i (note that s_i is a 2-splitting node of T_1).

We may find $U_i' \leq U_i$ such that U_i' and U_i have the same stem s_i and if s, t are the two 2-splitting nodes of U_i', then one of $(U_i')_s$, $(U_i')_t$ forces $\zeta \in \tau$ and the other forces $\zeta \notin \tau$.

Now let $T_2 = U_1' \cup U_2'$. Then T_2 and T_1 have the same 2-splitting nodes (the 1-splitting nodes of U_0' and U_1').

For the general construction: assume we arrived at T_m. For each $m+1$-splitting node t of T_m, let $U = (T_m)_t$. Find U_0', U_1' two extensions of $U_{t^\frown 0}$ and $U_{t^\frown 1}$ which force conflicting statements of $\zeta \in \tau$, for some ordinal $\zeta < \eta$. Let $U_t' = \bigcup U_0' \cup U_1'$. Then U_t' is an extension of $(T_m)_t$, with the same stem t. Finally, let T_{m+1} be the union of all these U_t', where t is ranging over all $m+1$-splitting nodes of T_m.

The sequence $T_0 \supset T_1 \supset T_2 \supset ...$ is a fusion sequence, and therefore $T = \bigcap_n T_n$ is a perfect tree. Also $T \leq T_0$.

In conclusion: we showed that the set of all trees T satisfying the following condition is dense in $\mathbb{S}$: for any splitting node t of T there is some ordinal $\zeta < \eta$ such that $T_{t^\frown 0}$ and $T_{t^\frown 1}$ decide conflicting statements about "$\zeta \in \tau$".

Since G is generic, there is a tree $T \in G$ is above. We can now define f from the set $X = \tau[G]$ as follows. First f extends the stem of T, t_0. There is some ζ such that $T_{t_0^\frown i}$ force conflicting information about $\zeta \in \tau$. Choose the unique i such that $T_{t_0^\frown i} \Vdash \zeta \in \tau \iff \zeta \in X$. Let t_1 be the next splitting node of T above $t_0^\frown i$. Continue this way, and define $f = \bigcup_n t_n$. Therefore f is in $L[X]$. By Lemma 10.5 it follows that G is in $L[X]$, and therefore $L[G] \subseteq L[X]$. $\square$