Riemann Surfaces and Theta Functions

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Contents

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Chapter 1

Riemann surfaces

1.1 Definition and examples

We begin with some general facts about topological spaces and differential geometry.

Definition 1.1.1 *A (real/complex) manifold of dimension n is a set M with a collection of pairs* $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ *where* $U_\alpha \subset \mathcal{M}$ *and* $\phi_\alpha : U_\alpha \to (\mathbb{R}/\mathbb{C})^n$ *on their respective images and such that*

1. $\phi_\alpha(U_\alpha)$ *is open in* $[\mathbb{R}/\mathbb{C}]^n$ *and* $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ *is one-to-one.*

2. *The sets* U_α *are a covering of* $\mathcal{M}$
$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \mathcal{M} \tag{1.1.1}$$

3. *If* $U_{\alpha,\beta} := U_\alpha \cap U_\beta \neq \emptyset$ *then both* $\phi_\alpha(U_{\alpha,\beta})$ *and* $\phi_\beta(U_{\alpha,\beta})$ *are open and*
$$G_{\alpha,\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha,\beta}) \to \phi_\alpha(U_{\alpha,\beta}) \tag{1.1.2}$$
are ($\mathcal{C}^k$*/analytic) functions of all the respective variables.*

The maps ϕ_α *are called* **local coordinates***, the sets* U_α *are called* **local charts***. The functions* $G_{\alpha,\beta}$ *are called* **transition functions***.*

Given two collections of local coordinate-charts $\{\phi_\alpha, U_\alpha\}_\alpha$ and $\{\psi_\beta, V_\beta\}_\beta$, we say that they are **equivalent** if their union still defines a (real/complex) manifold structure. The equivalence classes of local coordinate-charts $[\{(U_\alpha, \phi_\alpha)\}_\alpha]$ are called **atlases** (or **conformal structure** in the complex case).

Note that –interchanging $\alpha \leftrightarrow \beta$ in the last point of the definition– we have that $G_{\alpha,\beta}$ are invertible and the inverse is in the same class ($\mathcal{C}^k$ or analytic), $G^{-1}_{\alpha,\beta} = G_{\beta,\alpha}$.

A complex n-dimensional manifold is also a real $\mathcal{C}^\infty$ manifold of dimension $2n$. We will be concerned with manifolds of complex dimension 1 and hence the local charts $z_\alpha = \phi_\alpha(p)$ will be complex valued functions providing local identification of $\mathcal{M}$ with a domain in $\mathbb{C}$. The set $\mathcal{M}$ becomes immediately a topological space with the topology inherited via ϕ_α^{-1} from $[\mathbb{R}/\mathbb{C}]^n$; an open set U in $\mathcal{M}$ is a set such that $\phi_\alpha(U)$ is open $\forall\alpha$.

From now on we restrict the formulation to complex one–dimensional manifolds, but many definitions and statements are obvious specializations of more general ones where either we have more dimensions or we change the " category" of functions from "analytic" (holomorphic) to $\mathcal{C}^k$ or else.

Definition 1.1.2 *Let $\mathcal{M}$ be a complex one-dimensional manifold with atlas $\{(U_\alpha, z_\alpha)\}$. A function f : $\mathcal{M} \to \mathbb{C}$ is said to be* **holomorphic** *(***meromorphic***) if for each local chart we have*

$$\begin{array}{rcl} f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) & \to & \mathbb{C} \\ z_\alpha & \mapsto & f_\alpha(z_\alpha) := f(\phi_\alpha^{-1}(z_\alpha)) \end{array} \tag{1.1.3}$$

is holomorphic/meromorphic on the open set $\phi_\alpha(U_\alpha)$).

Note that on the intersection of charts $U_{\alpha,\beta}$ the notion of holomorphicity/meromorphicity in the different coordinates is the same since the transition functions are biholomorphic.

Theorem 1.1.1 *Let $\mathcal{M}$ be* **connected and compact** *in the topology of the atlas. Then the only holomorphic functions are constants.*

Proof. Since $|f|$ is continuous on the compact $\mathcal{M}$ then it takes on a maximum at $p \in \mathcal{M}$. Let $p \in U_\alpha$, then f_α has a maximum modulus in the interior of $\phi_\alpha(U_\alpha)$ and hence it is constant on U_α. Let $q \in \mathcal{M}$ and since $\mathcal{M}$ is connected it is also arcwise connected (**exercise**). Let γ be a continuous path from p to q: by compactness of γ it can be covered by a finite number of charts U_{α_j}, with $U_{\alpha_0} = U_\alpha$. By induction you can show that $f_{\alpha_k} \equiv C \Rightarrow f_{\alpha_{k+1}} \equiv C$ and hence $f_{\alpha_N} = C = f_{\alpha_0}$. Q.E.D.

Definition 1.1.3 *Let $\mathcal{M}$ and N be two complex one-dimensional manifolds with atlases respectively (U_α, ϕ_α) and (V_β, ψ_β). We say that a map*

$$\varphi : \mathcal{M} \to N \tag{1.1.4}$$

is **holomorphic** *if at any point $p \in \mathcal{M}$, $p \in U_\alpha$, $\varphi(p) \in V_\beta$ then $w_\beta = \psi_\beta(f(\phi_\alpha^{-1}(z_\alpha)$ is holomorphic in a small disk arount $\phi_\alpha(p)$.*

Remark 1.1.1 *It is customary to abuse the notation and identify a point $p \in U_\alpha$ with its* coordinate *$z_\alpha = z_\alpha(p) := \phi_\alpha(p)$. The above function then would be written as $w_\beta = f(z_\alpha)$.*

Definition 1.1.4 *Two complex manifolds $\mathcal{M}, N$ are* **biholomorphic** *(or* **biholomorphically equivalent***) if there exist two holomorphic bijections $\varphi : \mathcal{M} \to N$ and $\psi : N \to \mathcal{M}$ such that $\varphi \circ \psi = Id_N$ and $\psi \circ \varphi = Id_{\mathcal{M}}$. This defines an equivalence relation (***exercise***).*

When considering complex manifolds we do not distinguish between manifolds which are biholomorphically equivalent and hence we re-define a **complex manifold** to be the equivalence class of complex manifolds (as in the former definition).

Definition 1.1.5 *A holomorphic map $\varphi : \mathcal{M} \to \mathcal{M}$ which admits holomorphic inverse is called auto-biholomorphism or* **automorphism** *(for short). The set of automorphisms of a (complex) manifold $\mathcal{M}$ will be denoted by $Aut(\mathcal{M})$ and it is a group with respect to the composition of maps.*

Definition 1.1.6 *A map $\phi : \mathcal{M} \to N$ of Riemannn surfaces is said to be* **holomorphic** *(or* **analytic***) if in each local chart (of $\mathcal{M}$ and N) it is represented by a holomoprhic funciton.*

We have the easy

Theorem 1.1.2 *If $\varphi : \mathcal{M} \to N$ is a holomorphic mapping (nonconstant) between two connected RIemann surfaces then it is surjective*

Proof. Since φ is holomprhic, it is also open (**exercise**) and hence $\varphi(\mathcal{M})$ is open and closes in N, hence $\varphi(\mathcal{M}) = N$. **Q.E.D.**

1.1.1 Example: $\mathbb{C}P^1$

This is possibly the most famous example; it is also called the **Riemann's sphere**. It is the first of a sequence of spaces $\mathbb{C}P^n$ defined as follows

Definition 1.1.7 *The complex manifold $\mathbb{C}P^n$ is defined as $\mathbb{C}^{n+1} \setminus \{0\}/\sim$, where the equivalence relation $\sim$ is*

$$(Z_0, \dots, Z_n) \sim (Z'_0, \dots, Z'_n) \quad \Leftrightarrow \ \exists \lambda \in \mathbb{C}^\times \ s.t. \ Z_i = \lambda Z'_i \ \ \forall i = 0, \dots, n \tag{1.1.5}$$

Customarily there are $n+1$ charts that form an atlas:

$$U_k := \{\mathbf{Z} \text{ s.t. } Z_k \neq 0\}/\sim \tag{1.1.6}$$

with coordinates $z_j^{(k)} = Z_j/Z_k$, $j \neq k$. In the intersection $U_k \cap U_\ell$ one has

$$z_j^{(k)} = \frac{Z_j}{Z_k} = \frac{Z_j}{Z_\ell}\frac{Z_\ell}{Z_k} = \frac{z_j^{(\ell)}}{z_k^{(\ell)}} \ . \tag{1.1.7}$$

In the simplest case of $\mathbb{C}P^1$ we have only two charts

$$U_0 = \{(Z_0, Z_1) : \ Z_1 \neq 0\} \ , U_1 = \{(Z_0, Z_1) : \ Z_0 \neq 0\} \tag{1.1.8}$$

with the coordinates

$$z = \frac{Z_0}{Z_1} \ , \quad z' = \frac{1}{z} = \frac{Z_1}{Z_0} \ . \tag{1.1.9}$$

To put it differently, $\mathbb{C}P^1$ consists of the complex plane $\mathbb{C}$ with one added point ∞ (i.e. an Alexandrov's compactification). In a neighborhood of ∞ the local coordinate is declared to be $z' = \frac{1}{z}$, so that $z'(\infty) = 0$.

Exercise 1.1.1 *Prove that $\mathbb{C}P^n$ are compact complex manifolds.*

1.1.2 Algebraic functions and algebraic curves

Definition 1.1.8 *A function $f(z)$ defined on a domain $\mathcal{D}$ is called* **algebraic** *if there exists a polynomial function $P(w, z)$ such that*

$$P(f(z), z) \equiv 0, z \in \mathcal{D}. \tag{1.1.10}$$

The locus

$$\mathcal{C} := \{(w, z) \in \mathbb{C}^2 : \ P(w, z) = 0\} \tag{1.1.11}$$

is called an **algebraic curve**.

Sometimes it is useful to consider a rational function $R(x, y)$ instead of a polynomial and the definition requires a certain specification so as to "avoid" the zeroes of the denominator.

The second remark is that if $P(f(z), z) \equiv 0$ in $\mathcal{D}$ then so must be for **any analytic continuation** of f along any path: indeed if $\tilde{f}$ is the analytic continuation of f then the analytic continuation of $P(f(z), z)$ is $P(\tilde{f}(z), z)$ and since it is the continuation of the zero function it must be identically zero.

We now prove that a polynomial equation $P(w, z) = 0$ of degree n in w defines locally n (germs) of analytic functions. More precisely

Proposition 1.1.1 *Given the algebraic equation $P(w, z) = 0$ with*

$$P(w, z) = A_n(z)w^n + \ldots + A_0(z) \ , A_n(z) \not\equiv 0 \ , \tag{1.1.12}$$

and a point $(w_0, z_0) \in \mathbb{C}^2$ such that $\partial_w P\Big|_{(w_0,z_0)} \neq 0$ then there is a germ of analytic function $f(z) = w_0 + \sum_{n\geq 1} c_n(z - z_0)^n$ which satisfies the functional equation.

Sketch of proof. We regard the function $P(w, z) : \mathbb{C}^2 \to \mathbb{C}$ as a $\mathcal{C}^\infty$ function $\tilde{P} : \mathbb{R}^4 \to \mathbb{R}^2$. Then the condition $P_w \neq 0$ at (w_0, z_0) guarantees that the rank of the Jacobian of the function $\tilde{P} : \mathbb{R}^4 \to \mathbb{R}^2$ is maximal and can be solved locally for $\Re(w), \Im(w)$ yielding differentiable (actually infinitely differentiable) functions of $\Re(z), \Im(z)$. Then one has to check that these functions satisfy also Cauchy–Riemann equations. Q.E.D.

From now on we assume (to avoid trivial occurrences) that $P(w, z)$ is **irreducible** namely (def) cannot be written as the product of two non-constant polynomials. For simplicity in the discussions below we may also require that the **discriminant** of $P(w, z)$ (viewed as a polynomial in w with parameter z) is not the identically zero function of z.

Manifold structure on the locus $P(w,z) = 0$

We continue our consideration of the locus of point $(w,z) \in \mathbb{C}^2$ such that $P(w,z) = 0$. We wish to introduce a manifold structure on it. We restrict to the so–called smooth curves

Definition 1.1.9 *The set*

$$\mathcal{L} := \{(w,z) \in \mathbb{C}^2 \ : \ P(w,z) = 0\} \tag{1.1.13}$$

is called a **plane algebraic curve**. *We say that it is* **non-singular** *if the two complex partial derivatives* $P_w(w,z)$ *and* $P_z(w,z)$ *never vanish at the same point* $(w,z) \in \mathcal{L}$.

On non-singular curves we define the local coordinates as follows:

1. In a neighborhood U of a point (w_0, z_0) where $P_w(w_0, z_0) \neq 0$ we know from Prop. 1.1.1) that there is a unique holomorphic function $w(z)$ on a suitably small disk $D_{z_0}(\epsilon) : \{|z - z_0| < \epsilon\}$ which satisfies identically the equation $P(w(z), z) = 0$. In this disk we use z as coordinate.

2. In a neighborhood U of a point (w_0, z_0) where $P_z(w_0, z_0) \neq 0$ then, by the same arguments as before with interchange of the rôles, we use w as local coordinate.

In a neighborhood of a point where both derivatives P_w, P_z do not vanish, then we can use either w or z as coordinate. The derivatives are computed from the (complex) implicit differentiation theorem

$$\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{P_z(w,z)}{P_w(w,z)} , \qquad \frac{\mathrm{d}z}{\mathrm{d}w} = -\frac{P_w(w,z)}{P_z(w,z)} . \tag{1.1.14}$$

On the plane curve $\mathcal{L}$ we have the two functions $g(w,z) = w$ and $f(w,z) = z$ which are clearly holomorphic. Consider

$$f : \mathcal{L} \to \mathbb{C}(w,z) \mapsto z \tag{1.1.15}$$

Its ramification points are where the point z_j such that

$$P(w, z_j) = 0 \ \ P_w(w, z_j) = 0 . \tag{1.1.16}$$

and these are precisely the zeroes of the discriminant of P w.r.t. w.

Exercise 1.1.2 *Consider*

$$P(w,z) = (w - z^2)(w - 2z) . \tag{1.1.17}$$

Show that the construction above results in a non-connected topological space (P is not irreducible).

Instead of continuing within this level of abstraction, we take an instructive (and famous) class of plane curves.

Hyperelliptic curves

By definition they are plane curves of the form

$$w^2 = P(z)\ ,\qquad P(z) = c\prod_{j=1}^{n}(z - z_j)\ , c \neq 0 \tag{1.1.18}$$

We immediately assume $c = 1$ by possibly rescaling w.

Exercise 1.1.3 *This curve is nonsingular iff the z_j's are distinct.*

In this case there are only two sheets

$$w_\pm = \pm\sqrt{P(z)} \tag{1.1.19}$$

where the square root is defined on the simply connected domain described before. By analytic continuation then we have well–defined analytic functions on the domains $\mathcal{D}_\pm$. We claim (and this can be suitably generalized) that we can choose a different set of cuts in $\dot{\mathbb{C}} := \mathbb{C} \setminus \{z_1, \ldots, z_n\}$ where the new domain $\widetilde{\mathcal{D}}$ is **not simply–connected** but nevertheless the analytic continuation of $w_\pm$ gives bona-fide holomorphic functions.

Choose arcs of curves joining $[z_{2j-1}, z_{2j}]$ and –if n is odd– the last z_n to infinity in such a way that these arcs are simple and mutually avoiding. Take $\widetilde{\mathcal{D}}$ to be the plane $\mathbb{C}$ less these cuts.

We claim that

Theorem 1.1.3 *Given $z_0 \in \widetilde{\mathcal{D}}$ and $w_\pm$ the two germs of analytic functions defined at z_0 by the two square-roots of $P(z)$. Then they can be analytically continued to $\widetilde{\mathcal{D}}$ to holomorphic functions.*

Proof (sketch). Take a closed loop based at z_0 that encircles only one of the cuts say $[z_1, z_2]$; this contour must intersect the original cuts originating from z_1 and z_2. The analytic continuation "changes sign" twice and hence is analytically continued to the same function. Q.E.D.

We define a **compactification** of these plane curves; we distinguish the case of n even or odd and use the model given by the dissection $\widetilde{\mathcal{D}}$.

Even n. In this case no branchcut extends to ∞ in either sheet. We compactify $\mathcal{L}$ by adding two points $\infty_\pm$ to $\mathcal{D}_\pm$ with local coordinates $\zeta = \frac{1}{z}$.

Odd $\mathbf{n = 2k + 1}$. There is a branchcut extending to ∞ on both sheet. We compactify $\mathcal{L}$ by adding one points denoted by ∞ with the following local description: define $\eta = \frac{z^k}{w}$ and $\zeta = \frac{1}{z}$, then

$$\eta^2 = \frac{1}{\zeta^{2k}P(\frac{1}{\zeta})} = \frac{\zeta}{\zeta^n P(\frac{1}{\zeta})} = \frac{\zeta}{1 + \mathcal{O}(\zeta)} \tag{1.1.20}$$

The local coordinate can be chosen to be η near $\eta = 0, \zeta = 0$.

We will denote by $\overline{\mathcal{L}}$ the compactifications thus defined.

One has the following result

Theorem 1.1.4 *The meromorphic functions on $\overline{\mathcal{L}}$ are all functions of the form*

$$F = R_0(z) + yR_1(z) \tag{1.1.21}$$

with R_0, R_1 rational functions of z.

1.2 Holomorphic maps

Let $\mathcal{M}$ and N be two Riemann surfaces, both connected

Definition 1.2.1 *A nonconstant holomorphic map $\varphi : \mathcal{M} \to N$ is called a* **covering**.

Let $\varphi : \mathcal{M} \to N$ be a covering; let $P \in \mathcal{M}$ and z be a local coordinate near P, with $z(P) = 0$ (aka a **local parameter at** P). Let w be a local parameter at $\varphi(P) \in N$; then the map φ is locally represented as a function $w = f(z)$ defined on a neighborhood of $z = 0$ and covering a neighborhood of $w = 0$. Therefore it is representable as

$$w = f(z) = z^{b+1}(C + \mathcal{O}(z)) \ , \quad C \neq 0 \ , \quad b \in \mathbb{N} \tag{1.2.1}$$

The integer $b = b_\varphi(P)$ is called the **ramification number** of f at P and it **does not depend on the choice of local parameters**; it is simple to prove that we can make a change of coordinate $z \to \widetilde{z}$ such that f is locally represented by

$$w = \widetilde{z}^{b+1} \ , \tag{1.2.2}$$

and hence the map φ takes on the same value exactly $(b_\varphi(P) + 1)$ times in a neighborhood of P.

Definition 1.2.2 *The points $P \in \mathcal{M}$ with $b_\varphi(P) > 0$ are called* **ramification points**, *while the $Q \in N$ such that there is at least one ramification point in $\varphi^{-1}(Q)$ are called* **branch–points**.

We first have

Proposition 1.2.1 *Let $\varphi : \mathcal{M} \to N$ be a nonconstant holomorphic mapping between connected and* **compact** *Riemann surfaces. The number*

$$N_\varphi := \sum_{P \in \varphi^{-1}(Q)} (b_\varphi(P) + 1) \tag{1.2.3}$$

is independent of the point $Q \in N$ and it is called the **sheet number** *of the mapping.*

Proof (sketch). The number N_φ counts how many preimages there are of a point, including multiplicities. One needs to prove that the set $\Sigma := \{Q \in N : \ \sum_{P \in \varphi^{-1}(Q)} (b_\varphi(P) + 1) = N\}$ is both open and closed. The details are left as **exercise**. **Q.E.D.**

Definition 1.2.3 *The number $B = B_\varphi := \sum_{P \in \mathcal{M}} b_\varphi(P)$ is called the* **branching number**.

Chapter 2

Basic Topology

2.1 Fundamental group

The first observation is that any Riemann surface (i.e. complex 1–dimensional manifold) is also a **real surface**, namely a real 2–dimensional manifold (of class $\mathcal{C}^\infty$); the coordinates in a local chart are taken to be the real and imaginary parts of a complex local coordinate. As a topological model they are very simple, as the following theorem (without proof) shows.

Theorem 2.1.1 *Any compact Riemann surface is* homeomorphic *to a sphere with handles. The number of handles is called the* **genus** *of the surface.*
Riemann surfaces of different genera are not homeomorphic (and a fortiori *not biholomorphic).*
Any two (compact) Riemann surfaces of the same genus are homeomorphic but not necessarily biholomorphic.

The notion of *sphere with handles* is left to the common sense of the reader and to Fig. 2.1.

Let be given a (complex or real) connected manifold $\mathcal{M}$.

Definition 2.1.1 *A manifold $\mathcal{M}$ is said to be arc-connected if $\forall x, y \in \mathcal{M}$ there is a continuous curve $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = x, \gamma(1) = y$.*

For general topological spaces the two notions of connectedness are not equivalent, arc-connectedness being stronger than connectedness alone. However

Proposition 2.1.1 (Exercise) *A manifold $\mathcal{M}$ is connected iff it is arc-connected.*

Let $x \in \mathcal{M}$ be chosen arbitrarily and then fixed (the "basepoint"). We consider the collection of all closed curves starting and ending at x

$$\mathcal{L}(x, \mathcal{M}) := \{\gamma : [0,1] \to \mathcal{M}\ ,\ \gamma \in \mathcal{C}([0,1], \mathcal{M}),\ \gamma(0) = \gamma(1) = x\} \tag{2.1.1}$$

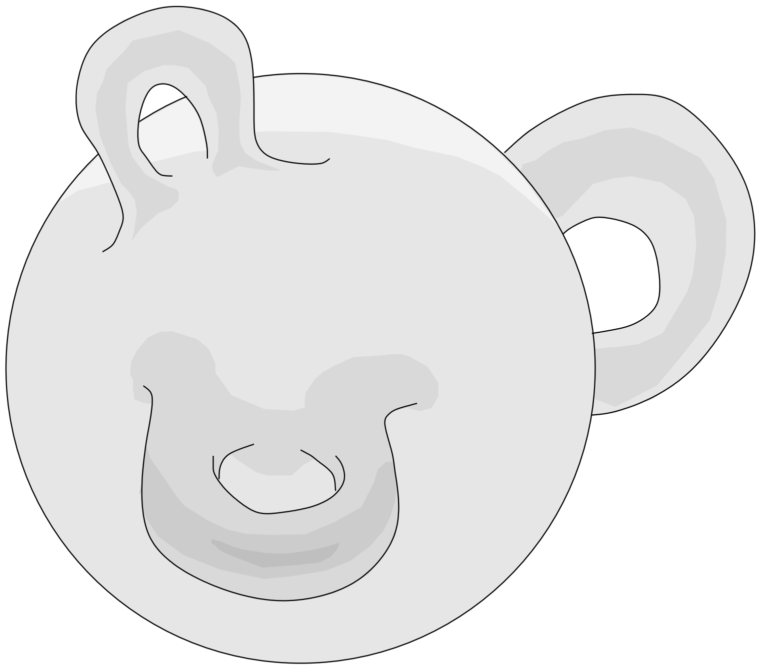

Figure 2.1: A sphere with handles.

We take the set-theoretical quotient of this set by the relation of **homotopy equivalence** at fixed end-points $\sim$

$$\pi_1(x, \mathcal{M}) := \mathcal{L}(x, \mathcal{M}) / \sim \tag{2.1.2}$$

This new set is called the **fundamental group of** $\mathcal{M}$ (or **first homotopy group**). The name is unjustified so far, since we did not define a group structure. We do it now: for any $[\gamma]$, $[\eta] \in \pi_1(x, \mathcal{M})$ with representatives $\gamma, \eta \in \mathcal{L}(x)$ we define the loop

$$\gamma \odot \eta(t) = \begin{cases} \eta(2t) & t \in \left[0, \frac{1}{2}\right) \\ \gamma(2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{2.1.3}$$

This defines a new loop that in the first half time runs along η and then along γ. The symbol $\odot$ here stands for the **concatenation** of contours and can be more generally defined for any two curves γ, η such that the endpoint of η is the starting point of γ. Then we define the product in the fundamental group

$$[\gamma] \cdot [\eta] = [\gamma \odot \eta] \tag{2.1.4}$$

The unit of the multiplication is the class of the constant loop $[0, 1] \to \{x\}$; the inverse of a loop γ is the class of the same loop run in the opposite sense.

Exercise 2.1.1 *Prove that this definition is well-posed (independent of the choice of representatives).*

Exercise 2.1.2 *Let $\mathcal{M}$ be an arc-connected manifold. Prove that $\pi_1(x, \mathcal{M})$ and $\pi_1(x', \mathcal{M})$ are naturally isomorphic and specify this isomorphism.*

This exercise implies that the fundamental group π_1 is "the same" no matter what basepoint is used in the definition and hence we can refer just to the manifold and omit the basepoint $\pi_1(x, \mathcal{M}) \equiv \pi_1(\mathcal{M})$.

Note that saying that $\pi_1(\mathcal{M}) = \{\mathbf{1}\}$ (the trivial group) is a rephrasing of saying that the arc-connected $\mathcal{M}$ is simply connected (and viceversa).

Exercise 2.1.3 *Let $\mathcal{M} = \{z : \ |z| = 1\}$ with the standard topology. Prove that $\pi_1(\mathcal{M}) \simeq \mathbb{Z}$ (the additive group of integers).*

2.2 Homology

Let $\mathcal{M}$ be a compact complex curve (Riemann surface).

Definition 2.2.1 *A* **triangulation** *of $\mathcal{M}$ is a (finite) collection $\mathcal{T}$ of triangles T_j (i.e. the image via a* smooth *mapping of an ordinary triangle respecting orientation), called $2-$***simplices***, such that*

$$\mathcal{M} = \bigcup T_j \ . \tag{2.2.1}$$

Definition 2.2.2 *A (simplicial) $0,1,2$***-chain** *is a formal sum of vertices P_j, edges γ_j or triangles T_j*

$$c_0 = \sum n_j P_j \quad c_1 = \sum m_j \gamma_j \quad c_2 = \sum k_j T_j \tag{2.2.2}$$
$$n_j, m_j, k_j \in \mathbb{Z}$$

The sets of p–chains $\mathcal{C}_p$ have the (natural) structure of free abelian groups (just by formal sums). The opposite of an edge γ is $-\gamma$, the edge in the opposite orientation. Ditto for triangles.

Definition 2.2.3 *We define the* **boundary operator** *δ acting on $0,1,2$–chains as follows: if γ is the oriented edge from vertex A to vertex B (denoted by (AB)) and T is the triangle with vertices (in order) (ABC) then*

$$\delta(A) = 0 \quad \delta\gamma = B - A \in \mathcal{C}_0 \quad \delta T = (AB) + (BC) + (CA) \tag{2.2.3}$$

and then extended by "linearity".

The **fundamental property** is that $\delta^2 \equiv 0$: indeed (we need to check this only for C_2)

$$\delta\delta(T) = \delta\left((AB) + (BC) + (CA)\right) = B - A + C - B + A - C = 0 \ . \tag{2.2.4}$$

Definition 2.2.4 *A p–chain c_p such that $\delta c_p = 0 \in \mathcal{C}_0$ is called a p–***cycle***. A chain which is the boundary of another chain is called a p–***boundary***. Clearly any p-boundary is a p-cycle, but not viceversa.*

In our case, being the manifold of real dimension 2, all the interesting information is contained in $\mathcal{C}_1$; the 1–cycles and 1–boundaries are two subgroups of $\mathcal{C}_1$. Let us denote by $\mathcal{Z}_1$ the subgroup of cycles in $\mathcal{C}_1$ and by $\mathcal{B}_1$ the subgroup of boundaries in $\mathcal{C}_1$; since $\mathcal{B}_1 = \delta\mathcal{C}_2 \subset \mathcal{Z}_1$ (and they are all Abelian groups) we can take the factor group. This is called

Definition 2.2.5 *The* **first homology group** *of $\mathcal{M}$ is denoted by $H_1(\mathcal{M}, \mathbb{Z})$ and is*

$$H_1(\mathcal{M}, \mathbb{Z}) := \frac{\mathcal{Z}_1(\mathcal{M})}{\mathcal{B}_1(\mathcal{M})} = \frac{\mathcal{Z}_1}{\delta\mathcal{C}_2} \ . \tag{2.2.5}$$

The homology groups can be shown to be independent of the choice of triangulation $\mathcal{T}$ (more precisely the homology groups corresponding to two triangulations are isomorphic).

A closed curve $\tilde{\gamma}$ can be homotopically deformed to a chain of edges in the triangulation $\mathcal{T}$ thus defining a cycle (**Exercise**: prove that it is a cycle!); this can be called a **simple cycle**.

One has

Proposition 2.2.1 *The first homology group $H_1(\mathcal{M}, \mathbb{Z})$ is isomorphic to the Abelianization of the first homotopy group, namely*

$$H_1(\mathcal{M}, \mathbb{Z}) \simeq \frac{\pi_1(\mathcal{M})}{[\pi_1(\mathcal{M}), \pi_1(\mathcal{M})]} \ . \tag{2.2.6}$$

It is a **free Abelian group** *with $2g$ generators and hence it is isomorphic to $\mathbb{Z}^{2g}$. These generators can be chosen as (classes of) simple cycles.*
Any cycle can be written as sum of simple cycles (with coefficients in $\mathbb{Z}$).

What this means can be simplified as follows: let $\Gamma_1, \ldots, \Gamma_n$ be generators of $\pi_1(\mathcal{M})$[1] and let γ be a closed loop. Then

$$[\gamma]_{\pi_1} = [\Gamma_{i_1}]_{\pi_1}^{j_1} \cdots [\Gamma_{i_k}]^{j_k}, \quad j_i \in \mathbb{Z} \tag{2.2.7}$$

If now $[\bullet]_{H_1}$ denotes the homology class, we have

$$[\gamma]_{H_1} = j_1[\Gamma_{i_1}]_{H_1} + \ldots + j_k[\Gamma_{i_k}]_{H_1}. \tag{2.2.8}$$

This in particular goes to showing that the homology is independent of the triangulation.

Intersection number

The notion of intersection number is more general than the one given here as it applies to any two submanifolds of complementary dimensions. In our case of complex one-dimensional manifold (i.e. real surface) two submanifolds of complementary dimension must have both dimension 1 (i.e. they must be curves) or 0 and 2 (points and domains). The latter case is rather degenerate (although not meaningless) and we focus only on the first case.

[1] We do not prove the fact that $\pi_1(\mathcal{M})$ is always **finitely generated**.

Given two simple cycles γ and η we represent them as smooth closed curves and we consider their intersection: again, possibly by a small deformation of one or both contours we can reduce to the situation that

(a) the intersection is finite and

(b) all intersections occur **transversally**, i.e. the tangents to γ and η at the point of intersection are not parallel.

Given $p \in \gamma \cap \eta$ one such point of intersection, we associate a number $\nu(p) \in \{+1, -1\}$ as follows. Let z be a local coordinate at p: the two (arcs) of γ and η now are arcs in a neighborhood of $z(p) = 0$ crossing each other transversally. We denote by $\dot{\gamma}_0$ and $\dot{\eta}_0$ the two tangent vectors at $z(p) = 0$; if the determinant of their compoments is positive we set $\nu(p) = 1$, if it is negative we set $\nu(p) = -1$. In other words the number $\nu(p)$ indicates the orientation of the axis spanned by $\dot{\gamma}_0$ and $\dot{\eta}_0$ (in this order!) relative to the orientation of the standard $\Re(z)$, $\Im(z)$ axes.

Definition 2.2.6 *The intersection number between γ and η is then defined by*

$$\gamma \sharp \eta := \sum_{p \in \gamma \cap \eta} \nu(p) \ . \tag{2.2.9}$$

It follows immediately from the definition that $\gamma \sharp \eta = -\eta \sharp \gamma$ and the intersection number is an integer. One can also prove that;

Proposition 2.2.2 *The intersection number is invariant under smooth homotopy deformations of γ and η.*

Therefore the intersection number depends only on the *homotopy classes* of γ and η, which we then denote by $[\gamma]\sharp[\eta]$.

In particular it makes sense to compute the **self-intersection** of a cycle

$$[\gamma]\sharp[\gamma] = 0 \ . \tag{2.2.10}$$

This makes sense because in the actual computation one chooses two different representatives in the same class of γ which intersect transversally: the fact that the result is zero then follows from the antisymmetry.

Note also that the intersection number depends on the orientation of the contours: if we reverse one contour the int. number changes sign

$$[\gamma]\sharp[\eta] = -[\gamma]^{-1}\sharp[\eta] \ . \tag{2.2.11}$$

Moreover

Lemma 2.2.1 *The intersection number of any boundary β with any cycle γ vanishes $\gamma \sharp \beta = 0$.*

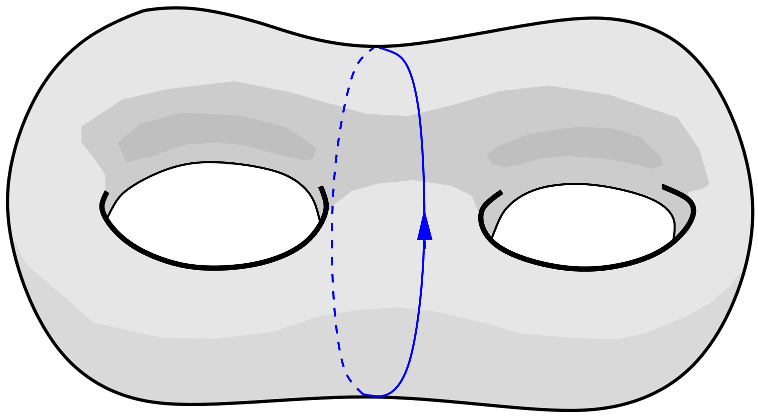

Figure 2.2: The blue contour is not homotopic to the trivial loop but it is homologous to zero because it separates the surface.

Proof. A boundary β is a collection of simple cycles that bound a domain (made of several triangles); if γ is a symple cycle it must traverse the boundary of this domain an even number of times, and two consecutive crossing count with opposite sign, hence cancel out. (You should make a picture). Q.E.D.

Remark 2.2.1 *A cycle may be* **homologous** *to the trivial cycle but* **not homotopic** *to a point, for example the one in Fig. 2.2.*

This lemma implies that the intersection number is well defined as a pairing on the first homology group. More in fact is true

Theorem 2.2.1 *The intersection pairing*

$$\sharp : H_1(\mathcal{M}, \mathbb{Z}) \times H_1(\mathcal{M}, \mathbb{Z}) \to \mathbb{Z} \tag{2.2.12}$$

is a bilinear skew–symmetric map. If $\mathcal{M}$ is a compact *Riemann surface then it is* **nondegenerate***.*

Remark 2.2.2 *The other homology groups are defined similarly: in particular $H_0(\mathcal{M}, \mathbb{Z})$ is made of the classes of points that cannot be joined by cycles. You should convince yourself that $H_0(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}^k$ where k is the number of connected components of $\mathcal{M}$ (hence for us $k = 1$). The generator is the class of any vertex.*

For $H_2(\mathcal{M}, \mathbb{Z})$ we have that H_2 is zero if $\mathcal{M}$ is compact and $\mathbb{Z}$ if it is not compact.

2.2.1 Homology of a compact Riemann surface of genus g

We have said that $H_1(\mathcal{M}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g}$ and that the intersection pairing is antisymmetric and nondegenerate. It can be shown that there are simple cycles

$$\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\} \tag{2.2.13}$$

that generate $H_1(\mathcal{M},\mathbb{Z})$ and such that

$$a_i \sharp a_j = 0,\ b_i \sharp b_j = 0\ ,\quad a_i \sharp b_j = \delta_{ij}\ . \tag{2.2.14}$$

Definition 2.2.7 *A basis of $H_1(\mathcal{M},\mathbb{Z})$ satisfying (2.2.14) is called a* **canonical basis***.*

A canonical basis exists but it is not unique: indeed suppose we make a transformation

$$\begin{pmatrix} \vec{a}' \\ \vec{b}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \tag{2.2.15}$$

where $\vec{a},\vec{b}$ denote the g generators and the $2g\times 2g$ matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is integer valued and nonsingular. The basis $\vec{a}',\vec{b}'$ will be a set of generators provided that S^{-1} is also integer–valued and hence the determinant of S must be ± 1.

Moreover if we want that the new basis is also canonical this forces

$$J := \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix} = \begin{pmatrix} \vec{a}' \\ \vec{b}' \end{pmatrix} \sharp(\vec{a}',\vec{b}') = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \sharp(\vec{a},\vec{b}) \tag{2.2.16}$$

so that

$$J = SJS^t \tag{2.2.17}$$

Matrices of dimension $2g\times 2g$ satisfying (2.2.17) form a group, the symplectic group, denoted by $Sp(g,\mathbb{Z})$.

2.2.2 Canonical dissection of a compact Riemann–surface

We take a basepoint P_0 and consider the homotopy group $\pi_1(\mathcal{M},P_0)$ of loops based at P_0. Amongst these there are $2g$ generators $a_1,b_1,\ldots,a_g,b_g$ whose *homology* classes form a canonical basis. Although these loops are only identified by their homotopy classes, we will think of them as concrete choices of (smooth) closed curves on the surface with basepoint P_0.

Definition 2.2.8 *The canonical dissection of $\mathcal{M}$ is the simply connected domain $\mathcal{L}$ obtained by removing the $2g$ generators identified above.*

The boundary of this domain consists of **both sides** of each generator and hence consists of $4g$ arcs (see Fig. 2.3). Viceversa we could start with a $4g$–gon with sides $a_1,b_1,a_1',b_1',\ldots$ and identify topologically the sides a_j,a_j', b_j,b_j' with opposite orientations. The result is a topological model of a Riemann surface of genus g.

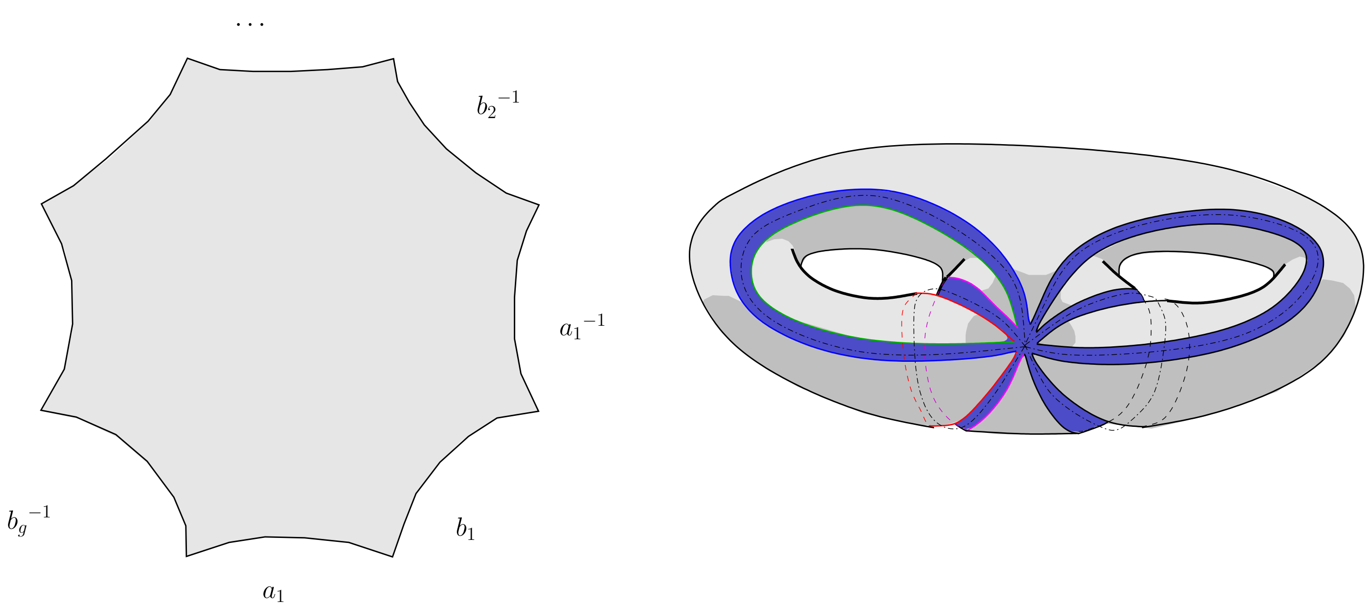

Figure 2.3: An example of a canonical dissection (genus 2)

Chapter 3

Differential and integral calculus

3.1 Differential forms

We always assume that $\left(\mathcal{M}, \{(U_\alpha, \phi_\alpha)\}_\alpha\right)$ is a complex one-dimensional variety with atlas of charts $z_\alpha = z_\alpha(p) = \phi_\alpha(p)$. Instead of introducing the abstract notion of tangent and cotangent space (which would be the "comme-il-faut" way) we give a "hands-on" definition of forms.

We need to consider also some non-holomorphic objects, and for this reason we will use the real coordinates x, y given by $z = x + iy$;

Definition 3.1.1 *A smooth, complex one-form ω is a collection of* smooth $\mathbb{C}$*–valued functions f_α, g_α such that*

$$\omega = f_\alpha \mathrm{d}z_\alpha + g_\alpha \mathrm{d}\overline{z_\alpha} \tag{3.1.1}$$

is independent of the chart*, namely*

$$f_\alpha(z_\alpha, \overline{z_\alpha}) = f_\beta(z_\alpha, \overline{z_\alpha}) \frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha} \ , \qquad g_\alpha(z_\alpha, \overline{z_\alpha}) = g_\beta(z_\alpha, \overline{z_\alpha}) \overline{\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}} \tag{3.1.2}$$

The form ω is said to be a $(1,0)$ form ($(0,1)$ respectively) if

$$\omega = f(z, \overline{z})\mathrm{d}z \qquad (\omega = g(z, \overline{z})\mathrm{d}\overline{z}) \tag{3.1.3}$$

with f, g **smooth** *(i.e. not holomorphic). The form ω is said to be* **holomorphic** *if f_α's are all holomorphic functions and $g_\alpha \equiv 0$, namely*

$$\omega = f(z)\mathrm{d}z \ , \tag{3.1.4}$$

in any local parameter $z = z_\alpha$.
It is called **antiholomorphic** *if viceversa*

$$\omega = g(\overline{z})\mathrm{d}\overline{z} \tag{3.1.5}$$

Note the distinction between $(1,0)$ forms and holomorphic forms, in that f is only smooth in the first case but holomorphic in the second.

In real coordinates $z_\alpha = z = z + iy$ this means

$$\omega = f\,(\mathrm{d}x + i\mathrm{d}y) + g\,(\mathrm{d}x - i\mathrm{d}y) = \big(f+g\big)\mathrm{d}x + i\big(f-g\big)\mathrm{d}y \tag{3.1.6}$$

Since the coordinates on $\mathcal{M}$ are local holomorphic equivalences, they are in particular conformal and thus they preserve angles; therefore the rotation by $\pi/2$ makes intrinsic sense. In terms of differentials this reads

Definition 3.1.2 *The* **Hodge dual** *of a one–form $\omega = f\mathrm{d}z + g\mathrm{d}\overline{z}$ is*

$$\star\,\omega := if\mathrm{d}z - ig\mathrm{d}\overline{z}. \tag{3.1.7}$$

Note that in real coordinates that reads

$$\star\,\omega = i\big(f-g\big)\mathrm{d}x - \big(f+g\big)\mathrm{d}y\ , \tag{3.1.8}$$

namely, precisely the "usual" rotation and that

$$\star\star\omega = -\omega. \tag{3.1.9}$$

Note also that if $u(x,y)$ is a (real–valued for the time being) harmonic function $((\partial_x^2+\partial_y^2)u(x,y)=0)$ then its **harmonic conjugate** is precisely the antiderivative of the Hodge dual of its differential, namely another harmonic function v such that

$$\mathrm{d}v = \star\mathrm{d}u\ . \tag{3.1.10}$$

Together they form a function $f = u + iv$ which is holomorphic (this is nothing but Cauchy–Riemann relations for a holomorphic function). We seek to generalize this to the case of forms on a Riemann surface a little later.

Definition 3.1.3 *A two–form is a collection of functions $f_\alpha(z_\alpha,\overline{z}_\alpha)$ such that*

$$\eta := f_\alpha\,\mathrm{d}z_\alpha \wedge \mathrm{d}\overline{z}_\alpha \tag{3.1.11}$$

is independent of the local coordinate.

In other words, if $\widetilde{z} = w(z)$ is another local coordinate and

$$\eta = f\mathrm{d}z \wedge \mathrm{d}\overline{z} = \widetilde{f}\mathrm{d}\widetilde{z} \wedge \mathrm{d}\overline{\widetilde{z}} \tag{3.1.12}$$

then

$$f(z,\overline{z}) = \widetilde{f}(\widetilde{z},\overline{\widetilde{z}})\,|w'(z)|^2 \tag{3.1.13}$$

Finally we have the exterior derivative d, which is the sum of two operators $\partial, \overline{\partial}$ as follows; les $f, \omega = f\mathrm{d}z + g\mathrm{d}\overline{z}, \eta = h\mathrm{d}z \wedge \mathrm{d}\overline{z}$ be a function, $1,2$ forms

$$\mathrm{d}f := \partial f + \overline{\partial} f := \partial_z f \mathrm{d}z + \partial_{\overline{z}} f \mathrm{d}\overline{z} \tag{3.1.14}$$

$$\mathrm{d}\omega := \mathrm{d}f \wedge \mathrm{d}z + \mathrm{d}g \wedge \mathrm{d}z = \big(\partial_{\overline{z}} f - \partial_z g\big)\mathrm{d}z \wedge \mathrm{d}\overline{z} \tag{3.1.15}$$

$$\mathrm{d}\eta \equiv 0. \tag{3.1.16}$$

Note the fundamental property of the exterior derivative that

$$\mathrm{d}^2 \equiv 0. \tag{3.1.17}$$

(which is very similar to the property of the boundary operator on chains... except that d increases the degree of forms, whereas δ decreases the dimension of chains...).

Definition 3.1.4 *A smooth k–differential ω if called* **exact** *if there is a* smooth $k-1$ *differential η such that*

$$\omega = \mathrm{d}f \tag{3.1.18}$$

A k–differential ω is called **closed** *if $\mathrm{d}\omega \equiv 0$. The vector space of k–differentials being denoted by Ω^k, the space of closed differentials being denoted by $\mathcal{Z}^k$ we define the k–th* **de–Rham cohomology group** *as the vector–space quotient*

$$H^k(\mathcal{M}, \mathbb{C}) := \frac{\mathcal{Z}^k}{\mathrm{d}\Omega^{k-1}} = \frac{\{\text{Closed differentials}\}}{\{\text{Exact differentials}\}} \tag{3.1.19}$$

The most important for us will be $H^1(\mathcal{M}, \mathbb{C})$; the relation between H^1 and H_1 will be explained in the next section.

Remark 3.1.1 *All holomorphic and antiholomorphic differentials are closed.*
A $(1,0)$ differential is closed iff it is holomorphic (and similarly for $(0,1)$ forms).

3.1.1 Integration formulæ

By generalization one calls ordinary functions **0-forms**; this is useful and consistent. Indeed we can integrate 1-forms on curves, 2–forms on domains (pieces of surfaces) and 0–forms on 0-dimensional submanifolds, namely points.

In other words **we can integrate k–forms on k–chains.**; if c_0 is the 0–chain

$$c_0 = \sum k_i P_i \tag{3.1.20}$$

and f is any (continuous) function we can write

$$\int_{c_0} f = \sum k_j f(P_i)\ . \tag{3.1.21}$$

Similatly if c_1 is a 1–chain $c_1 = \sum k_j \gamma_j$ we can write

$$\int_{c_1} \omega = \sum k_j \int_{\gamma_j} \omega \tag{3.1.22}$$

where the integrals on the RHS are the usual line–integrals. And so on and so forth.

The interplay between the boundary operator $\delta : \mathcal{C}_k \mapsto \mathcal{C}_{k-1}$ and k–forms is contained in

Theorem 3.1.1 (Stokes) *Let $\mathcal{D} \subset \mathcal{M}$ be a domain with piecewise smooth boundary $\delta\mathcal{M}$ (endowed with the natural orientation); then, for any one–form ω (sufficiently regular) we have*

$$\int_{\mathcal{D}} \mathrm{d}\omega \tag{3.1.23}$$

Note that Stokes' theorem applies also to 1–chains and differentials of functions

$$\int_{c_1} \mathrm{d}f = \int_{\delta c_1} f \ . \tag{3.1.24}$$

We have here a glimpse of the duality between d and δ.

More importantly Stokes' theorem implies that the integral of a closed form on a cycle **does not depend on its representative in the homology class**. Indeed if $\delta\mathcal{D} = \gamma_1 - \gamma_2$ and ω is closed then

$$0 = \int_{\mathcal{D}} \mathrm{d}\omega = \int_{\delta D} \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega. \tag{3.1.25}$$

In addition the integral of a closed form ω on a cycle is independent of its representative on the **cohomology class** for

$$\int_{\gamma} \omega = \int_{\gamma} \omega' + \mathrm{d}f = \int_{\gamma} \omega' + \int_{\delta\gamma} f = \int_{\gamma} \omega' \tag{3.1.26}$$

This shows immediately that

Proposition 3.1.1 *The integration is a pairing between the first homology group and the first cohomology group*

$$\int : H_1(\mathcal{M}, \mathbb{Z}) \times H^1(\mathcal{M}, \mathbb{C}) \to \mathbb{C} \ . \tag{3.1.27}$$

The relevance of the statement lies in that the result of the integration is independent of the choices of representatives in the respective classes.

Remark 3.1.2 *We will see that $H^1(\mathcal{M}, \mathbb{C})$ for a compact Riemann surface of genus g is a vector space of dimension $2g$ and that a basis can be chosen formed by g holomorphic differentials and g antiholomorphic ones.*

Remark 3.1.3 *This explains in part how the homology is the abelian part of the homotopy group. Indeed, fix any closed one–form ω: then one knows that the integral $\int_\gamma \omega$ depends only on the homotopy class of the closed loop γ. In other words one has a map*

$$\begin{array}{rl} \int_\bullet \omega : \pi_1(\mathcal{M}) & \mapsto \mathbb{C} \\ [\gamma] & \mapsto \int_\gamma \omega \end{array} \tag{3.1.28}$$

which is well-defined and also a **group homomorphism** *into* $(\mathbb{C}, +)$*. It is obvious that the kernel of this map contains the commutator subgroup* $[\pi_1(\mathcal{M}), \pi_1(\mathcal{M})]$*, in other words this map descends to a homomorphism for the homotopy group.*

3.1.2 Riemann Bilinear identity

This is a fundamental theorem which is also relatively easy to prove. We start with the

Definition 3.1.5 *Given a closed differential ω and a cycle γ the integral $\oint_\gamma \omega$ is called the* **period** *of ω along γ.*

Theorem 3.1.2 *Let ω, η be two* **closed differentials** *and $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be* any *canonical basis on the Riemann surface $\mathcal{M}$ of genus g. Then*

$$\int_{\mathcal{M}} \omega \wedge \eta = \sum_{j=1}^{g} \oint_{a_j} \omega \oint_{b_j} \eta - \oint_{a_j} \eta \oint_{b_j} \omega \tag{3.1.29}$$

Proof. We start by observing that the RHS is independent of the choice of canonical basis (**exercise**). The quantities entering the identity are the periods of the differentials on the basis; they clearly depend only on the homology class of the cycles, so for our computation we can choose them represented by convenient contours.

We take them as generators of the **homotopy** group of $\mathcal{M}$, $\pi_1(\mathcal{M})$ with basepoint $P_0 \in \mathcal{M}$ and then cut the surface open along these canonical cycles so as to obtain a canonical simply connected domain $\mathcal{L}$. Since $\mathcal{M} \setminus \mathcal{L}$ has measure zero we then have

$$\int\!\!\int_{\mathcal{M}} \omega \wedge \eta = \int\!\!\int_{\mathcal{L}} \omega \wedge \eta. \tag{3.1.30}$$

Now the two–form $\omega \wedge \eta$ is closed and since $\mathcal{L}$ is simply connected it is actually the exterior derivative of the following one-form

$$\omega \wedge \eta(p) = \mathrm{d}(F(p)\eta) \tag{3.1.31}$$

where

$$F(p) = \int_{P_0}^{p} \omega. \tag{3.1.32}$$

In this integration the path of integration is a path lying entirely in $\mathcal{L}$, and since $\mathcal{L}$ is simply connected there is only one homotopy class (at fixed endpoints). In other words, $F(p)$ is a well–defined **smooth** function on $\mathcal{L}$.

If $p, p' \in \partial\mathcal{L}$ are points on the boundary of the domain $\mathcal{L}$ that correspond to the same point in $\mathcal{M}$ on the cuts, accessed from the two sides, then there is a unique homotopy class of contour $\gamma_{pp'} \subset \mathcal{L}$ joining them; this contour correspond to a homotopy class $[\gamma] \in \pi_1(\mathcal{M})$ and

$$F(p) = F(p') + \oint_\gamma \omega \tag{3.1.33}$$

Given a point $p \in \partial\mathcal{L}$ and a homotopy class γ as above we will denote $p' = p + [\gamma]$.

Now the boundary $\partial\mathcal{L}$ consists of the two sides of each cut, and using Stokes' theorem on the domain $\mathcal{L}$ we have

$$\begin{aligned}
&\iint_{\mathcal{L}} \omega \wedge \eta = \iint_{\mathcal{L}} \mathrm{d}(F(p)\eta) = \oint_{\partial\mathcal{L}} F(p)\eta(p) = \\
&= \sum_{j=1}^g \int_{P_0}^{P_0+a_j} F(p)\eta(p) - \int_{P_0+b_j}^{P_0+b_j+a_j} F(p)\eta(p) + \int_{P_0}^{P_0+b_j} F(p)\eta(p) - \int_{P_0+a_j{}^{-1}}^{P_0+b_j+a_j{}^{-1}} F(p)\eta(p) = \\
&= \sum_{j=1}^g \int_{P_0}^{P_0+a_j} (F(p) - F(p+b_j))\eta(p) + \int_{P_0}^{P_0+b_j} (F(p) - F(p-a_j))\eta(p) = \\
&= \sum_{j=1}^g \oint_{a_j} \omega \oint_{b_j} \eta - \oint_{b_j} \omega \oint_{a_j} \eta \ .
\end{aligned} \tag{3.1.34}$$

Q.E.D.

We can now start estimating the dimension of the vector space of holomorphic differentials.

Definition 3.1.6 *Let $\mathcal{H}^1$ be the space of holomorphic differentials.*

Proposition 3.1.2 *Let ω be a* **holomorphic** *differential and let A_j, B_j be its periods on a canonical basis for the first homology group. Then*

$$\Im \sum_{j=1}^g A_j \overline{B}_j \le 0 \ , \tag{3.1.35}$$

with the equality being valid only if $\omega \equiv 0$.

Proof. We have

$$\int_{\mathcal{M}} i\omega \wedge \overline{\omega} = \int_{\mathcal{M}} i|h|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z} = 2\int_{\mathcal{M}} |h(z)|^2 \mathrm{d}x \wedge \mathrm{d}y \ge 0 \tag{3.1.36}$$

where the equality is valid iff $h(z) \equiv 0$ (in local coordinates in all charts).

On the other hand Riemann's bilinear relations with $\eta = i\overline{\omega} = \star\omega$ gives

$$\int_{\mathcal{M}} i\omega \wedge \overline{\omega} = i\sum_{j=1}^g A_j\overline{B}_j - \overline{A}_j B_j = -2\Im\sum_{j=1}^g A_j\overline{B}_j. \tag{3.1.37}$$

This concludes the proof. **Q.E.D.**

Corollary 3.1.1 *If $\omega \in \mathcal{H}^1$ and all a–periods (or all b–periods) vanish (in any symplectic canonical basis) then $\omega \equiv 0$.*
If $\omega \in \mathcal{H}^1$ and all (a, b)–periods are purely real (or purely imaginary) then $\omega \equiv 0$.

Corollary 3.1.2 *The dimension of $\mathcal{H}^1$ does not exceed $g = \mathrm{genus}(\mathcal{M})$.*

Proof. If it were $\dim \mathcal{H}^1 \geq g+1$ we could find $K = \dim \mathcal{H}^1$ linearly independent $\omega_1, \ldots \omega_K$. Consider the $g \times K$ matrix

$$A_{j,k} := \oint_{a_j} \omega_k \ ; \tag{3.1.38}$$

such matrix cannot have rank greater than $\min(g, K) = g$. Hence there must be a linear combination of said ω_j's such that all a–periods vanish. By the previous corollary such a combination would be zero, thus contradicting the linear independence of $\omega_1, \ldots, \omega_K$. **Q.E.D.**

3.2 Zeroes, poles and residues: Abelian differentials of the three kinds

A holomorphic/meromorphic differential ω is said to have a pole at a point P if in a local coordinate z near P (w.l.o.g. $z(P) = 0$) $\omega = f(z)\mathrm{d}z$ with $f(z)$ meromorphic with a pole at $z = 0$. The order of the pole of ω at P is the order of the pole of f at $z = 0$. Similarly the notion of order of zero of a differential is defined. These notions do not depend on the local coordinate chosen.

Definition 3.2.1 *Given a meromorphic differential ω we denote $ord_{\,\omega}(P)$ the order of ω at P namely the multiplicity of the zero if P is a zero, minus the order of the pole if P is a pole or 0 if P is a regular point.*

Definition 3.2.2 *Given a meromorphic differential ω with a pole at the point P we denote*

$$\operatorname*{res}_P \omega := \frac{1}{2i\pi} \oint_{|z(P)|=\epsilon} \omega = \operatorname*{res}_{z=0} f(z)\mathrm{d}z = f_{-1} \tag{3.2.1}$$

and call it the **residue** *of ω at the pole P, where f_{-1} is the coefficient of the Laurent expansion of $f(z)$ at $z(P) = 0$ of the power -1. Alternatively the residue is the result of an integration of ω around a small positively oriented loop around the point*[1]

We immediately have

Theorem 3.2.1 *Let $\mathcal{M}$ be a* **compact complex curve**. *Let ω be a meromorphic one–form. Then the sum of all residues is zero*

$$\sum_{p=pole\ of\ \omega} \operatorname*{res}_p \omega = 0 \tag{3.2.2}$$

[1] Here "small" simply means that the loop is homotopic to the point; typically one chooses a circle in a local coordinate.

Proof. We take the domain $\mathcal{D} = \mathcal{M} \setminus \bigcup_{p=\text{ poles of }\omega} D_p(\epsilon)$ where $D_p(\epsilon)$ stands for small disks centered at p (in a local coordinate). Then the sum of residues is precisely (minus) the integral of ω on the boundary $\partial\mathcal{D}$

$$\sum_{p=\text{pole of }\omega} \operatorname*{res}_p \omega = -\int_{\partial\mathcal{D}} \omega = -\iint_{\mathcal{D}} \mathrm{d}\omega = 0 \tag{3.2.3}$$

by Stokes' theorem. Q.E.D.

Definition 3.2.3 *An Abelian differential is any (holo/mero)morphic differential on the Riemann surface* $\mathcal{M}$*.*

- *A* holomorphic *Abelian differential is said to be of the* **first kind***.*
- *A* meromorphic *Abelian differential is said to be of the* **second kind** *if it has poles but the residues at each pole are zero.*
- *A meromorphic Abelian differential is said to be of the* **third kind** *if it has only simple poles (and hence with nonzero residues).*

According to this definition a meromorphic differential is a sum of differentials of the three kinds.

3.3 Existence Theorems

We define the Hilbert space $L^2(\mathcal{M}, \Omega^1)$ to be the closure of the space of smooth differentials on $\mathcal{M}$ with the inner product

$$(\omega, \eta) := \int_{\mathcal{M}} \omega \wedge \star\overline{\eta} \tag{3.3.1}$$

Exercise 3.3.1 *Check that this inner product satisfies the properties of a Hilbert inner product. Check also that* $\star$ *is an isometry.*

We introduce two subspaces in $L^2(\mathcal{M})$

Definition 3.3.1 *The space of* **exact** *differentials is the closure of the subspace of differentials of compactly supported* **smooth** *functions*

$$E := \overline{\{\mathrm{d}f : \ f \in \mathcal{C}_0^\infty(\mathcal{M})\}} \ . \tag{3.3.2}$$

Similarly we define the space of **co-exact** *differentials as the closure of the subspace of Hodge-duals of exact differentials*

$$E^\star := \overline{\{\star\mathrm{d}f : \ f \in \mathcal{C}_0^\infty(\mathcal{M})\}} \tag{3.3.3}$$

or, with obvious meaning $E^\star = \star E$*.*

We point out immediately that if $\mathcal{M}$ is not compact, there might be exact differentials which do not belong to E; this is the case if they are the differentials of non-compactly supported functions.

We need to characterize the orthogonal complements of these two spaces

Proposition 3.3.1 *Let* $\omega \in \mathcal{C}^1(\mathcal{M})$*. Then*

- $\omega \in E^\perp$ *iff* ω *is co-closed,* $\mathrm{d}\star\omega = 0$*.*
- $\omega \in (E^\star)^\perp$ *iff* ω *is closed,* $\mathrm{d}\omega = 0$*.*

Proof. The form ω is in $E^\perp$ iff it is orthogonal to all smooth **exact** differentials (by density). Therefore for all $f \in \mathcal{C}_0^\infty(\mathcal{M})$

$$\int_{\mathcal{M}} \mathrm{d}f \wedge \star\overline{\omega} = \int_{\mathcal{M}} \mathrm{d}(f \star \overline{\omega}) - \int_{\mathcal{M}} f \mathrm{d} \star \overline{\omega} \ . \tag{3.3.4}$$

The first term vanishes by Stokes' theorem because f is compactly supported. Therefore we have $\int_{\mathcal{M}} f \mathrm{d}\star \overline{\omega} = 0$ for all $f \in \mathcal{C}_0^\infty(\mathcal{M})$. Therefore $\mathrm{d}\star\overline{\omega}$ must vanish identically (the conjugation does not change the result). Similarly one proves the second statement. **Q.E.D.**

Corollary 3.3.1 *The two spaces* $E, E^\star$ *are orthogonal to each other.*

Exercise 3.3.2 *Prove Corollary 3.3.1.*

Exercise 3.3.3 *Let* W, V *be two mutually orthogonal closed subspaces of a Hilbert space. Prove that*

$$(W \oplus V)^\perp = W^\perp \cap V^\perp. \tag{3.3.5}$$

Let us now introduce

$$H := E^\perp \cap (E^\star)^\perp = (E \oplus E^\star)^\perp. \tag{3.3.6}$$

We aim to prove that H consists of harmonic differentials, where the definition is

Definition 3.3.2 *A differential* ω *is harmonic if it is smooth* ($\mathcal{C}^\infty$) *and both closed and co-closed.*

According to this definition

$$\omega = f_\alpha \mathrm{d}z_\alpha + g_\alpha \mathrm{d}\overline{z}_\alpha \tag{3.3.7}$$

is harmonic iff f is holomorphic and g antiholomorphic. Indeed

$$\mathrm{d}\omega = (\partial_{\overline{z}} f - \partial_z g)\mathrm{d}z \wedge \mathrm{d}\overline{z} \tag{3.3.8}$$

$$\mathrm{d}\star\omega = i(\partial_{\overline{z}} f + \partial_z g)\mathrm{d}z \wedge \mathrm{d}\overline{z} \ . \tag{3.3.9}$$

Equivalently ω is locally $\mathrm{d}h$ with $\partial_z\partial_{\overline{z}} h = 0$.

It is clear from the characterization of the orthogonal complements of $E, E^\star$ that H contains all harmonic differentials: the key point is to show that any $\eta \in H$ is (almost everywhere equal to) a harmonic differential.

The proof is technical and requires dropping smoothness assumptions; it is based on the following lemma (see [1]).

Lemma 3.3.1 (Weyl's lemma) *Let F be square–integrable on $\mathbb{D} := \{|z| < 1\}$. Then F is holomorphic if and only if*

$$\int_{\mathbb{D}} f \partial_{\overline{z}} \eta \mathrm{d}z \wedge \mathrm{d}\overline{z} = 0 \tag{3.3.10}$$

for all functions $\eta \in \mathcal{C}_0^\infty(\mathbb{D})$.

Exercise 3.3.4 *Prove that H consists of only harmonic differentials (using Weyl's lemma).*

Summarizing these results we have the orthogonal decomposition

$$L^2(\mathcal{M}, \Omega) = E \overset{\perp}{\oplus} E^\star \overset{\perp}{\oplus} H \tag{3.3.11}$$

3.3.1 Holomorphic differentials

Given a harmonic differential

$$\eta = f \mathrm{d}z + g \mathrm{d}\overline{z} \tag{3.3.12}$$

we know f is holomorphic and g is antiholomorphic. Hence

$$\omega := \eta - i \star \eta = 2f \mathrm{d}z \tag{3.3.13}$$

is holomorphic. This gives a tool to construct holomorphic differentials from harmonic ones (and viceversa).

We now **assume $\mathcal{M}$ is compact** and find the dimension of the space of holomorphic differentials (and the dimension of the first cohomology group H^1_{dR}).

To this end we introduce the notion of harmonic differential associated to a closed curve.

Definition 3.3.3 *Let γ be a smooth closed curve in $\mathcal{M}$ that does not separate $\mathcal{M}$ (i.e. $\mathcal{M} \setminus \gamma$ is still connected). Let f_γ be the (discontinuous) function defined as follows: it is identically equal to 1 on a small left neighborhood of the curve γ, it vanishes on a bigger left neighborhood that strictly contains the previous one, and it is smooth ($\mathcal{C}^\infty$) on $\mathcal{M} \setminus \gamma$ (see Fig. 3.1).*

Then the differential $G_\gamma := \mathrm{d}f_\gamma$ is closed (but not exact because f_γ is not smooth!). According to the decomposition of the Hilbert space,

$$G_\gamma = \mathrm{d}h + \eta_\gamma \tag{3.3.14}$$

where η_γ is harmonic and h is smooth. This is the harmonic differential associated to γ.

Exercise 3.3.5 *For a curve γ as in Def. 3.3.3 show that there exists a closed curve $\gamma^\star$ that intersects γ only at one point.*

If $\gamma^\star$ is a closed curve crossing γ at only one point P: then it is easy to show (using the fundamental theorem of calculus!)

$$\int_{\gamma^\star} G_\gamma = \int_{\gamma^\star} \mathrm{d}h + \eta_\gamma = \int_{\gamma^\star} \eta_\gamma = \gamma \sharp \gamma^\star \ . \tag{3.3.15}$$

where $\gamma \sharp \gamma^\star$ is the intersection number (which is ± 1 depending on the relative orientations).

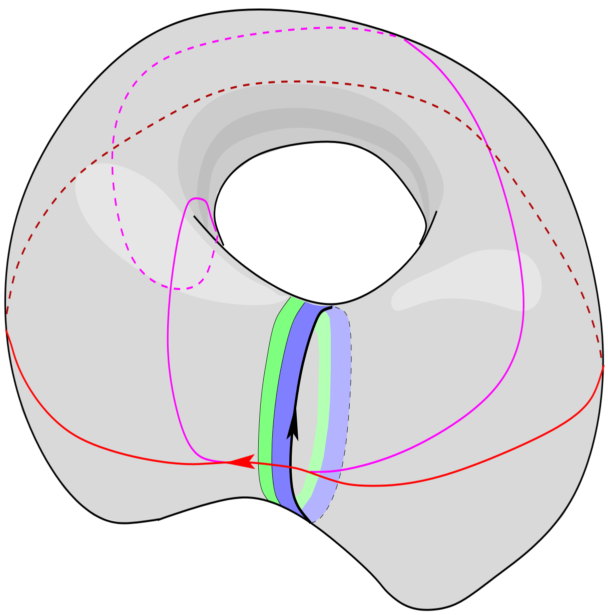

Figure 3.1: The curve γ (black) used in Def. 3.3.3: the function f_γ is 1 on the blue region, vanishes outside the colored belts and connects smoothly 1 to 0 on the green belt. Shown is also a curve $\gamma^\star$ (red); there are, however several choices, the only requirement is that it connects a point on the right of γ to a point outside the green belt on the left of γ, and how it then connects these two points is immaterial. For example the purple contour serves the same purpose, althoug it is neither homologically nor homotopically equivalent to the red one.

Proposition 3.3.2 *The dimension of H is $2g$, where g is the genus of $\mathcal{M}$. The dimension of the first cohomology group is $2g$. The dimension of the space of holomorphic differentials is g,* $\dim \mathcal{H}^1 = g$.

Proof. Choosing a basis of cycles $a_1, b_1, \ldots, a_g, b_g$ in the homology $H_1(\mathcal{M}, \mathbb{Z})$ one can then construct $2g$ linearly independent harmonic differentials; this proves that $\dim H \geq 2g$. On the other hand, any harmonic differential is the sum of a holomorphic and antiholomorphic,

$$H \subseteq \mathcal{H}^1 \overset{\perp}{\oplus} \overline{\mathcal{H}^1} \tag{3.3.16}$$

and hence $\dim H = 2 \dim \mathcal{H}^1$ (note that the above sum is an orthogonal sum). We have seen in Corollary 3.1.2 that $\dim \mathcal{H}^1 \leq g$. The proof follows combining the two inequalities.

Finally, the first cohomology is the quotient of closed modulo exact smooth differentials; since closed differentials are orthogonal to co-exact, the quotient is isomorphic to H by our decomposition theorem. **Q.E.D.**

We can now draw several important but easy conclusions, all of which are left as **exercises**

Proposition 3.3.3 *With the notations of this section we have*

$$\gamma \sharp \gamma' = \int_\gamma \eta_{\gamma'} = \int_{\mathcal{M}} \eta_{\gamma'} \wedge \eta_\gamma. \tag{3.3.17}$$

Proposition 3.3.4 *Fixing a symplectic canonical basis in* $H_1(\mathcal{M}, \mathbb{Z}) = \mathbb{Z}\{a_1, b_1, \ldots, a_g, b_g\}$ *we can always find*

1. *a basis of H consisting of $2g$ harmonic forms η_j such that*

$$\oint_{a_j} \eta_k = \delta_{jk} = \oint_{b_j} \eta_{g+k} \tag{3.3.18}$$

$$\oint_{a_j} \eta_{g+k} = \oint_{b_j} \eta_k = 0 \ , \qquad j,k = 1, \ldots, g. \tag{3.3.19}$$

2. *a basis of holomorphic differentials in* $\mathcal{H}^1 = \mathbb{C}\{\omega_1, \ldots, \omega_g\}$ *such that*

$$\oint_{a_i} \omega_j = \delta_{ij} \ , \qquad i,j = 1, \ldots, g \tag{3.3.20}$$

3.3.2 Differentials of second and third kind

We want to prove existence of differentials of 2nd and 3rd kind; in principle we will provide explicit formulæ in terms of Theta functions later on. However, to rigor, those formulæ rely on Jacobi inversion theorem and Abel theorem which are proven using the existence of these differentials.

Second kind differentials

Let $P \in \mathcal{M}$ be a point and z a local coordinate (arbitrarily chosen but fixed in the following) such that $z(P) = 0$. Let $\mathbb{D}_\epsilon$ be the coordinate disk of radius ϵ Let $\rho : \mathcal{M} \to \mathbb{R}$ be a function that is identically 1 on $\mathbb{D}_\epsilon$, identically zero on $\mathcal{M} \setminus \mathbb{D}_{2\epsilon}$ and $\mathcal{C}_0^\infty(\mathcal{M})$ (we assume that ϵ is small enough so that both disks belong to the chart of the local coordinate z).

Define

$$G(z, \overline{z}) := \left\{ \begin{array}{cc} -\frac{z^{-K}}{K}\rho(z,\overline{z}) & z \in \mathbb{D}_{2\epsilon} \\ 0 & \in \mathcal{M} \setminus \mathbb{D}_{2\epsilon} \end{array} \right. \tag{3.3.21}$$

Then G is holomorphic in the punctured disk $\mathbb{D}_\epsilon \setminus \{P\}$ and $\mathcal{C}^\infty(\mathcal{M} \setminus \{P\})$.

Now $\mathrm{d}G = \partial_z G \mathrm{d}z + \partial_{\overline{z}} G \mathrm{d}\overline{z}$ is a closed differential on $\mathcal{M} \setminus \{P\}$: its $(0,1)$ part is smooth on $\mathcal{M}$ (**exercise**) and

$$\alpha := 2\partial_{\overline{z}} G \mathrm{d}\overline{z} = \mathrm{d}G + i \star \mathrm{d}G = \mathrm{d}f + \star \mathrm{d}g + \eta \tag{3.3.22}$$

is the decomposition into exact, co-exact and harmonic differentials (all of which smooth). Then

Exercise 3.3.6 *The differential $\rho = \mathrm{d}G - \mathrm{d}f$ is harmonic in $\mathcal{M} \setminus \{P\}$ and $\rho - \frac{\mathrm{d}z}{z^{K+1}}$ is harmonic in $\mathbb{D}_\epsilon$.*

Proposition 3.3.5 *The differential*

$$\eta = \frac{1}{2}(\rho + i \star \rho) \tag{3.3.23}$$

is meromorphic on $\mathcal{M}$ with a pole only at P of order $K+1$; the differential

$$\eta - \frac{\mathrm{d}z}{z^{K+1}} \tag{3.3.24}$$

is **holomorphic** *in $\mathbb{D}_\epsilon$.*

This proposition means that η is a differential such that in the local coordinate z near P it has the expansion

$$\eta = \left(z^{-K-1} + \mathcal{O}(1) \right) \mathrm{d}z \tag{3.3.25}$$

Remark 3.3.1 *It is important to point out that the construction* **depends on the choice of the local parameter**. *However there is a class of changes of local parameter (***Exercise: determine this class***) such that the ensuing second kind differential is independent of the choice in this class.*

Third kind differentials

In a very similar way one can construct third kind differentials; we start from two points in the same coordinate chart P, Q and use the function $G = \ln \frac{z-z(P)}{z-z(Q)}$ in the above construction (one should also slit the curve along a path joining P to Q). The resulting differential η_{PQ} is holomorphic in $\mathcal{M} \setminus \{P, Q\}$ and has simple poles at P and Q with residues $+1$ and -1 respectively.

In order to construct third kind differentials with poles at two arbitrary points (not necessarily on the same coordinate chart) we "add and subtract" into a telescopic sum

$$\eta_{PQ} = \eta_{PS_1} + \eta_{S_1S_2} + \ldots + \eta_{S_kQ} \tag{3.3.26}$$

where $S_\ell, S_{\ell+1}$ belong to the same coordinate chart (and P, S_1 and S_k, Q as well).

3.3.3 Normalized differentials of the second and third kind

For several applications later on it is necessary to "normalize" the differentials above constructed. First of all any differential of the second kind is a linear combination of the second kind differentials constructed above, for different choices of K and P. Similarly for third kind differentials.

Now it should be clear that if η is an Abelian differential of the second or third kind and ω is of the first kind (i.e. holomorphic) then $\eta + \omega$ is still of the same kind as η.

Definition 3.3.4 *An Abelian differential of the second(third) kind is said to be* **normalized** *with respect to the canonical basis $a_1, b_1, \ldots, a_g, b_g$ if*

$$\oint_{a_j} \eta = 0 \ , \forall j = 1, \ldots, g. \tag{3.3.27}$$

Exercise 3.3.7 *Prove that if η is not normalized, it is always possible to add a first kind differential ω so that $\eta + \omega$ is normalized.*

Harmonic functions with prescribed singularities

The construction of differentials of the second and third kind allows us to construct also harmonic functions with prescribed singularities.

First of all if H is a harmonic function so are its real and imaginary parts; so we can simply construct real–valued harmonic functions. To do this consider an Abelian differential of the second kind η with a pole at P of order K and local expansion

$$\eta = (z^{-K-1} + \mathcal{O}(1))\mathrm{d}z \tag{3.3.28}$$

Using the basis of harmonic differentials (note that they are real valued) $\eta_j, j = 1, \ldots, 2g$ in Prop. 3.3.3 we can always add a linear combination of them such that

$$\rho := \eta + \sum_{j=1}^{2g} c_j \eta_j \tag{3.3.29}$$

has all imaginary periods $\oint_\gamma \rho \in i\mathbb{R}$ for all cycle γ. It follows that the locally defined function

$$H(P) := \Re\left(\int_{P_0}^{P} \rho\right) \tag{3.3.30}$$

extends to a single valued harmonic function on the whole $\mathcal{M} \setminus \{P\}$; near P it has the behavior

$$H \sim |z|^{-K} + \mathcal{O}(1) \tag{3.3.31}$$

Doing the same for the third kind differential η_{PQ} yields a harmonic function that has logarithmic singularities at P, Q.

3.4 Reciprocity theorems

Suppose that the surface has been cut open along the basis; the resulting simply connected domain $\mathcal{L}$ is topologically a $4g$–gon with identifications on the sides. We first establish another form of Riemann–bilinear relations.

Proposition 3.4.1 (Bilinear relation 2) *Let η be a meromorphic Abelian differential and ω an Abelian differential of the* **second** *kind (i.e. residueless). Let $\mathfrak{u}(P) := \int_{P_0}^{P} \omega$ be the meromorphic function on $\mathcal{L}$ defined by integration using contours within $\mathcal{L}$ (which is simply–connected)*[2] *Then*

$$\sum_{Q=pole\ of\ \eta,\omega} \operatorname*{res}_{Q} \mathfrak{u}\,\eta = \frac{1}{2i\pi}\sum_{j=1}^{g} \oint_{b_j} \omega \oint_{a_j} \eta - \oint_{b_j} \eta \oint_{a_j} \omega\ . \tag{3.4.1}$$

Proof. We take the differential (**one**–form) $\mathfrak{u}\,\eta$ on the simply connected domain $\mathcal{L}$ and use residue's theorem. The integration along the boundary produces the RHS exactly as in the proof of the previous version of Riemann's bilinear relations (Thm. 3.1.2). **Q.E.D.**

Consider now an Abelian differential of the second kind η subordinated to the local parameter z, $z(P) = 0$

$$\eta = (z^{-K-1} + \mathcal{O}(1))\mathrm{d}z \tag{3.4.2}$$

Suppose this differential has been **normalized** with respect to a canonical basis in $H_1(\mathcal{M}, \mathbb{Z})$. As a simple consequences we have

Proposition 3.4.2 (Reciprocity theorems) • *The b–periods of the differential η are given by*

$$\oint_{b_j} \eta = \frac{2i\pi}{K} \operatorname*{res}_{P} z^{-K}\omega_j \tag{3.4.3}$$

[2]This function has just poles at the poles of ω but no logarithmic singularities thanks to the condition that all residues vanish.

- *Let η_{PQ} be the normalized third kind differential with simple poles at P, Q then*

$$\oint_{b_j} \eta_{PQ} = \int_Q^P \omega_j \tag{3.4.4}$$

Proof. Using the Riemann bilinear relations above with $\omega = \omega_j$ (the normalized Abelian differentials of the first kind) one obtains immediately this result. **Q.E.D.**

Chapter 4

Compact Riemann surfaces

As the title of the chapter says, we are now exclusively considering compact Riemann surfaces; we recall the basic facts that we have already established

- The first homology group is isomorphic to $\mathbb{Z}^{2g}$ and the generators can be chosen in a canonical way

$$a_j \sharp b_k = \delta_{jk} = -b_k \sharp a_j \ , \quad j,k \leq g \tag{4.0.1}$$

- There are g linearly independent holomorphic differentials $\omega_1, \dots, \omega_g$; they can be uniquely normalized w.r.t. the chosen canonical basis

$$\oint_{a_j} \omega_k = \delta_{jk} \tag{4.0.2}$$

- The space of harmonic differentials is spanned by the holomorphic and antiholomorphic differentials.

4.1 Divisors and the Riemann–Roch theorem

Definition 4.1.1 *A divisor is a finite formal sum of points with integer coefficients*

$$\mathcal{D} = \sum_j k_j P_j \tag{4.1.1}$$

For a given divisor its **degree** *is the sum of all its coefficients*

$$\deg(\mathcal{D}) = \sum_j k_j \tag{4.1.2}$$

Sometimes one uses a multiplicative notation for divisors (e.g. [1]); it is a matter of taste. Divisors are used mainly as "book-keeping devices" to encode the position and order of zeroes/poles of a function or a differential.

Definition 4.1.2 *Given a meromorphic function f (or a meromorphic differential ω) we denote its divisor*

$$(f) = \sum_{P \in \mathcal{M}} ord_{\,f}(P)P \tag{4.1.3}$$

$$(\omega) = \sum_{P \in \mathcal{M}} ord_{\,\omega}(P)P \tag{4.1.4}$$

Note that the above sums are finite (formal) sums.

I recall that the order of a function or differential at a point is 0 if it is a regular point, it is the multiplicity of the zero if P is a zero or minus the order of the pole if P is a pole (by the compactness of $\mathcal{M}$ and the meromorphicity it is an **exercise** to show that there are only finitely many zeroes and poles and all of finite orders).

Exercise 4.1.1 *For a meromorphic function f prove that* $\deg((f)) = 0$.

Let us consider a divisor $\mathcal{D} = \sum k_j P_j$ (finite sum!), $k_j \in \mathbb{Z}$. Consider the following vector space

$$\mathfrak{R}(\mathcal{D}) := \{f \in Mero(\mathcal{M}) : \quad \text{ord}_{\,f}(P_j) \geq k_j \ , \ \ \forall j\} \tag{4.1.5}$$

$$r(\mathcal{D}) := \dim \mathfrak{R}(\mathcal{D}) \ . \tag{4.1.6}$$

It is a simple verification that this is a vector space of meromorphic functions; in words, these are meromorphic functions such that

1. if $k_j > 0$ then they have a zero of order **at least** k_j at P_j;

2. if $k_j < 0$ then they have a pole of order **at most** $|k_j| = -k_j$ at P_j.

Similarly one defines

$$\mathfrak{I}(\mathcal{D}) := \{\omega \in \ \{\text{Meromorphic Abelian differential}\} : \quad \text{ord}_{\,\omega}(P_j) \geq k_j\} \tag{4.1.7}$$

$$i(\mathcal{D}) := \dim \mathfrak{I}(\mathcal{D}) \tag{4.1.8}$$

where the word description is entirely similar to the above.

The Riemann–Roch theorem interrelates the **dimensions** of these two spaces and gives some other tools to study their dimensions.

Definition 4.1.3 *A divisor is called* **(strictly) positive** *(or* **(strictly) integral***) and denote this property by $\mathcal{D} \geq 0$ ($\mathcal{D} > 0$) if $\mathcal{D} = \sum k_j P_j$ with $k_j \geq 0$ ($k_j > 0$). This induces a partial order on the group of divisors $\mathcal{D}' \geq \mathcal{D}$ iff $\mathcal{D}' - \mathcal{D} \geq 0$.*

Definition 4.1.4 (Linear equivalence) *Two divisors $\mathcal{D}_1, \mathcal{D}_2$ are said to be* **linearly equivalent** *if there is a meromorphic function such that $(f) = \mathcal{D}_1 - \mathcal{D}_2$ (or viceversa, using $1/f$). The divisors of meromorphic functions are called* **principal**.

Proposition 4.1.1 *Two linearly equivalent divisors have the same degree.*

Exercise 4.1.2 *Prove Prop. 4.1.1.*

Definition 4.1.5 *The* **divisor class** *of a divisor is the equivalence class modulo linear equivalence.*

Definition 4.1.6 (Canonical class) *The divisor class of any (meromorphic or holomorphic) Abelian differential is denoted by $\mathcal{K}$ and it is called the* **canonical class**.

Note that if ω_1, ω_2 are two differentials then $\frac{\omega_1}{\omega_2}$ is a meromorphic function; indeed it is independent of the choice of local coordinate. This implies immediately (by definition) that there is only one canonical class

Proposition 4.1.2 *Let $\mathcal{D}_1, \mathcal{D}_2$ be linearly equivalent. Then*

- $r(\mathcal{D}_1) = r(\mathcal{D}_2)$
- $i(\mathcal{D}_1) = i(\mathcal{D}_2)$
- $r(\mathcal{D}) = i(\mathcal{D} + \mathcal{K})$ *for any divisor (class) $\mathcal{D}$.*

Proof. Since $\mathcal{D}_1, \mathcal{D}_2$ are linearly equivalent there is a meromorphic function f such that

$$(f) = \mathcal{D}_1 - \mathcal{D}_2 \ . \tag{4.1.9}$$

Let $g \in \mathfrak{R}(\mathcal{D}_1)$. Then $g/f \in \mathfrak{R}(\mathcal{D}_2)$; viceversa if $h \in \mathfrak{R}(\mathcal{D}_2)$ then $hf \in \mathfrak{R}(\mathcal{D}_1)$. Thus we have a bijection

$$f_* : \mathfrak{R}(\mathcal{D}_1) \mapsto \mathfrak{R}(\mathcal{D}_2) \tag{4.1.10}$$

$$g \mapsto f_* g := g/f \tag{4.1.11}$$

which instates a isomorphism of vector spaces. Thus $r(\mathcal{D}_1) = r(\mathcal{D}_2)$. The case of differentials it is entirely parallel

The last equality is proven as follows; let $g \in \mathfrak{R}(\mathcal{D})$ and let ω be any Abelian differential of the first kind (holomorphic) chosen and fixed. Then $\eta := g\,\omega \in \mathfrak{I}(\mathcal{K} + \mathcal{D})$. Viceversa if $\eta \in \mathfrak{I}(\mathcal{K} + \mathcal{D})$ then $\eta/\omega \in \mathfrak{R}(\mathcal{D})$. These two maps are clearly linear and inverse to each other, hence the two spaces are isomorphic. **Q.E.D.**

Proposition 4.1.3 *If* deg $\mathcal{D} > 0$ *then* $r(\mathcal{D}) = 0$.

Proof. If $f \in \mathfrak{R}(\mathcal{D})$ then $(f) - \mathcal{D} \geq 0$; but (using that deg is a homomorphism)

$$0 \leq \deg((f) - \mathcal{D}) = -\deg\ \mathcal{D}\ , \tag{4.1.12}$$

and if $\deg \mathcal{D} > 0$ then this is impossible. (To put it differently, the divisor $\mathcal{D}$ has too many zeroes for the poles). **Q.E.D.**

Proposition 4.1.4 *The following properties hold (and are left as exercise)*

- *If* $\mathcal{D}_1 \geq \mathcal{D}_2$ *then* $r(\mathcal{D}_1) \leq r(\mathcal{D}_2)$.
- *If* $\mathcal{D} = \mathcal{D}_+ - \mathcal{D}_-$ *with both* $\mathcal{D}_\pm$ *strictly positive then*

$$\mathfrak{R}(\mathcal{D}) \subset \mathfrak{R}(-\mathcal{D}_-)\ . \tag{4.1.13}$$

- *If* $\mathbf{0}$ *is the trivial divisor then* $\mathfrak{R}(\mathbf{0}) = \mathbb{C}\{1\}$ *(the span of the constant function).*
- $i(\mathbf{0}) = g$ *because* $\mathfrak{I}(\mathbf{0}) = \mathcal{H}^1$ *(holomorphic differentials).*

4.1.1 Writing meromorphic functions

Given a meromorphic function $F : \mathcal{M} \to \mathbb{C}$ then clearly $\mathrm{d}F$ is a meromorphic differential of the second kind (i.e. without any residue). Suppose we want to study $\mathfrak{R}(-\mathcal{D})$, assuming that $\deg(-\mathcal{D}) \leq 0$ (for otherwise the space is trivial, see Prop. 4.1.3).

We **assume at first** that $\mathcal{D}$ is a positive divisor. We start by constructing all functions in $\mathfrak{R}(-\mathcal{D})$. Let

$$\mathcal{D} = \sum_{j=1}^{N} k_j P_j, \quad k_j \geq 1 \tag{4.1.14}$$

Now, the meromorphic differential $\mathrm{d}F$ satisfies

$$(\mathrm{d}F) \geq -\widetilde{\mathcal{D}} \tag{4.1.15}$$

with

$$\widetilde{\mathcal{D}} := \sum_{j=1}^{N} (k_j + 1) P_j \tag{4.1.16}$$

This simply means that if F has a pole at P of order k its differential has a pole of order $k+1$ at the same point.

Thus we have a map

$$\begin{array}{rcl} \mathrm{d} : \mathfrak{R}(-\mathcal{D}) & \longrightarrow & \mathfrak{I}(-\widetilde{\mathcal{D}}) \\ F & \mapsto & \mathrm{d}F \end{array} \tag{4.1.17}$$

which has one–dimensional kernel consisting of the constant functions. The image of d consists of those meromorphic differentials which are exact, namely those differentials whose periods vanish (all of them). Indeed if η is an Abelian differential of the second kind whose periods vanish

$$\oint_{a_j} \eta = \oint_{b_j} \eta = 0 \tag{4.1.18}$$

then $\int \eta$ is a well–defined meromorphic function (the integration does not depend on the class of the contour of integration by the vanishing of the periods).

We have just proved

Lemma 4.1.1 *The image of* $\mathrm{d} : \mathfrak{R}(-D) \to \mathfrak{I}(-\widetilde{\mathcal{D}})$ *consists of the subspace of meromorphic differentials in the target space that are of the second kind and whose periods vanish.*

The next key tool is using **reciprocity formulæ** (Thm. 3.4.2).

Let us denote the space of second kind differentials as follows

$$\mathfrak{I}_{II}(\mathcal{D}) := \{\omega \in \mathfrak{I}(\mathcal{D}),\ \omega \text{ a } \textbf{normalized } 2^{\text{nd}} \text{ kind differential}\} \tag{4.1.19}$$

$$i_{II}(\mathcal{D}) := \dim \mathfrak{I}_{II}(\mathcal{D}) \ . \tag{4.1.20}$$

It is very easy to compute $i_{II}(-\widetilde{\mathcal{D}})$; for each point $P_j \in \mathcal{D}$ we construct all second kind differentials (using the procedure of Sec. 3.3.2) with poles of order not more than $k_j + 1$. If they are normalized along the a–cycles there are k_j of them. Taking linear combination for all points $P_j \in \mathcal{D}$ we obtain

$$i_{II}(-\widetilde{\mathcal{D}}) = \sum_{j=1}^{N} k_j = \deg \mathcal{D} \tag{4.1.21}$$

(If we had considerer non-normalized differentials then we would have the freedom to add any holomorphic differential and hence the dimension would increase by g.)

Now d maps $\mathfrak{R}(-\mathcal{D})$ into a proper subspace of $\mathfrak{I}_{II}(-\widetilde{\mathcal{D}})$; this subspace, by our discussion above, is characterized by the vanishing of all b–periods (the a–periods automatically vanish because the space we consider is of normalized 2-nd kind differentials), and they can be expressed in terms of reciprocity theorems.

Let z_j be the local parameters near $P_j \in \mathcal{D}$ (i.e. $z_j(P_j) = 0$) used to construct the Abelian differentials of the second kind; then any $\eta \in \mathfrak{I}_{II}(-\widetilde{\mathcal{D}})$ has the local expansion

$$\eta = \left(\sum_{\ell=1}^{k_j} t_\ell^{(j)} {z_j}^{-\ell-1} + \mathcal{O}(1) \right) \mathrm{d}z_j \tag{4.1.22}$$

$$\oint_{a_j} \eta = 0 \ , \quad j = 1, \ldots, g \ . \tag{4.1.23}$$

Note the absence of the $1/z$ term in the expansion (since the differentials are residueless). By the **reciprocity theorem** (Thm. 3.4.2) we have

$$\frac{1}{2i\pi}\oint_{b_n}\eta = \sum_{j=1}^{N}\sum_{\ell=1}^{k_j}\frac{t_\ell^{(j)}}{\ell}\operatorname*{res}_{P_j} {z_j}^{-\ell}\omega_n \ , \tag{4.1.24}$$

where ω_n are the normalized first–kind differentials. The residues that appear above form a matrix Π of dimension $\deg(\mathcal{D}) \times g$ representing the "period mapping"

$$\Pi^t := \left[\begin{array}{cccc}
\omega_1(P_1) & \omega_2(P_1) & \dots & \omega_g(P_1) \\
\omega_1'(P_1) & \omega_2'(P_1) & \dots & \omega_g'(P_1) \\
\vdots & & & \vdots \\
\omega_1^{(k_1)}(P_1) & \omega_2^{(k_1)}(P_1) & \dots & \omega_g^{(k_1)}(P_1) \\
\hline
\omega_1(P_2) & \omega_2(P_2) & \dots & \omega_g(P_2) \\
\vdots & & & \vdots \\
\omega_1^{(k_2)}(P_2) & \omega_2^{(k_2)}(P_2) & \dots & \omega_g^{(k_2)}(P_2) \\
\hline
\vdots & & & \vdots \\
\hline
\omega_1(P_N) & \omega_2(P_N) & \dots & \omega_g(P_N) \\
\vdots & & & \vdots \\
\omega_1^{(k_N)}(P_N) & \omega_2^{(k_N)}(P_N) & \dots & \omega_g^{(k_N)}(P_N)
\end{array}\right] \tag{4.1.25}$$

where by the evaluations above we have used a short-cut notation

$$\omega^{(\ell)}(P_j) := \frac{1}{\ell}\operatorname*{res}_{P_j} {z_j}^{-\ell}\omega \ , \quad \ell \geq 1 \ . \tag{4.1.26}$$

Since $\Im(\mathrm{d})$ is the kernel of the period mapping Π

$$\Im(\mathrm{d}) = \ker(\Pi) \ , \tag{4.1.27}$$

we have

$$\operatorname{rank}(\mathrm{d}) = \dim\ker(\Pi) = \deg(\mathcal{D}) - \operatorname{rank}(\Pi) \tag{4.1.28}$$

On the other hand the map Π^t is the "residue" map

$$\Pi^t : \mathcal{H}^1 \to \mathbb{C}^{\deg(\mathcal{D})} \tag{4.1.29}$$

that associates to $\omega \in \mathcal{H}^1$ its "residues" $\frac{1}{\ell}\operatorname*{res}_{P_j} {z_j}^{-\ell}\omega$. The kernel of this transposed map consists of all differentials which vanish **at least of order** k_j at all points $P_j \in \mathcal{D}$, in other words

$$\ker(\Pi^t) = \mathfrak{I}(\mathcal{D}) \ . \tag{4.1.30}$$

Finally we have

$$i(\mathcal{D}) = \dim\ker(\Pi^t) = g - \operatorname{rank}(\Pi^t) = g - \operatorname{rank}(\Pi) = g - \deg(\mathcal{D}) + \operatorname{rank}(\mathrm{d}). \tag{4.1.31}$$

Rearranging terms

$$\mathrm{rank(d)} = i(\mathcal{D}) - g + \deg(\mathcal{D}) \tag{4.1.32}$$

Recalling that $r(-\mathcal{D}) = \mathrm{rank(d)} + 1$ we have proved

Theorem 4.1.1 (Riemann–Roch theorem for positive divisors) *Let $\mathcal{D}$ be a positive divisor; then*

$$r(-\mathcal{D}) = i(\mathcal{D}) - g + \deg(\mathcal{D}) + 1 \tag{4.1.33}$$

At this point we want to extend the theorem to an arbitrary divisor: there are a few steps

Lemma 4.1.2 (Degree of $\mathcal{K}$) *The degree of the canonical class is $2g-2$.*

Proof For $g = 0$ one computes the degree of $\mathrm{d}z$ on the Riemann–sphere. For $g > 0$ we want to use R.R.

$$\deg(\mathcal{K}) = r(-\mathcal{K}) - i(\mathcal{K}) + g - 1 \stackrel{\text{by Prop. 4.1.2}}{=} \overbrace{i(\mathbf{0})}^{=g} - i(\mathcal{K}) + g - 1 = 2g - 1 - i(\mathcal{K}) \tag{4.1.34}$$

Now, if $\mathcal{K}$ is the divisor of the holomorphic differential ω then $i(\mathcal{K}) = 1$ for if there were another independent holomorphic differential $\eta \in \mathfrak{I}(\mathcal{K})$ then η/ω would be a meromorphic function without poles, hence a constant (contradiction). The proof is complete. **Q.E.D.**

Lemma 4.1.3 *The Riemann–Roch theorem holds for all divisors that satisfy one or the other of the following conditions*

1. *$\mathcal{D}$ is linearly equivalent to a positive divisor.*
2. *$-\mathcal{D} + \mathcal{K}$ is linearly equivalent to a positive divisor.*

Proof. The proof of 1 is immediate since all quantities depend only on the class. To prove the second assertion we rearrange the terms

$$\begin{aligned} r(-\mathcal{D}) = i(-\mathcal{D} + \mathcal{K}) &\stackrel{\text{Thm. 4.1.1}}{=} r(\mathcal{D} - \mathcal{K}) + g - \deg(-\mathcal{D} + \mathcal{K}) - 1 = \\ = i(\mathcal{D}) + g - (2g-2) - 1 + \deg(\mathcal{D}) &= i(\mathcal{D}) - g + \deg(\mathcal{D}) + 1 \ \textbf{Q.E.D.} \end{aligned} \tag{4.1.35}$$

Lemma 4.1.4 *If $r(-\mathcal{D}) > 0$ then $\mathcal{D}$ is equivalent to a positive divisor.*

Proof. Indeed if $f \in \mathfrak{R}(-\mathcal{D})$ then $(f) + \mathcal{D} \geq -\mathcal{D} + \mathcal{D} = \mathbf{0}$. **Q.E.D.**

Now we can prove the full version of Riemann–Roch theorem; the cases that are left out after Lemma 4.1.4 and Lemma 4.1.3 is the following:
neither the divisor $\mathcal{D}$ nor the divisor $-\mathcal{D}+\mathcal{K}$ are equivalent to a positive divisor, and hence also $r(-\mathcal{D}) = 0$ (by Lemma 4.1.4).

Theorem 4.1.2 (Riemann–Roch theorem) *For any divisor $\mathcal{D}$ on a compact $\mathcal{M}$ we have*

$$r(-\mathcal{D}) = i(\mathcal{D}) - g + \deg \mathcal{D} + 1 \tag{4.1.36}$$

Proof. As we have said it remains only the case $r(-\mathcal{D}) = 0$ for a divisor that **(a)** $\mathcal{D}$ is not equivalent to a positive one and **(b)** $\mathcal{K} - \mathcal{D}$ is not equivalent to a positive one. So we have $r(-\mathcal{D}) = 0 = r(\mathcal{D} - \mathcal{K})$. Suppose $\deg \mathcal{D} \geq g$ and $\mathcal{D} = \mathcal{D}_+ - \mathcal{D}_-$ where $\mathcal{D}_\pm$ are positive divisors. Then

$$r(-\mathcal{D}_+) = i(\mathcal{D}_+) - g + \deg(\mathcal{D}_+) + 1 \geq \deg(\mathcal{D}_+) + 1 - g = \deg(\mathcal{D}) - g + 1 + \deg(\mathcal{D}_-) \geq \deg(\mathcal{D}_-) + 1. \tag{4.1.37}$$

This implies (by linear algebra) that we can find in $\mathfrak{R}(-\mathcal{D}_+)$ a nonzero function that vanishes to the correct order at $\mathcal{D}_-$ (because this imposes $\deg(\mathcal{D}_-)$ linear constraints). Thus $r(-\mathcal{D}) = r(\mathcal{D}_- - \mathcal{D}_+) \geq 1$ which is a contradiction.

Thus we must have $\deg(\mathcal{D}) < g$; but since $\mathcal{K} - \mathcal{D}$ is not linearly equivalent to a positive divisor, the computation above (replacing $\mathcal{D}$ by $\mathcal{K} - \mathcal{D}$) also shows that $\deg(\mathcal{K} - \mathcal{D}) < g$. But then

$$g > \deg(\mathcal{K} - \mathcal{D}) = 2g - 2 - \deg(\mathcal{D}) \ \Rightarrow \deg(\mathcal{D}) > g - 2 \quad \Rightarrow \deg(\mathcal{D}) = g - 1. \tag{4.1.38}$$

Therefore

$$r(-\mathcal{D}) = 0 = i(\mathcal{D}) - g + g - 1 + 1 = i(\mathcal{D}). \tag{4.1.39}$$

Therefore we conclude the proof if we can prove that $i(\mathcal{D}) = 0$. But again

$$i(\mathcal{D}) = r(\mathcal{D} - \mathcal{K}) = 0\ . \tag{4.1.40}$$

This concludes the proof. **Q.E.D.**

4.1.2 Consequences of Riemann–Roch theorem

Proposition 4.1.5 *There is no point $P \in \mathcal{M}$ for which all the holomorphic differentials vanish.*

Proof If this were the case then $i(P) = g$ and hence

$$r(-P) = g - g + 1 + 1 = 2 \tag{4.1.41}$$

One of the functions $(f) > -P$ is the constant function, the other is a nonconstant meromorphic function with only one pole. Such a function would be a univalent map of $\mathcal{M}$ into $\mathbb{C}P^1$, and hence $\mathcal{M}$ would be of genus 0, in which case there are no holomorphic differentials. **Q.E.D.**

Corollary 4.1.1 *If there is a point P such that $r(-P) \geq 2$ then the genus is zero.*

Let us consider a point $P \in \mathcal{M}$; we want to study the dimensions $r(-kP)$ for $k \geq 1$. We have some obvious observations

- For $k=1$ $r(-P)=1$ and hence $i(P)=g-1$ $(g>0)$.
- For $k\geq 2g-1$ $i(kP)=0$ and hence $r(-kP)=k-g+1$.
- $i(kP)$ is the nullity of the $k\times g$ matrix

$$T_k(P) := \begin{bmatrix} \omega_1(P) & \dots & \omega_g(P) \\ \vdots & & \vdots \\ \omega_1^{(k-1)}(P) & \dots & \omega_g^{(k-1)}(P) \end{bmatrix} \tag{4.1.42}$$

(where the derivatives are taken w.r.t. any chosen local parameter at P) because if $\omega = \sum c_j \omega_j$ is such that $T\vec{c}=0$ then this means that ω has a zero of the desired order at P. Therefore $i(kP)\geq g-k$ for $k\leq g$.

Definition 4.1.7 *For a given and fixed $P\in\mathcal{M}$ the integers $k\in\mathbb{N}$ for which $r(-kP)=1$ (i.e. there are* **no nontrivial meromorphic functions***) is called a* **Weierstrass gap***).*

Clearly the notion of gap depends on the chosen point. By the third bulleted item above the rank of $T_k(P)$ is generically k for $k<g$ and g for $k\geq g$, **unless P is chosen in some special position**. In particular

Definition 4.1.8 *A point $P\in\mathcal{M}$ for which $r(-gP)\geq 2$ (or equivalently $i(gP)\geq 1$) is called a* **Weierstrass point**.

More generally

Definition 4.1.9 *A positive divisor $\mathcal{D}$ of degree $\deg(\mathcal{D})\leq g$ is called a* **special divisor** *if $i(\mathcal{D})>g-\deg(\mathcal{D})$ or equivalently if $r(\mathcal{D})>1$.*

Remark 4.1.1 *In [1] the definition is different; $\mathcal{D}$ is special according to [1] if there is another positive divisor $\mathcal{D}'$ such that $\mathcal{D}+\mathcal{D}'$ is canonical. In particular according to Farkas-Kra's book, any divisor of degree $\leq g-1$ is special. I am not sure if I am breaking any law here, but I prefer to call an arrangement of points special if it does not occur for any arrangement. Hence the definition I gave.*

Thus Weierstrass' points are points that give a special divisor $\mathcal{D}=gP$.

We ask the general question as if all divisors of degree g are special.

Proposition 4.1.6 *For any positive divisor $\widetilde{\mathcal{D}}$ of degree g there is a non-special divisor $\mathcal{D}$ of the same degree and made of points close to the points of $\widetilde{\mathcal{D}}$ that is non-special: MORE CLEAR . This divisor can always be chosen consisting of g distinct points.*

Proof. Let $\widetilde{\mathcal{D}} = \sum_{j=1}^{g} \widetilde{P}_j$ (possibly repeated).

We show that we can construct a sequence of divisors $\mathcal{D}_k$ of degrees k and non-special which contains only points chosen close to the points of $\widetilde{\mathcal{D}}$.

We start with $\mathcal{D}_1 = \widetilde{P}_1$ which is certainly non-special ($r(-\widetilde{P}_1) = 1$ for $g > 0$). Consider $\mathcal{D}_1 + \widetilde{P}_2$; if it is nonspecial we keep $\mathcal{D}_2 = \mathcal{D}_2 + P_2 \widetilde{P}_2$. If it is special then $\imath i(\mathcal{D}_1 + \widetilde{P}_2) > g - 2$ and hence $i(\mathcal{D}_1 + \widetilde{P}_2) = i(\mathcal{D}_1) = g - 1$. In a neighborhood of $P_2 \widetilde{P}_2$ there must be a point where not all differentials in $\mathfrak{I}(\mathcal{D}_1 + \widetilde{P}_2)$ vanish; for example choose $\omega \in \mathfrak{I}(\mathcal{D}_1 + \widetilde{P}_2)$ and certainly near $\widetilde{P}_2$ (since $\omega \not\equiv 0$) there is a point P_2 where $\omega \neq 0$. Then we define $\mathcal{D}_2 = \mathcal{D}_1 + P_2$ which must be non-special because $i(\mathcal{D}_2) < i(\mathcal{D}_1) = g - 1$. Continuing so forth, we get at the last stage with a nonspecial divisor $\mathcal{D}_{g-1}$, $i(\mathcal{D}_{g-1}) = 1$. If $\mathcal{D}_{g-1} + \widetilde{P}_g$ is special then we replace $\widetilde{P}_g$ as before with a suitably generic P_g. Clearly we can also require that all the points P_j are pairwise distinct. **Q.E.D.**

Definition 4.1.10 *A holomorphic/meromorphic q–differential is an expression $\omega = f(z)\mathrm{d}z^q$ which is invariant under changes of coordinates, with $f(z)$ holomorphic/meromorphic (for $q = 1$ these are simply Abelian differentials).*

Proposition 4.1.7 *The set of Weierstrass points is finite or, equivalently, $\det T_g(P)$ is not identically zero.*

Proof. First of all we note that $\det T_g(P)$ is naturally a $g(g+1)/2$–differential; indeed if $\omega_j = f_j(z)\mathrm{d}z$ (in a local coordinate) then $\det T_g(P)$ in this local parameter is nothing but the Wronskian of these functions. If we change parameter $w = w(z)$ then (**exercise**) this determinants transforms as $(dw/dz)^{g(g+1)/2}$ hence the assertion. Moreover its zeroes correspond (by the above bulleted list) to the Weierstrass points. To rephrase

$$\det T_g(P) = W(f_1, \ldots, f_g)\mathrm{d}z^{\frac{g(g+1)}{2}} \tag{4.1.43}$$

is invariantly defined. Clearly this is a holomorphic $q = g(g+1)/2$–differential and hence either it vanishes identically or it has (by compactness of $\mathcal{M}$) a finite number of zeroes. We rule out that it is identically zero and this is the main point. We fix a local coordinate $z(P) = 0$; it is sufficient to show that $W(f_1, \ldots, f_g)$ is not identically zero in a neighborhood of P.

To this end we make an upper–triangular change of basis of $\mathbb{C}\{f_1, \ldots, f_g\}$ (which changes W only by a nonzero constant) so that

$$\text{ord}_{f_1}(P) < \text{ord}_{f_2}(P) < \ldots < \text{ord}_{f_g}(P) . \tag{4.1.44}$$

This is accomplished by induction by taking f_1 to be a function with the minimum order of vanishing at P; subtracting from $f_2, \ldots$ a multiple of f_1 we can assume that $\text{ord}_{f_j}(P) > \text{ord}_{f_1}(P)$, $j > 1$. Continuing in this fashion we obtain the desired basis. Denoting by $\nu_j := \text{ord}_{f_j}(P)$ in this basis we have that $\nu_j \geq j$ and

$$f_j = c_j z^{\nu_j}(1 + \mathcal{O}(z)), \quad c_j \neq 0 . \tag{4.1.45}$$

Then the Wronskian is

$$W(f_1, \ldots, f_g) = \prod c_j z^{\sum_j \nu_j - j + 1}(1 + \mathcal{O}(1)). \tag{4.1.46}$$

This proves that W is not identically zero. **Q.E.D.**

Finally we can compute the dimensions of the spaces of holomorphic q–differentials

Definition 4.1.11 *The space of holomorphic q–differentials is denoted by $\mathcal{H}^q = \mathcal{H}^q(\mathcal{M})$*

Note that $\mathcal{H}^{-1}$ is the space of holomorphic vector-fields (which is actually trivial for $g > 1$ as we will see.)

In order to compute the dimensions of $\mathcal{H}^q$ (we know that it is g for $q = 1$) we first estabilsh

Lemma 4.1.5 *The space $\mathcal{H}^q$ is isomorphic to the space $\mathfrak{R}(-q\mathcal{K})$ for any $q \in \mathbb{Z}$.*

Proof. Let $\omega \in \mathcal{H}^1$ and $\mathcal{K} = (\omega)$ be chosen and fixed. For any $\eta \in \mathcal{H}^q$; then

$$F := \frac{\eta}{\omega^q} \tag{4.1.47}$$

is a meromorphic function in $\mathfrak{R}(-q\mathcal{K})$. Viceversa for any $F \in \mathfrak{R}(-q\mathcal{K})$ then $F\omega^q \in \mathcal{H}^q$. **Q.E.D.**

Proposition 4.1.8 *The dimensions $h_q := \dim \mathcal{H}^q$ are given by*

$g = 0$ *We have $h_q = 0$ if $q > 0$ and $h_q = 1 - 2q$ for $q \leq 0$.*

$g = 1$ *We have $h_q = 1,\ \forall q \in \mathbb{Z}$.*

$g \geq 2$ *We have $h_q = \delta_{q1} + (2q-1)(g-1),\ q \geq 1,\ h_0 = 1$ and $h_q = 0$ for $q < 0$.*

Proof. By Lemma 4.1.5 we need to compute $r(-q\mathcal{K})$.

$$r(-q\mathcal{K}) = i(q\mathcal{K}) - g + q(2g-2) + 1 = r((q-1)\mathcal{K}) + (2q-1)(g-1) \tag{4.1.48}$$

Genus 0: is left as exercise.

Genus 1 The unique holomorphic differential has no zeroes, hence $\mathcal{K} = \mathbf{0}$ and there is little information in the above equation. However, $\eta \in \mathcal{H}^q$ does not have any zero because the degree of $q\mathcal{K}$ is zero. If $\omega \in \mathcal{H}^1$ (it has no zeroes) it is easy to see that $\mathcal{H}^q = \mathbb{C}\{\omega^q\}$ and hence $h_q = 1$ for all q.

Genus $g > 2$ The divisor $\mathcal{K}$ is positive, so

$$r((q-1)\mathcal{K}) = \delta_{q1}\ ,\quad q \geq 1 \tag{4.1.49}$$

$$r(-q\mathcal{K}) = 0\ ,\quad q < 0. \tag{4.1.50}$$

Thus

$$r(-q\mathcal{K}) = \left\{ \begin{array}{ll} \delta_{q1} + (2q-1)(g-1) & q \geq 1 \\ 1 & q = 0 \\ 0 & q < 0\ . \end{array} \right. \tag{4.1.51}$$

Q.E.D.

Corollary 4.1.2 *There are $g^3 - g$ Weierstrass points counted with multiplicities (which is called the* **weight** *of the point and is the order of vanishing of the $\frac{1}{2}g(g+1)$–form $\det T_g(P)$).*

4.1.3 Riemann–Hurwitz formula

We derive the famous Riemann–Hurwitz formula as a consequence of Riemann–Roch via the formula for the degree of a canonical divisor; note that however there are purely topological derivation (and much more elementary) see [1]. Let $\varphi : \mathcal{M} \to N$ be a nonconstant holomorphic mapping between compact Riemann surfaces (recall the setting of section 1.2). Let $g = genus(\mathcal{M})$ and $\gamma = genus(N)$. Given any Abelian differential η on N we can pull it back to a differential $\omega = \varphi_\star \eta$. We next count the degrees of both divisors.

The explicit form of the pullback is as follows, in suitable local coordinates near a point $P \in \mathcal{M}$

$$\eta = g(w)\mathrm{d}w \ , \quad w = f(z) = z^{b_\varphi(P)+1} \ , \tag{4.1.52}$$

$$\varphi_* \eta = g(w(z)) f'(z) \mathrm{d}z = (b_\varphi(P) + 1) g(w(z)) z^{b_\varphi(P)} \mathrm{d}z \ . \tag{4.1.53}$$

This shows that if η has a a zero of order k ad at $\varphi(P)$ then $\varphi_* \eta$ has a zero of order $k + b_\varphi(P)$. Moreover each zero of η appears N times in the divisor of zeroes of $\varphi_* \eta$, where N is the degree of φ (i.e. the sheet number).

We know also that $\deg(\eta) = 2\gamma - 2$, $\deg \varphi_* = 2g - 2$ and we have just proven

Theorem 4.1.3 (Riemann–Hurwitz) *The following relation holds between the genera and the branching number of any map from $\mathcal{M}$ to N*

$$2g - 2 = N(2\gamma - 2) + B \tag{4.1.54}$$

or –equivalently but more commonly–

$$g = N(\gamma - 1) + \frac{B}{2} + 1 \tag{4.1.55}$$

Note that the theorem implies that

- The branching number is always even.
- $g = 0$ implies $\gamma = 0$ (and $B = 2N - 2$)/
- If $\gamma = 0$ then $g = \frac{B}{2} - N + 1$
- If φ is **unramified** (i.e. $B = 0$) then
  - $g = 0$ implies $\gamma = 0$ and $N = 1$;
  - if $\gamma = 1$ then N is anything and $g = 1$;
  - if $g > 1$ then $g = \gamma$ for $N = 1$ and $\gamma - 1$ divides $g - 1$ for $N \geq 2$.

4.2 Abel Theorem and Jacobi inversion theorem

Definition 4.2.1 *A* **Torelli marked** *compact Riemann surface is a $\mathcal{M}$ with a choice of canonical homology basis $H_1(\mathcal{M},\mathbb{Z}) = \mathbb{Z}\{a_1, b_1, \dots, a_g, b_g\}$.*

For a given Torelli-marked surface we choose the corresponding normalized basis of holomorphic differentials

$$\oint_{a_j} \omega_k = \delta_{jk} \tag{4.2.1}$$

We also assume that the cycles a_j, b_j are realized as loops in the homotopy based at the point P_0 (the **basepoint**) and that the surface $\mathcal{M}$ has been cut open along these cycles to form a simply connected domain $\mathcal{L}$ (a $4g$–gon).

For a given germ of analytic function $f(P)$ we denote the analytic continuation along the (homotopy class of) a cycle γ by

$$\widetilde{f}(P) = f(P+\gamma). \tag{4.2.2}$$

We then define

Definition 4.2.2 (Abel map) *Given a point $P \in \mathcal{M}$ we define the* **Abel map** $\mathfrak{u}$ *as follows*

$$\begin{array}{lcc} \mathfrak{u} : \mathcal{L} & \longrightarrow & \mathbb{C}^g \\ P & \mapsto & \mathfrak{u}(P) := (\int_{P_0}^P \omega_1, \dots, \int_{P_0}^P \omega_g)^t \end{array} \tag{4.2.3}$$

where the contour of integration is taken to lie within the simply connected domain $\mathcal{L}$.

The Abel map is extended to arbitrary divisors $\mathcal{D} = \sum k_j P_j$ as follows

$$\mathfrak{u}(\mathcal{D}) := \sum k_j \mathfrak{u}(P_j)\ . \tag{4.2.4}$$

The components of the Abel map are holomorphic functions that can be analytically continued to the universal cover of $\mathcal{M}$; their behaviour under analytic continuation is specified by the following relations

$$\begin{aligned} \mathfrak{u}_j(P+a_k) &= \int_{P_0}^{P+a_k} \omega_j = \int_{P_0}^P \omega_j + \oint_{a_k} \omega_j = \mathfrak{u}_j(P) + \delta_{jk} \\ \mathfrak{u}_j(P+b_k) &= \mathfrak{u}_j(P) + \oint_{b_k} \omega_j\ . \end{aligned} \tag{4.2.5}$$

It is clear that the nontrivial information is contained in the b–periods of the normalized holomorphic differentials

Definition 4.2.3 *The period matrix of the Torelli marked surface $\mathcal{M}$ is defined to be*

$$\tau_{jk} = \oint_{b_j} \omega_k\ . \tag{4.2.6}$$

There are a few simple but important properties of the period matrix.

Proposition 4.2.1 **(1)** *The period matrix is symmetric* $\tau_{jk} = \tau_{kj}$. **(2)** *The imaginary part of the period matrix* $\mathbb{B} := \Im\tau$ *is a positive definite real symmetric matrix.*

Proof. Using the Riemann bilinear relations (Prop. 3.4.1)

$$0 = 2i\pi \sum_{P=\text{pole}} \operatorname*{res}_{P} \mathfrak{u}_j \omega_k = \sum_{\ell=1}^{g} \oint_{a_\ell} \omega_k \oint_{b_\ell} \omega_j - \oint_{a_\ell} \omega_j \oint_{b_\ell} \omega_k = \oint_{b_k} \omega_j - \oint_{b_j} \omega_k \ . \tag{4.2.7}$$

This proves the symmetry.

Similarly, using the other form of the bilinear relations (Thm. 3.1.2) we have

$$\omega = \sum_{j=1}^{g} c_j \omega_j$$
$$0 < i \int_{\mathcal{M}} \omega \wedge \overline{\omega} = 2 \sum_{j,k=1}^{g} c_j \overline{c_k} \mathbb{B}_{jk} \tag{4.2.8}$$

which is valid for any numbers c_j. This proves the positive definiteness of $\mathbb{B}$. **Q.E.D.**

4.2.1 Complex Tori and Jacobi variety

From the periodicity properties of $\mathfrak{u}$ (4.2.5) it follows that the Abel map is well defined if we take it modulo those periods. Consider the space

$$J(\mathcal{M}) := \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g) := \mathbb{C}^g/\Lambda(\mathcal{M}) \tag{4.2.9}$$

called the **Jacobian variety** (or simply the Jacobian) of the complex curve $\mathcal{M}$. We have then

Proposition 4.2.2 *The Abel map* $\mathfrak{u} : \mathcal{M} \to J(\mathcal{M})$ *is an immerion (i.e. locally injective or maximal rank)*

Proof. The fact that it is a well–defined map follows from the fact that by analytic continuation

$$\mathfrak{u}(P+\gamma) = \mathfrak{u}(P) + \begin{pmatrix} m_1 \\ \vdots \\ m_g \end{pmatrix} + \tau \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_g \end{pmatrix} \ ,$$
$$\gamma = \sum m_j a_j + n_j b_j \ . \tag{4.2.10}$$

and hence the image differs only by an element of the lattice $\Lambda(\mathcal{M}) = 2i\pi\mathbb{Z}^g + 2i\pi\tau \cdot \mathbb{Z}^g$.

Now consider the Jacobian of the Abel map

$$\mathrm{d}\mathfrak{u}(P) = \begin{pmatrix} \omega_1(P) \\ \vdots \\ \omega_g(P) \end{pmatrix} \tag{4.2.11}$$

Since there is no point for which all the ω_j's vanish simultaneously (Prop. 4.1.5) then the rank is always maximal). **Q.E.D.**

Remark 4.2.1 *We will see later that in fact this is an embedding (i.e. globally injective).*

Proposition 4.2.3 *The Jacobian variety $J(\mathcal{M})$ is topologically a $2g$ torus; it has a natural structure of Abelian commutative group.*

Proof. The statement is purely topological; let $\mathfrak{u} = \vec{x} + i\vec{y}$ The rank of the lattice $\mathbb{Z} + \tau \cdot \mathbb{Z}$ is $2g$ because $\Im(\tau)$ is positive definite

$$\mathbb{C}^g \sim \mathbb{R}^{2g}$$

$$\mathfrak{u} = \vec{x} + i\vec{y}$$

The lattice $\mathbb{Z}^g \oplus \mathbb{Z}^g$ is embedded in $\mathbb{R}^{2g}$ as

$$\left[\begin{array}{cc} \mathbf{1}_g & \Re(\tau) \\ 0 & \Im(\tau) \end{array}\right] \left[\begin{array}{c} n_1 \\ \vdots \\ n_g \\ m_1 \\ \vdots \\ m_g \end{array}\right] . \tag{4.2.12}$$

This proves that the quotient is a $2g$ real torus. Hence $J(\mathcal{M})$ is also compact. The group structure is entirely obvious. **Q.E.D.**

Remark 4.2.2 *We will see that the group structure of $J(\mathcal{M})$ corresponds to a group structure on the divisor classes of degree g.*

Theorem 4.2.1 (Abel Theorem) *Let $\mathcal{D}$ be a divisor of degree 0; then $\mathcal{D}$ is the divisor of a meromorphic function (i.e. it is principal) if and only if $\mathfrak{u}(\mathcal{D}) = 0 \in J(\mathcal{M})$.*

Proof. Let F be a meromorphic function and $\mathcal{D} = (F)$. Let $\mathcal{L}$ be the polygonization of $\mathcal{M}$ along the Torelli marking. Using Riemann bilinear relations and elementary complex calculus we have

$$\mathfrak{u}(\mathcal{D}) = \sum_{P=\{\text{ poles and zeroes of } F\ \}} \operatorname*{res}_P \frac{\mathrm{d}F}{F}\mathfrak{u} = \frac{1}{2i\pi}\sum_{j=1}^{g} \oint_{a_j} \frac{\mathrm{d}F}{F} \oint_{b_j} \vec{\omega} - \oint_{b_j} \frac{\mathrm{d}F}{F} \oint_{a_j} \vec{\omega} \tag{4.2.13}$$

The quantities $\oint_\gamma \mathrm{d}F/F$ are all integers times $2i\pi$ since they count the number of times $\ln(F)$ winds around $F = 0$ as the point of integration runs along γ. Hence $\mathfrak{u}(\mathcal{D})$ is in the lattice $\Lambda(\mathcal{M})$ and so it is zero in the Jacobian. This proves the necessity of the condition.

To prove sufficiency we note that $\mathrm{d}\ln F = \frac{\mathrm{d}F}{F}$ is a third–kind differential with residues at the poles and zeroes of F equal to $\operatorname{ord}_F(P)$ and this gives the main idea.

So let

$$\mathcal{D} = \sum k_j P_j - \sum h_j Q_j\ , \qquad k_j, h_j \in \mathbb{N} \tag{4.2.14}$$

be our divisor with $\sum k_j = \sum h_j$; for simplicity we will write

$$\mathcal{D} = \sum_{\ell=1}^{M} (P_\ell - Q_\ell) \tag{4.2.15}$$

where in the sums the points may be repeated. Let ω_{PQ} be the normalized Abelian differential of the third kind with two poles at P, Q and residues $1, -1$ respectively. Define

$$\eta := \sum_{\ell=1}^{M} \omega_{P_\ell Q_\ell} \ . \tag{4.2.16}$$

This differential has vanishing a–periods because of our normalization; the b–periods are, by the **reciprocity theorem** (Thm. 3.4.2)

$$\oint_{b_k} \eta = 2i\pi \sum_{\ell=1}^{M} \int_{Q_\ell}^{P_\ell} \omega_k = 2i\pi \mathfrak{u}_k(\mathcal{D}) = 2i\pi m_k + 2i\pi \sum_{j=1}^{g} \tau_{kj} n_j \tag{4.2.17}$$

Now consider the periods of

$$\widetilde{\eta} := \eta - 2i\pi \sum_{j=1}^{g} n_j \omega_j$$

$$\oint_{a_k} \widetilde{\eta} = -2i\pi n_k \tag{4.2.18}$$

$$\oint_{b_k} \widetilde{\eta} = 2i\pi m_k \ . \tag{4.2.19}$$

and this is still a third kind differential as before; since all the periods are multiple of $2i\pi$ it follows that

$$F := \exp(\int \widetilde{\eta}) \tag{4.2.20}$$

(originally defined on a simply connected domain obtained by cutting $\mathcal{L}$ from P_0 to all the poles of $\widetilde{\eta}$) is a single valued meromorphic function with the prescribed divisor. Indeed near a pole P of $\widetilde{\eta}$ of residue k one has in a local parameter $z(P) = 0$

$$\widetilde{\eta} = \frac{k}{z} \mathrm{d}z + \mathcal{O}(1)$$
$$F = \exp(k \ln(z) + \mathcal{O}(1)) = z^k \mathcal{O}(1). \tag{4.2.21}$$

that is $\mathrm{ord}_F(P) = k$. The analytic continuation around the a, b–cycles yields the same function because of the integrality of the periods of $\frac{1}{2i\pi}\widetilde{\eta}$. This proves sufficiency. **Q.E.D.**

As promised earlier we have

Corollary 4.2.1 *If $g > 0$ the Abel map is an embedding of $\mathcal{M}$ into $J(\mathcal{M})$.*

Proof. Suppose $\mathfrak{u}(P) = \mathfrak{u}(Q) \in J(\mathcal{M})$ for $P \neq Q$. Then $\mathcal{D} = P - Q$ is a divisor with vanishing Abel map, hence by Abel's Theorem it is principal. This would imply that there is a meromorphic function with only one simple pole and one simple zero, a contradiction with the assumption $g > 0$.**Q.E.D.**

Corollary 4.2.2 *Let $\mathcal{D}$ be an arbitrary divisor; then its Abel map $\mathfrak{u}(\mathcal{D})$ depends only on its divisor class.*

4.3 Jacobi Inversion theorem

The dimension of $J(\mathcal{M})$ as a complex manifold is clearly g; hence $\mathfrak{u}(\mathcal{M})$ cannot be surjective. However the extension of the Abel map to divisor allows to have higher dimensional submanifolds. In particular if we choose g points (i.e. a positive divisor of degree g) we can expect the Abel map to be surjective. This is in essence Jacobi inversion theorem.

We first introduce the notations

Definition 4.3.1 *We denote by $\mathcal{M}_n$ the symmetric product n–times of $\mathcal{M}$ with itself, i.e. the manifold of dimension n obtained by quotienting $\mathcal{M} \times \mathcal{M} \cdots \times \mathcal{M}$ by the symmetric group. It is equivalent to the positive divisors of degree n. By W_n we denote the image of $\mathcal{M}_n$ under the Abel map.*

Consider now $\mathcal{M}_g$ and its image $\mathfrak{u}(\mathcal{M}_g) = W_g$. We have

Theorem 4.3.1 (Jacobi Inversion theorem) *We have the tautologically equivalent statements;*

- *Every $\mathbf{z} \in J(\mathcal{M})$ is the image of a positive divisor of degree g*
- *$W_g = J(\mathcal{M})$ (set-theoretically).*
- *Let $\mathcal{D}_1, \mathcal{D}_2$ be two positive* **nonspecial** *divisors of degree g. Then $\mathcal{D}_1$~??=$\mathcal{D}_2$ if and only if their image in $J(\mathcal{M})$ is the same, $\mathfrak{u}(\mathcal{D}_1) = \mathfrak{u}(\mathcal{D}_2)$ (i.e. $\mathcal{M}_g \setminus \Delta \simeq W_g \setminus \mathfrak{u}(\Delta)$, where Δ are the special divisors).*

Remark 4.3.1 *We will see that $\mathfrak{u}(\Delta)$ coincides with the zero-level set of Θ.*

Proof. We know from Prop. 4.1.6 that we can choose a nonspecial divisor $\mathcal{D}$ and that these divisors are *generic* in the sense that in any neighborhood of any special divisor there is a nonspecial divisor. Let $\mathcal{D}$ be non-special and of degree g, namely $i(\mathcal{D}) = 0$. We know also that we can assume it to consist of g pairwise distinct points $P_1, \dots, P_g$. Choosing local coordinates z_j near P_j the polydisk $\mathbb{D}_\epsilon \times \cdots \times \mathbb{D}_\epsilon$ in $\mathbb{C}^g$ parametrizes a neighborhood $\mathcal{U}_0$ of $\mathcal{D}$ in $\mathcal{M}_g$. With respect to these coordinates $\vec{z} = (z_1, \dots, z_g)$ the Jacobian of $\mathfrak{u}$ at $\mathcal{D}$ is

$$\frac{\partial \mathfrak{u}_j}{\partial z_k} = \operatorname*{res}_{z_j=0} \frac{1}{z_j} \omega_k(P_j). \tag{4.3.1}$$

The ensuing $g \times g$ matrix is precisely the matrix that enters the proof of RR theorem and the nonspeciality is the statement that the determinant of this matrix is nonzero. Hence the Jacobian of $\mathfrak{u}$ is nonsingular precisely at all non-special divisors, which are an open set in the variety of all divisors of degree g. At the same time this shows, by the inversion theorem, that

$$\mathfrak{u} : \mathcal{U}_0 \to \mathfrak{u}(\mathcal{D}) + V_0 \tag{4.3.2}$$

is a bijection, where V_0 is a small neighborhood of $0 \in J(\mathcal{M})$ (which is also identifiable as a neighborhood of the origin in $\mathbb{C}^g$).

Let now $\vec{c} \in \mathbb{C}^g$ be an arbitrary vector; then $\vec{c}/N \in V_0$ for $N \in \mathbb{N}$ large enough. Therefore there is $\mathcal{D}' \in \mathcal{U}_0$ (also consisting of pairwise distinct points) such that

$$\mathfrak{u}(\mathcal{D}') = \mathfrak{u}(\mathcal{D}) + \frac{1}{N}\vec{c} \quad \Leftrightarrow \quad \vec{c} = \mathfrak{u}(N\mathcal{D}' - N\mathcal{D}) \ . \tag{4.3.3}$$

Take now the basepoint for the Abel map P_0 and consider the divisor of degree g

$$\widehat{\mathcal{D}} := N\mathcal{D}' - N\mathcal{D} + gP_0 \tag{4.3.4}$$

Then by Riemann–Roch theorem

$$r(-\widehat{\mathcal{D}}) = i(\widehat{\mathcal{D}}) - g + \deg \widehat{\mathcal{D}} + 1 = i(\widehat{\mathcal{D}}) + 1 \geq 1. \tag{4.3.5}$$

Since $-\widehat{\mathcal{D}}$ has some positive part, there cannot be any constant function (it would have to vanish at some points), hence there is at least one nontrivial meromorphic function $F \in \mathfrak{R}(-\widehat{\mathcal{D}})$. Thus $\widehat{\mathcal{D}}$ must be linearly equivalent to a **positive** divisor

$$(F) + \widehat{\mathcal{D}} = \widetilde{\mathcal{D}} > 0 \ , \quad \deg \widetilde{\mathcal{D}} = g. \tag{4.3.6}$$

This implies that

$$\mathfrak{u}(\widetilde{\mathcal{D}}) = \mathfrak{u}(N\mathcal{D}' - N\mathcal{D} + gP_0) = \mathfrak{u}(N\mathcal{D}' - N\mathcal{D}) = \mathfrak{u}(\widetilde{\mathcal{D}}) = \vec{c} \tag{4.3.7}$$

but then $\mathfrak{u}(\mathcal{D}\widetilde{\mathcal{D}}) = \vec{c}$ solves the Jacobi inversion problem.

The last assertion is proven as follows: suppose $\mathcal{D}_1, \mathcal{D}_2$ have the same Abel map. Hence they are linearly equivalent and one is special iff the other is. Suppose one (and hence both) divisors are nonspecial, $i(\mathcal{D}_j) = 0$; if they were different then there would be a function f with zeroes at $\mathcal{D}_1$ and poles at $\mathcal{D}_2$. We show that there is no such function by the nonspeciality.

Indeed then

$$r(-\mathcal{D}_2) = i(\mathcal{D}_2) - g + g + 1 = 1 \tag{4.3.8}$$

and hence there is only the constant function in $\mathfrak{R}(-\mathcal{D}_2)$. The function that puts in equivalence $\mathcal{D}_1, \mathcal{D}_2$ would have also zeroes at $\mathcal{D}_1$, clearly impossible. **Q.E.D.**

Corollary 4.3.1 *Suppose $\mathcal{D}$ is such that $1 \leq i(\mathcal{D}) = s \leq g$. Then there is a variety of dimension s of divisors with the same Abel map. Viceversa if $\mathcal{D}$ has the following property then $i(\mathcal{D}) \geq s$: for any positive $\mathcal{D}'$ of degree $\leq s$ there is another positive $\widetilde{\mathcal{D}}$ with $\mathcal{D} \sim \mathcal{D}' + \widetilde{\mathcal{D}}$.*

Proof. The proof is an elaboration of the last point above. If $i(\mathcal{D}) = s$ then

$$r(-\mathcal{D}) = i(\mathcal{D}) - g + \deg \mathcal{D} + 1 = s + 1. \tag{4.3.9}$$

Within $\mathfrak{R}(-\mathcal{D})$ there is certainly the constant function f_0 and then s nonconstant meromorphic functions, $f_1, \ldots, f_s$. We show that the matrix $T_{\mathcal{D}_s} := \{f_j(Q_j)\}_{i,j \le s}$ (where $\mathcal{D}_s = \sum Q_j$) is not identically degenerate for any choice of Q_j's; indeed in this case

$$0 \equiv F(Q) := \det \begin{bmatrix} f_0 & f_0 & \ldots & f_0 \\ f_1(Q) & f_1(Q_1) & \ldots & f_1(Q_s) \\ f_2(Q) & f_2(Q_1) & \ldots & f_2(Q_s) \\ \vdots & & & \vdots \\ f_s(Q) & f_s(Q_1) & \ldots & f_s(Q_s) \end{bmatrix} = C_0 f_0(Q) + \ldots + C_s f_s(Q) \tag{4.3.10}$$

(one can easily show that not all C_j's are zero) and this violates linear independence[1]

Then the $F(Q)$ constructed above has zeroes at $Q_1, \ldots, Q_s$ and $(F)+\mathcal{D} \ge Q_1 + \ldots + Q_s$ is a positive divisor. Clearly the points Q_j can be chosen in a open set of $\mathcal{M}_s$. Then all divisors $\mathcal{D}$ and $\mathcal{D}+(F)$ have the same Abel map because they are linearly equivalent.

To prove the "viceversa" part we pick an arbitrary positive divisor $\mathcal{D}'$ of degree s. Then we suppose $r(-DD\mathcal{D}) = k$ and show $k \ge s$. We construct a $k \times s$ matrix with maximal rank as before. If $k < s$ then there would not exist any nontrivial function F with $(F) = \mathcal{D}' + \widetilde{\mathcal{D}} - \mathcal{D}$, contrary to the assumption. **Q.E.D.**

Corollary 4.3.2 *The Jacobian variety is isomorphic as a group to the group of divisors of degree* 0 *modulo principal divisors.*

Proof. It is essentially a tautology: first of all the divisors of degree 0 form naturally a group and the principal divisors are a subgroup of that. The quotient is an Abelian group.

We must prove that any point of $J(\mathcal{M})$ is the image of a unique class of divisors of degree 0.

Suppose that $\mathcal{D}_1, \mathcal{D}_2$ both of degree zero but not equivalent have the same image

$$\mathfrak{u}(\mathcal{D}_1) = \mathfrak{u}(\mathcal{D}_2)\ . \tag{4.3.11}$$

Immediately by Abel's theorem $\mathcal{D}_1 - \mathcal{D}_2$ is principal. **Q.E.D.**

.

[1] To construct the above matrix we take $f_0 = 1$ and we find Q_1 such that the nonconstant function f_1 forms a matrix $\{f_j(Q_k)\}_{0 \le k, j \le 1}$ of maximal rank (this must be possible by the independence). We keep going this way until we have the above matrix, with $Q = Q_0$.

Chapter 5

Theta Functions

5.1 Definition in general

Let τ be a symmetric $g \times g$ matrix with positive definite imaginary part (it does not necessarily come from the period matrix of a Riemann surface).

Definition 5.1.1 *The space of such matrices τ is denoted by $\mathfrak{S}_g$ and called the* **Siegel upper half space of genus** *g.*

The **Theta function** associated to τ is the following function of g complex variables $\mathbf{z} = (z_1, \dots, z_g)$

$$\Theta(\mathbf{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp 2i\pi \left(\frac{1}{2} \vec{n}^t \cdot \tau \cdot \vec{n} + \vec{n}\mathbf{z} \right) \tag{5.1.1}$$

Since $\Im\tau > 0$ (is positive definite) it is an **exercise** to show that the series is convergent for any value of $\mathbf{z}$ and that defines a holomorphic function on $\mathbb{C}^g$.

We can express the main properties of $\Theta(\mathbf{z}, \tau)$ in the next proposition, whose proof is left as an exercise (a direct manipulation of the series).

Proposition 5.1.1 *The Theta function has the following properties*

1. $\Theta(\mathbf{z}, \tau) = \Theta(-\mathbf{z}, \tau)$ *(parity).*

2. *For any $\lambda, \lambda' \in \mathbb{Z}^g$ we have*

$$\Theta(\mathbf{z} + \lambda' + \tau\lambda, \tau) = \exp 2i\pi \left[-\lambda^t \mathbf{z} - \frac{1}{2} \lambda^t \tau \lambda \right] \Theta(\mathbf{z}, \tau) \tag{5.1.2}$$

   In particular Θ is periodic in each z_j of period 1.

3. It satisfies the **heat equation** *(in several variables)*

$$\frac{\partial\Theta(\mathbf{z},\tau)}{\partial\tau_{jk}} = \frac{1}{2i\pi}\frac{\partial^2\Theta(\mathbf{z},\tau)}{\partial z_j\partial z_k}, \quad j\neq k$$
$$\frac{\partial\Theta(\mathbf{z},\tau)}{\partial\tau_{jj}} = \frac{1}{4i\pi}\frac{\partial^2\Theta(\mathbf{z},\tau)}{\partial z_j^2} \ . \tag{5.1.3}$$

If we translate the $\mathbf{z}$ argument by a vector $\mathbf{e}\in\mathbb{C}^g$ the periodicity properties become (we suppress the dependence on τ)

$$\Theta(\mathbf{z}+\mathbf{e}+\lambda'+\tau\lambda) = \exp 2i\pi\left[-\lambda^t(\mathbf{z}+\mathbf{e}) - \frac{1}{2}\lambda^t\tau\lambda\right]\Theta(\mathbf{z}+\mathbf{e}) \tag{5.1.4}$$

In order to construct meromorphic functions on the quotient $\mathbb{C}^g/(\mathbb{Z}^g+\tau\mathbb{Z}^g)$ we can take for example any two vectors $\mathbf{e}_1,\mathbf{e}_2$ and consider

$$F(\mathbf{z}) := \frac{\Theta(\mathbf{z}+\mathbf{e}_1)\Theta(\mathbf{z}-\mathbf{e}_1)}{\Theta(\mathbf{z}+\mathbf{e}_2)\Theta(\mathbf{z}-\mathbf{e}_2)} \tag{5.1.5}$$

For practical reasons it is convenient to introduce special translates of Θ; first of all we note that any $\mathbf{e}\in\mathbb{C}^g$ can be uniquely written as

$$\mathbf{e} = \frac{1}{2}\vec{\epsilon}' + \frac{1}{2}\tau\vec{\epsilon}, \quad \vec{\epsilon},\vec{\epsilon}'\in\mathbb{R}^g \tag{5.1.6}$$

since the matrix $(\mathbf{1},\tau)$ injects $\mathbb{R}^{2g}$ into $\mathbb{C}^g$ (**exercise**).

Then we have

Definition 5.1.2 *For any* $\mathbf{e}$ *the vectors* ϵ,ϵ' *are called the (half) characteristics of* $\mathbf{e}$*.*

We now define

Definition 5.1.3 *The* Θ *function with characteristics* ϵ,ϵ' *is defined and denoted as*

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\mathbf{z}) := \exp\left[2i\pi\left(\frac{1}{8}\epsilon^t\tau\epsilon + \frac{1}{2}\epsilon^t\mathbf{z} + \frac{1}{4}\epsilon^t\epsilon'\right)\right]\Theta\left(\mathbf{z}+\frac{\epsilon'}{2}+\frac{\tau}{2}\epsilon\right) = \tag{5.1.7}$$

Proposition 5.1.2 *The Theta function with* **integer** *half-characteristics* $\epsilon,\epsilon'\in\mathbb{Z}^g$ *has the properties*

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\mathbf{z}+\lambda'+\tau\lambda) = \exp 2i\pi\left(\frac{1}{2}(\epsilon^t\lambda'-\lambda^t\epsilon') - \lambda^t\mathbf{z} - \frac{1}{2}\lambda^t\tau\lambda\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\mathbf{z}) \tag{5.1.8}$$

$$\Theta\begin{bmatrix}\epsilon+2\nu\\ \epsilon'+2\nu'\end{bmatrix}(\mathbf{z}) = \exp\left(i\pi\epsilon^t\nu'\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\mathbf{z})\ , \ \nu,\nu'\in\mathbb{Z}^g \tag{5.1.9}$$

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-\mathbf{z}) = \exp\left(i\pi\epsilon^t\epsilon'\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\mathbf{z}) \tag{5.1.10}$$

The first and second properties hold also if ϵ,ϵ' *are arbitrary* **complex** *vectors.*

Definition 5.1.4 *A characteristics* $\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right]$ *is called a* **odd half integer characteristics** *if* $\epsilon,\epsilon'\in\mathbb{Z}^g$ *and* $\epsilon^t\epsilon'$ *is odd.*

Remark 5.1.1 *Since we are using by construction half-characteristics the half-integer characteristics are obtained out of* integer ϵ, ϵ'. *The reason of the definition is then simply that (from eq. 5.1.10) in this case* $\Theta[]$ *is odd. Since* $\Theta\left[{\epsilon \atop \epsilon'}\right](\mathbf{z})$ *is a nonzero multiple of* $\Theta(\mathbf{z}+\mathbf{e})$ *(with* $2\mathbf{e} = \epsilon' + \tau\epsilon$*) we see that if* $\mathbf{e}$ *is an odd half-integer characteristics then* $\Theta(\mathbf{e}) = 0$ *(from the oddity of* $\Theta[\mathbf{e}](\mathbf{z})$*).*

5.2 Theta functions associated to compact Riemann surfaces

We now assume that τ is the period matrix of a Torelli–marked Riemann surface: as usual we set

- ω_i the normalized Abelian differentials of the first kind (holomorphic)

$$\oint_{a_j} \omega_k = \delta_{jk} \ . \tag{5.2.1}$$

- $\mathcal{L}$ the polygonization of $\mathcal{M}$ along a choice of representatives of the Torelli marking with basepoint P_0.

- $\mathfrak{u}$ the Abel map with basepoint P_0

$$\mathfrak{u}(P) = \int_{P_0}^{P} \vec{\omega} \tag{5.2.2}$$

- ω_{PQ} the normalized third kind differential with residues ± 1

$$\operatorname*{res}_P \omega_{PQ} = 1 = -\operatorname*{res}_Q \omega_{PQ}, \quad \oint_{a_j} \omega_{PQ} = 0 \ , \quad \frac{1}{2i\pi}\oint_{b_j} \omega_{PQ} = \mathfrak{u}_j(P) - \mathfrak{u}_j(Q) = \int_Q^P \omega \tag{5.2.3}$$

Consider now, for an arbitrary $\mathbf{e} \in \mathbb{C}^g$ the function

$$\begin{array}{rcl} \vartheta_{\mathbf{e}} : \mathcal{M} & \longrightarrow & \mathbb{C} \\ P & \mapsto & \Theta(\mathfrak{u}(P) - \mathbf{e}) \end{array} \tag{5.2.4}$$

Because of the periodicity of $\mathfrak{u}$ and of Θ this function has the properties under analytic continuation

$$\begin{gathered} \vartheta_{\mathbf{e}}(P + a_j) = \vartheta_{\mathbf{e}}(P) \\ \vartheta_{\mathbf{e}}(P + b_j) = \exp 2i\pi \left[-\mathfrak{u}_j(P) + \mathbf{e}_j - \frac{1}{2}\tau_{jj} \right] \vartheta_e(P) \ . \end{gathered} \tag{5.2.5}$$

and hence it is not a single–valued function. Nonetheless its **zeroes** are well defined because the multivaluedness is multiplicative with a non-vanishing factor. Therefore we can talk about the **divisor** of $\vartheta_{\mathbf{e}}$ (i.e. the set of points in $\mathcal{M}$ where it vanishes).

Two questions are in order now

- What is the degree of this divisor (i.e. how many points are there)?

- What is the Abel map of this divisor.

Proposition 5.2.1 *Provided that* $\vartheta_{\mathbf{e}}$ *does not vanish identitcally we have* $\deg(\vartheta_{\mathbf{e}}) = g$ *.*

Proof. We integrate $\mathrm{d}\ln\vartheta_{\mathbf{e}}$ along the boundary of $\mathcal{L}$

$$
\begin{aligned}
&\frac{1}{2i\pi}\oint_{\partial\mathcal{L}}\frac{\mathrm{d}\vartheta_{\mathbf{e}}(P)}{\vartheta_{\mathbf{e}}(P)} = \\
&= \frac{1}{2i\pi}\sum_{j=1}^{g}\left(\int_{P_0}^{P_0+a_j} + \int_{P_0+a_j}^{P_0+a_j+b_j} + \int_{P_0+a_j+b_j}^{P_0+b_j} + \int_{P_0+b_j}^{P_0}\right)\mathrm{d}\ln\vartheta_{\mathbf{e}}(P) = \\
&= \frac{1}{2i\pi}\sum_{j=1}^{g}\left(\int_{P_0}^{P_0+a_j} - \int_{P_0+b_j}^{P_0+b_j+a_j}\right. \\
&\qquad\left. + \int_{P_0+b_j}^{P_0} - \int_{P_0+a_j+b_j}^{P_0+a_j}\right)\mathrm{d}\ln\vartheta_{\mathbf{e}}(P) = \\
&= \sum_{j=1}^{g}\int_{P_0}^{P_0+a_j}\mathrm{d}\mathfrak{u}_j = g\ .
\end{aligned}
\tag{5.2.6}
$$

where we have used the definition $\mathrm{d}\mathfrak{u}_j = \omega_j$ and the normalization of ω_j. This concludes the proof. **Q.E.D.**

Proposition 5.2.2 *Let* $\mathcal{D} = (\vartheta_e)$ *for a* $\mathbf{e}$ *such that* $\vartheta_{\mathbf{e}} \not\equiv 0$*: then*

$$
\mathfrak{u}(\mathcal{D}) = \mathbf{e} - \mathcal{K} \tag{5.2.7}
$$

where $\mathcal{K}$ *is a vector called* **Riemann constants** *and defined as*

$$
\mathcal{K}_j = \frac{\tau_{jj}}{2} - \sum_{k=1}^{g}\left[\int_{P_0}^{P_0+a_k}\mathfrak{u}_j\mathrm{d}\mathfrak{u}_k\right] \tag{5.2.8}
$$

Remark 5.2.1 *The vector of Riemann constants depends on the Torelli marking and on the basepoint* P_0 *(in the last integral). The differential of* $\mathcal{K}(P_0)$ *w.r.t.* P_0 *is*

$$
\mathrm{d}\mathcal{K}(P_0) = (g-1)\vec{\omega}(P_0) \tag{5.2.9}
$$

[Check!]

Proof. Similarly to the previous computation we integrate $\mathfrak{u}\mathrm{d}\ln\vartheta_{\mathbf{e}}$ along $\partial\mathcal{L}$ taking care of the analytic continuations.

$$
\begin{aligned}
&\frac{1}{2i\pi}\oint_{\partial\mathcal{L}}\mathfrak{u}_k\mathrm{d}\ln\vartheta_{\mathbf{e}} = \\
&\qquad = \frac{1}{2i\pi}\sum_{j=1}^{g}\left(\int_{P_0}^{P_0+a_j} + \int_{P_0+a_j}^{P_0+a_j+b_j} + \int_{P_0+a_j+b_j}^{P_0+b_j} + \int_{P_0+b_j}^{P_0}\right)\mathfrak{u}_k\mathrm{d}\ln\vartheta_{\mathbf{e}}(P) =
\end{aligned}
$$

$$= \frac{1}{2i\pi}\sum_{j=1}^{g}\left(\int_{P_0}^{P_0+a_j} - \int_{P_0+b_j}^{P_0+b_j+a_j}\right.$$

$$\left. + \int_{P_0+b_j}^{P_0} - \int_{P_0+a_j+b_j}^{P_0+a_j}\right)\mathfrak{u}_k(P)\mathrm{d}\ln\vartheta_{\mathbf{e}}(P) =$$

$$= \frac{1}{2i\pi}\sum_{j=1}^{g}\int_{P_0}^{P_0+a_j}\Big(\mathfrak{u}_k\mathrm{d}\ln\vartheta_{\mathbf{e}} - (\mathfrak{u}_k+\tau_{kj})(\mathrm{d}\ln\vartheta_{\mathbf{e}} - 2i\pi\omega_j)\Big) +$$

$$+\frac{1}{2i\pi}\sum_{j=1}^{g}\int_{P_0+b_j}^{P_0}\Big(\mathfrak{u}_k\mathrm{d}\ln\vartheta_{\mathbf{e}} - (\mathfrak{u}_k+2i\pi\delta_{jk})\mathrm{d}\ln\vartheta_{\mathbf{e}}\Big) = \qquad (5.2.10)$$

$$= \sum_{j=1}^{g}\int_{P_0}^{P_0+a_j}\mathfrak{u}_k\omega_j - \frac{\tau_{kj}}{2i\pi}\overbrace{\int_{P_0}^{P_0+a_j}\mathrm{d}\ln\vartheta_{\mathbf{e}}}^{=0} + \tau_{kj}\overbrace{\oint_{a_j}\omega_j}^{=1} - \delta_{kj}\int_{P_0}^{P_0+b_j}\mathrm{d}\ln\vartheta_{\mathbf{e}} = \qquad (5.2.11)$$

$$= \sum_{j=1}^{g}\int_{P_0}^{P_0+a_j}\mathfrak{u}_k\omega_j - \overbrace{\sum_{j=1}^{g}\tau_{kj}}^{\equiv 0\in J(\mathcal{M})} - \frac{\tau_{kk}}{2} + \mathbf{e}_k \qquad (5.2.12)$$

where we have used that

$$\int_{P_0}^{P_0+\gamma}\mathrm{d}\ln\vartheta_{\mathbf{e}} = \ln\frac{\vartheta_{\mathbf{e}}(P_0+\gamma)}{\vartheta_{\mathbf{e}}(P_0)} \qquad (5.2.13)$$

and the periodicity properties (5.2.5) of $\vartheta_{\mathbf{e}}$.

Q.E.D.

Corollary 5.2.1 *Let $\mathcal{D}$ be a positive, nonspecial divisor of degree g. The function*

$$\vartheta_{\mathcal{D}}(P) = \Theta(\mathfrak{u}(P) - \mathfrak{u}(\mathcal{D}_g) - \mathcal{K}) \qquad (5.2.14)$$

provided does not vanishes identically[1] *then its divisor of zeroes coincides precisely with $\mathcal{D}$.*

Proposition 5.2.3 (Theta divisor 1) *The function Θ vanishes at $\mathbf{e}\in\mathbb{C}^g$ if and only if $\mathbf{e} = \mathfrak{u}(\mathcal{D}_{g-1}) + \mathcal{K}$ for some positive divisor of degree $g-1$ i.e. Θ vanishes on a $g-1$–dimensional variety parametrized by arbitrary $g-1$ points on $\mathcal{M}$, or $W_{g-1}+\mathcal{K}$.*

Proof. Suppose $\mathbf{e} = \mathfrak{u}(\mathcal{D}_{g-1}) + \mathcal{K}$, where $\mathcal{D}_{g-1} = P_1 + \ldots + P_{g-1}$ (not necessarily distinct); choose another point P_g and augment the divisor by it $\mathcal{D} := \mathcal{D}_{g-1} + P_g$. We assume that $\mathcal{D}$ is non-special so that its Abel map uniquely determines it (remember Corollary 4.3.1); this is an open condition because it correspond to the nonvanishing of the determinant of the $g\times g$ matrix of holomorphic differentials at $\mathcal{D}$ in some choice of local parameters (and hence for all choices).

[1]This cannot happen for all divisors since from Jacobi inversion theorem we could choose a divisor of degree g whose Abel map can be any $\mathbf{e}\in\mathbb{C}^g$ and Θ is not identically zero on $J(\mathcal{M})$.

Consider $\vartheta_{\mathcal{D}}(P) := \Theta(\mathfrak{u}(P) - \mathfrak{u}(\mathcal{D}) - \mathcal{K})$ for some arbitrary point Q P. If $\vartheta \equiv 0$ then $\vartheta_{\mathcal{D}}(P_g) = 0 = \Theta(-\mathbf{e}) = \Theta(\mathbf{e})$ (the last equality follows from parity). If $\vartheta_{\mathcal{D}}(P)$ is not identically zero then however it has g zeroes which coincide (by the nonspecialty of $\mathcal{D}$) with $\mathcal{D}$. Hence, again $\vartheta_{\mathcal{D}}(P_g) = 0$ as before. Since nonspecial divisors form an open and dense set amongst all divisors (with the natural topology of $\mathcal{M}_g = \mathcal{M} \times \ldots \times \mathcal{M}/S_g$) then the statement follows.

Viceversa suppose $\Theta(\mathbf{e}) = 0 = \Theta(-\mathbf{e})$. Consider the integer s with the property: **(P)** for all divisors $\mathcal{D}', \mathcal{D}''$ of degree $\leq s$ then $\Theta(\mathfrak{u}(\mathcal{D}' - \mathcal{D}'') - \mathbf{e}) \equiv 0$, but for some (and hence an open-dense set) divisors $\widehat{\mathcal{D}}, \widetilde{\mathcal{D}}$ of degree $s+1$ then $\Theta(\mathfrak{u}(\widehat{\mathcal{D}} - \widetilde{\mathcal{D}}) - \mathbf{e}) \not\equiv 0$. By Jacobi inversion, $s \leq g-1$.

Let such $\widehat{\mathcal{D}} = P_1 + P_2 + \ldots + P_{s+1}$ and $\widetilde{\mathcal{D}} = Q_1 + \ldots + Q_{s+1}$ for which $\Theta(..) \neq 0$; then, as a function of P, it is not identically zero

$$\psi(P) := \Theta(\mathfrak{u}(P) + \mathfrak{u}(P_2 + \ldots + P_{s+1} - Q_1 - \ldots - Q_{s+1}) - \mathbf{e}). \tag{5.2.15}$$

Clearly $\psi(Q_j) = 0$ are $s+1$ zeroes (because then it is $\Theta(\mathfrak{u}(\mathcal{D}' - \mathcal{D}'') - \mathbf{e})$ for divisors of degree s); since it has g zeroes there are points $T_{s+2}, \ldots, T_g$ such that

$$(\psi) = Q_1 + \ldots + Q_s + T_{s+1} + \ldots + T_g = \mathcal{D}_0. \tag{5.2.16}$$

Then, by Prop. 5.2.2,

$$\mathfrak{u}(Q_1 + \ldots + Q_{s+1} + T_{s+2} + \ldots + T_g) = -\mathfrak{u}(P_2 + \ldots + P_{s+1}) + \mathfrak{u}(Q_1 + \ldots + Q_{s+1}) + \mathbf{e} - \mathcal{K} \tag{5.2.17}$$

and hence

$$\mathbf{e} = \mathfrak{u}(P_2 + \ldots + P_{s+1} + T_{s+2} + \ldots + T_g) + \mathcal{K} \tag{5.2.18}$$

namely $\mathbf{e} - \mathcal{K}$ is the Abel map of a divisor of degree $g-1$. **Q.E.D.**

Corollary 5.2.2 (Theta divisor 1bis) *The vector* $\mathbf{e}$ *belongs to the Theta divisor* (Θ) *(the zero-set in* $J(\mathcal{M})$*) if and only if*

$$\mathbf{e} = \mathfrak{u}(P_1 + \ldots + P_{g-1}) + \mathcal{K}\ . \tag{5.2.19}$$

The divisor $\mathcal{D} := P_1 + \ldots + P_{g-1}$ *(of degree* $g-1$*) is a divisor with index of specialty* $s \geq 1$ *(*$i(\mathcal{D}) = s$*) if and only if* $\Theta(\mathfrak{u}(\mathcal{D}') - \mathfrak{u}(\mathcal{D}'') - \mathbf{e}) \equiv 0$ *for all divisors* $\mathcal{D}', \mathcal{D}''$ *of degree* $\leq s$ *and* $\Theta(\mathfrak{u}(\widehat{\mathcal{D}} - \widetilde{\mathcal{D}}) - \mathbf{e})$ *is not identically zero for divisors of degree* $s+1$ *(*P_0 *is the basepoint of the Abel map)*

Proof. If we examine the proof of the above Theorem Proposition 5.2.3 we see that the $g-s-1$ points $T_{s+2}, \ldots, T_g$ are determined by the $Q_1, \ldots, Q_{s+1}$ and the $P_2, \ldots, P_{s+1}$. If we consider the Q_j's as parameters of the problem then we may write that

$$T := T_{s+2} + \ldots + T_g = T(P_2, \ldots, P_{s+1}). \tag{5.2.20}$$

This also means that (at least in a small neighborhood) we can move the $P_2, \ldots, P_{s+1}$ freely. Also, by eq. (5.2.18) the Abel maps of

$$\mathcal{D}(\vec{P}) := P_1 P_2 + \ldots + P_{s+1} + T(\vec{P}) \tag{5.2.21}$$

is independent of $\vec{P}$. By Abel's theorem we can then find meromorphic F such that

$$(F) = \mathcal{D}(\vec{P}') - \mathcal{D}(\vec{P}) \tag{5.2.22}$$

for any choices of points P_j, P'_j. This implies that $r(-\mathcal{D}(\vec{P})) \geq s$ by Coroll. 4.3.1. Since $\deg(\mathcal{D}(\vec{P})) = g-1$ then $i(\mathcal{D}(\vec{P})) \geq s-1$ (by Riemann–Roch).

We now show that, in fact, $i(\mathcal{D}(\vec{P})) = s-1$. Indeed, again by Coroll. 4.3.1, if it were $i(\mathcal{D}) \geq s+1$ then $r(-\mathcal{D}) \geq s+21$ and then a bigger manifold of divisors would share the same Abel map, which contradicts the hypothesis. **Q.E.D.**

Note that the above corollary also implies the much weaker (but maybe clearer)

Corollary 5.2.3 *The function $\Theta(\mathfrak{u}(P) - \mathbf{e})$ vanishes* **identically** *if and only if $\mathbf{e} = \mathfrak{u}(\mathcal{D}_{g-1}) + \mathcal{K}$ and $i(P_0 + \mathcal{D}_{g-1}) \geq 1$ (i.e. it is special[2]) where P_0 is the basepoint of the Abel map and $\mathcal{D}$ is a positive divisor of degree $g-1$. SO THAT THE PROOF BELOW DOES NOT NEED TO CHANGE EVERY $\mathcal{D}$ INTO $\mathcal{D}_{g-1}$.*

Proof. Suppose that $\Theta(\mathfrak{u}(P) - \mathfrak{u}(\mathcal{D}) - \mathcal{K}) \equiv 0$; since $\mathfrak{u}(P_0) = 0$, there is another divisor $\mathcal{D}'$ of degree $g-1$ such that

$$\mathfrak{u}(P) - \mathfrak{u}(P_0) - \mathfrak{u}(\mathcal{D}) = -\mathfrak{u}(\mathcal{D}') \quad \Leftrightarrow \quad \mathfrak{u}(P_0 + \mathcal{D}) = \mathfrak{u}(P + \mathcal{D}') \ . \tag{5.2.23}$$

By Abel's theorem then there is a nontrivial meromorphic function F such that

$$(F) = P + \mathcal{D}' - (P_0 + \mathcal{D}) \tag{5.2.24}$$

and hence in particular $r(-P_0 - \mathcal{D}) \geq 2 \Rightarrow \quad i(P_0 + \mathcal{D}) \geq 1$, namely it is special.

Viceversa, if $P_0 + \mathcal{D}$ is special, then $r(-P_0 + -\mathcal{D}) \geq 2$ and hence there is a nontrivial and nonconstant meromorphic function f with divisor of poles $0 < \mathcal{D}_\infty \leq P_0 + \mathcal{D}$ and vanishing at any $P \in \mathcal{M}$ (take $F(Q) - F(P)$); let

$$P_0 + \mathcal{D} = \mathcal{D}_\infty + \mathcal{D}^c_\infty \tag{5.2.25}$$

and $\mathcal{D}' := (f) + \mathcal{D}_\infty - P$ so that

$$\mathfrak{u}(P+\mathcal{D}') = \mathfrak{u}(\mathcal{D}_\infty) \ \Leftrightarrow \ \mathfrak{u}(P+\mathcal{D}'+\mathcal{D}^c_\infty) = \mathfrak{u}(P_0+\mathcal{D}) \ \Rightarrow \Theta(\mathfrak{u}(P)-\mathfrak{u}(P_0+\mathcal{D})-\mathcal{K}) = \Theta(-\mathfrak{u}(\underbrace{\mathcal{D}' + \mathcal{D}^c_\infty}_{\deg = g-1})-\mathcal{K}) = 0 \, . \tag{5.2.26}$$

This concludes the proof. **Q.E.D.**

[2] A divisor of degree $k \leq g$ is special if $i(\mathcal{D}) > g - k$, Def. 4.1.9.

Corollary 5.2.4 *Let* $\deg \mathcal{D}_g = g$*; then* $\mathfrak{u}(\mathcal{D}_g) + K$ *is in the Theta divisor iff* $\mathcal{D}_g$ *is special.*

Proof. Exactly as above. **Q.E.D.**

It would then take a little more effort to prove the following complete characterization of the Theta divisor

Theorem 5.2.1 (Riemann Theorem) *Let* s *be the least integer such that* $\Theta(\mathfrak{u}(\mathcal{D}_{s-1} - \mathcal{D}'_{s-1}) - \mathbf{e}) \equiv 0$ *but* $\Theta(\mathfrak{u}(\mathcal{D}_s - \mathcal{D}'_s) - \mathbf{e}) \not\equiv 0$*. Then*

- $\mathbf{e} = \mathfrak{u}(\mathcal{D}) + \mathcal{K}$ *with* $\deg \mathcal{D} = g - 1$*,* $\mathcal{D} > 0$*;*
- $i(\mathcal{D}) = s$*;*
- *All partial derivatives of* Θ *at* $\mathbf{e}$ *of order* $\leq s-1$ *vanish but at least one partial of order* s *does not.*

Viceversa the above properties characterize the image in $J(\mathcal{M})$ *of the special divisor of degree* $g-1$ *and index* $i(\mathcal{D}) = s$*.*

We conclude this chapter with a proposition that explains the meaning of the vector of Riemann constants $\mathcal{K}$

Proposition 5.2.4 *The vector* $-2\mathcal{K}$ *is the Abel map of the divisor of a differential. Viceversa any divisor* $\mathcal{C}$ *of degree* $2g-2$ *is canonical if and only if* $\mathfrak{u}(\mathcal{C}) = -2\mathcal{K}$*.*

Proof. Let $\xi = P_1 + \ldots + P_{g-1}$. Then $\mathbf{e} := \mathfrak{u}(\xi) + \mathcal{K}$ is a zero of Θ (Prop. Cor. 5.2.3). By symmetry, $\Theta(-\mathbf{e}) = 0$ and hence for some other divisor $\deg \eta = g-1$

$$-\mathbf{e} = \mathfrak{u}(\eta) + \mathcal{K} \quad \Rightarrow \quad \mathfrak{u}(\eta + \xi) = -2\mathcal{K}\ . \tag{5.2.27}$$

We now prove that $\eta + \xi$ is the divisor of a first–kind differential. By Corollary 4.3.1, since ξ was arbitrary: MORE KINDLY if for an arbitrary positive divisor ξ of degree $g-1$ there exists $\eta > 0$ such that $\mathfrak{u}(\eta + \xi) = -2\mathcal{K}$ then $r(-\xi - \eta) \geq g$ and hence

$$r(-\xi - \eta) = i(\xi + \eta) - g + (2g-2) + 1 = i(\xi + \eta) + g - 1 \geq g \quad \Leftrightarrow \quad i(\xi + \eta) \geq 1. \tag{5.2.28}$$

Then at least one $\omega \in \mathfrak{I}(\xi + \eta)$ exists.

For the second part, suppose $\mathfrak{u}(\mathcal{C}) = -2\mathcal{K}$; we know that there is a holomorphic differential ω with $\mathfrak{u}((\omega)) = -2\mathcal{K}$. Hence $\mathfrak{u}(\mathcal{C}) = \mathfrak{u}((\omega))$, so there is a meromorphic function (by Abel's theorem) F with $(F) = \mathcal{C} - (\omega)$. Then $\widetilde{\omega} := F\omega$ is the desired differential (holmorphic) and $(\widetilde{\omega}) = \mathcal{C}$. **Q.E.D.**

Chapter 6

Writing functions and differentials with Θ

This chapter is devoted to one of the most practical aspects of the theory of Theta functions (at least in my limited experience). For example we will see that once the normalized first kind Abelian differentials are given, then the second and third kind differentials can be easily written in terms of Θ functions and derivative thereof. Also we will be able of writing any meromorphic function (up to multiplicative constant) if we know its divisor. One of the basic ideas is contained in the following

Lemma 6.0.1 *Let* $\mathbf{e}$ *be in the nonsingular part of the* Θ*–divisor, namely (Thm. 5.2.1)*

$$\mathbf{e} = \mathfrak{u}(P_1 + \ldots + P_g P_{g-1}) + \mathcal{K} =: \mathfrak{u}(\mathcal{D}_{g-1}) -+ \mathcal{K}$$
$$i(\mathcal{D}_{g-1}) = 1 \ . \tag{6.0.1}$$

Then

$$F(P;Q) := \Theta(\mathfrak{u}(P-Q) - \mathbf{e}) \tag{6.0.2}$$

vanishes at $P = Q$ *and at* $P \in \mathcal{D}_{g-1}$*, where the position of the last* $g-1$ *zeroes is* **independent of** Q*.*

Proof. It follows from Prop. 5.2.2 that, as a function of P F has zeroes at the divisor $Q + \mathcal{D}_{g-1} = \mathcal{D}_g$; since $i(\mathcal{D}_{g-1}) = 1$ then, generically $i(\mathcal{D}_g) = 0$. **Q.E.D.**

We remark the importance of the nonspecialty of $\mathcal{D}_{g-1}$ (and also of $\mathcal{D}_g$, although we can choose Q in an open and dense set).

Let now f be a meromorphic function with divisor $f = \sum P_j - \sum Q_j$; then

$$f(P) = c\frac{\prod \Theta(\mathfrak{u}(P-P_j) - \mathbf{e})}{\prod \Theta(\mathfrak{u}(Q-Q_j) - \mathbf{e})} \ , \quad c \in \mathbb{C}^\times . \tag{6.0.3}$$

To check the assertion we need to check that the RHS defines a single–valued function with the desired properties; the poles and zeroes being evident then one has to check the periodicities around the a, b

cycles. This is an exercise using Prop. 5.1.1. The only care is in the choice of $\mathbf{e}$ in such a way that none of the divisors $P_j + \mathcal{D}_{g-1}, Q_j + \mathcal{D}_{g-1}$ is special (for in this case one of the Theta's vanishes identically). This can always be accomplished (**why?**).

In order to get more refined tools we need to step into Fay's book (for instance) [2]

6.1 The odd nonsingular characteristics

Let Δ denote a **odd, half integer, nonsingular characteristics**; I recall that this means that Δ is a half–period

$$\Delta = \frac{1}{2}\epsilon' + \frac{1}{2}\tau \cdot \epsilon \ , \ \epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon \cdot \epsilon' \in 2\mathbb{Z} + 1. \tag{6.1.1}$$

We denote by Θ_Δ the Theta funciton with that characteristics (Def. 5.1.3) and we know that $\Theta_\Delta(\mathbf{z})$ is odd, hence $\Theta_\Delta(0) = \Theta(\Delta) = 0$.

In particular for $\Theta_\Delta(\mathfrak{u}(P))$ is valid all that was said in the previous chapter and in particular Thm. 5.2.1; we know that $\Delta = \mathfrak{u}(\mathcal{D}^\Delta_{g-1}) + \mathcal{K}$ and that $\Theta_\Delta(\mathfrak{u}(P))$ does not vanish identically iff $i(\mathcal{D}^\Delta_{g-1}) = 1$. For this reason we need to request that Δ be non-singular.

Theorem 6.1.1 *There exist nonsingular odd half–integer characteristics.*

The proof can be found in [3]. From now on, we suppose $\mathcal{D}^\Delta_{g-1}$ is non-singular

Consider now the same (or almost) function used in Lemma. 6.0.1

$$F_\Delta(P, Q) := \Theta_\Delta(\mathfrak{u}(P - Q)) \ . \tag{6.1.2}$$

This function is antisymmetric $F(P, Q) = -F(Q, P)$; as a function of P it has zeroes at Q and $\mathcal{D}^\Delta_{g-1}$.

Lemma 6.1.1 *For no point $Q \in \mathcal{M} \setminus \mathcal{D}^\Delta_{g-1}$ the divisor $Q + \mathcal{D}^\Delta_{g-1}$ is singular. Hence $F_\Delta(P, Q)$ has zeroes at $P = Q$ and $(P, Q) \in \mathcal{D}^\Delta_{g-1} \times \mathcal{M} \cup \mathcal{M} \times \mathcal{D}^\Delta_{g-1}$.*

Proof. Suppose Q_0 is such that $i(QQ_0 + \mathcal{D}_{g-1}\mathcal{D}^\Delta_{g-1}) = 1$ (it can't be bigger than that because $i(\mathcal{D}_{g-1}\mathcal{D}^\Delta_{g-1}) = 1$). Then $F(P, Q_0) \equiv 0$ as a function of P; hence $F(Q_0, P) = 0$ identically (by anti-symmetry, something we did not have in Lemma 6.0.1). But $F(Q, P)$ is not identically zero (at least for an open-dense set of P's) and has zeros $P, \mathcal{D}^\Delta_{g-1}$. This means that $Q_0 \in \mathcal{D}^\Delta_{g-1}$, a contradiction. The last assertion follows immediately. **Q.E.D.**

Lemma 6.1.2 *The divisor $2\Delta\mathcal{D}^\Delta_{g-1}$ is the divisor of a holomorphic differential for* **any odd** *half-period Δ (singular or not).*

Proof. By Prop. 5.2.4 we need to prove that $\mathfrak{u}(2\mathcal{D}^{\Delta}_{g-1}) = -2\mathcal{K}$. Indeed

$$\mathfrak{u}(\mathcal{D}^{\Delta}_{g-1}) = \Delta - \mathcal{K} \quad \Rightarrow \quad \mathfrak{u}(2\mathcal{D}^{\Delta}_{g-1}) = -2\mathcal{K} \tag{6.1.3}$$

since $\Delta = -\Delta$ is a half–period. **Q.E.D.**

The next technically important object is contained in the next proposition

Proposition 6.1.1 *Let Δ be a nonsingular, odd half–characteristics. The holomorphic differential*

$$\omega_\Delta := \sum_{j=1}^{g} \partial_{z_j}\Theta_\Delta(0)\omega_j \tag{6.1.4}$$

has double zeroes at $\mathcal{D}^{\Delta}_{g-1}$, or, precisely

$$(\omega_\Delta) = 2\mathcal{D}^{\Delta}_{g-1} \ . \tag{6.1.5}$$

namely it is the differential advocated in Lemma 6.1.2.

Proof. [Check!] Consider

$$F_\Delta(P,Q) := \Theta_\Delta(\mathfrak{u}(P) - \mathfrak{u}(Q)) \ , \tag{6.1.6}$$

where Q is chosen generically so that Theta is not identically zero (and that means $Q \not\in \mathcal{D}^{\Delta}_{g-1}$). The differential w.r.t. P is (using the chain rule)

$$\mathrm{d}_P F_\Delta(P,Q) = \sum_{j=1}^{g} \partial_{z_j}\Theta_\Delta(\mathfrak{u}(P) - \mathfrak{u}(Q))\omega_j(P) \tag{6.1.7}$$

If we set $P = Q$ then we have

$$\omega_\Delta := \mathrm{d}_P F_\Delta(P,Q)\big|_{P=Q} = \sum_{j=1}^{g} \partial_{z_j}\Theta_\Delta(0)\omega_j(Q) \ . \tag{6.1.8}$$

Since $F_\Delta(P,Q)$ has a zero for $Q \in \mathcal{D}^{\Delta}_{g-1}$, then so must be for the differential above, so that $(\omega_\Delta) \geq \mathcal{D}^{\Delta}_{g-1}$. This is confirmed by a computation in local coordinates. Let $R \in \mathcal{M}$ appear in $\mathcal{D}^{\Delta}_{g-1}$ with multiplicity k; let z be a local coordinate, $z(R) = 0$. Let $z = z(P),\ z' = z(Q)$, then

$$F_\Delta(P,Q) = f(z,z') = (z - z')(C(z') + \mathcal{O}((z - z'))) \ , \tag{6.1.9}$$

where the $\mathcal{O}$ is uniform in z, z'. Indeed $f(z.,z')$ has a simple zero for $z = z'$, so that $f(z,z')/(z - z') = H(z,z')$ is an even function (in the exchange $z \leftrightarrow z'$) such that $C(z') = H(z',z')$ is not identically zero and vanishes of order k at $z = z(R) = 0$. Then

$$\partial_z f(z,z')\big|_{z=z'} = C(z') \tag{6.1.10}$$

has the desired property.

On the other hand, since $\mathcal{D}^{\Delta}_{g-1}$ is nonsingular, i.e. $i(\mathcal{D}^{\Delta}_{g-1}) = 1$, its complementary in the canonical divisor $\mathcal{K}$ is uniquely determined, and since $2\Delta = 0$ it follows that

$$\mathfrak{u}(\mathcal{D}^{\Delta}_{g-1} + \xi) = -2\mathcal{K} \quad \Leftrightarrow \quad \xi = \Delta^{\Delta}_{g-1}. \tag{6.1.11}$$

Therefore $(\omega_\Delta) = 2\mathcal{D}^{\Delta}_{g-1}$.

Or, more mundanely, since $H(z, z')$ above must vanish of order k **both** in z and z' at $z = 0$ or $z' = 0$ it follows that actually $C(z') = H(z', z')$ necessarily vanishes of order $2k$. **Q.E.D.**

6.2 The Prime form

Much of the work has been already done.

We consider Δ a odd-nonsingular half–integer characteristics and all that was used in the previous section.

Definition 6.2.1 *A* **spinor** *or* **half–differential** *is an assignment of locally holomorphic functions f_α on an atlas U_α for $\mathcal{M}$ such that*

$$f_\alpha(z) = \sqrt{\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}} f_\beta(z) \tag{6.2.1}$$

or, equivalently, such that the expression

$$f_\alpha \sqrt{\mathrm{d}z_\alpha} = f_\beta \sqrt{\mathrm{d}z_\beta} \ , \tag{6.2.2}$$

where the square–root in eq. (6.2.1) is chosen consistently i.e. so as to satisfy the cocycle condition

$$\sqrt{\frac{\mathrm{d}z_\beta}{\mathrm{d}z_\alpha}} \sqrt{\frac{\mathrm{d}z_\alpha}{\mathrm{d}z_\gamma}} \sqrt{\frac{\mathrm{d}z_\gamma}{\mathrm{d}z_\beta}} = 1 \tag{6.2.3}$$

in all triple intersections.

The natural question would be "how many ways are there to choose the square–roots?". The answer is 4^g, i.e. one for each half–period.

We note immediately that if $\mathfrak{s}$ is a half–form (for some choice of square–roots) then $\mathfrak{s}^2$ is a differential, independent of the choice of square–roots. This implies that

$$\mathfrak{u}(2(\mathfrak{s})) = -2\mathcal{K} \ . \tag{6.2.4}$$

However, in particular, $\mathfrak{s}^2$ has clearly only **double** (or –more generally– even) zeroes).

Viceversa if ω is a differential with only even zeroes, then $\sqrt{\omega}$ is a half–differential (spinor)

Recalling the differential ω_Δ of Lemma 6.1.2 or Prop. 6.1.1, we **define the spinor**

$$h_\Delta := \sqrt{\sum_{j=1}^{g} \partial_{z_j} \Theta_\Delta(0) \omega_j} \ . \tag{6.2.5}$$

Definition 6.2.2 *The prime form (of Weil) is the bi-half-differential*

$$E(P,Q) := \frac{\Theta_\Delta(\mathfrak{u}(P)-\mathfrak{u}(Q))}{h_\Delta(P)h_\Delta(Q)} \ . \tag{6.2.6}$$

Proposition 6.2.1 *The prime form has the properties*

1. *It is skew–symmetric* $E(P,Q) = -E(Q,P)$
2. *It has the periodicity properties*

$$\begin{aligned} E(P+a_j,Q) &= E(P,Q) \\ E(P+b_j,Q) &= \exp\left[-2i\pi\left(\frac{\tau_{jj}}{2}+\int_P^Q \omega_j\right)\right]E(P,Q) \end{aligned} \tag{6.2.7}$$

3. *It vanishes to first order at* P,Q *and nowhere else; in a local coordinate chart* z *containing both* P,Q *we have*

$$E(P,Q) = \frac{(z-z')}{\sqrt{\mathrm{d}z}\sqrt{\mathrm{d}z'}}(1+\mathcal{O}((z-z')^2)) \tag{6.2.8}$$

where the coefficient 1 *is well–defined, independently of the choosen coordinate (***verify it!***)*

4. *The prime form is independent of the choice of* Δ *provided it is an odd, nonsingular, half–integer characteristics.*

Proof. Properties 1,2 are obvious or straightforward. Property 3 follows from the fact that $\Theta_\Delta(\mathfrak{u}(P) - \mathfrak{u}(Q)$ vanishes for $P = Q$ and $P \in \mathcal{D}^\Delta_{g-1}$; these last $g-1$ zeroes cancel with the zeroes (of the same exact multiplicity) of $h_\Delta(P)$ in the denominator. The normalization of the "residue" comes from expanding the numerator near the diagonal, which gives $h_\Delta(P)^2$, cancelling with the denominator.

The last properties follows from the fact that $E_\Delta, E_{\Delta'}$ would have the same periodicities and same behavior so that the ratio would be a holomorphic function, hence constant. The constant is one because of the behaviour on the diagonal. **Q.E.D.**

6.3 The fundamental bidifferential

Closely related to the prime form is the fundamental normalized bidifferential;

Definition 6.3.1 *The fundamental normalized bidifferential* $\Omega(P,Q)$ *is a differential in both* P *and* Q *with the following properties*

1. *Symmetry* $\Omega(P,Q) = \Omega(Q,P)$
2. *Normalization:* $\displaystyle\oint_{P\in a_j} \Omega(P,Q) \equiv 0$

3. *Meromorphicity: $\Omega(P,Q)$ is meromorphic in P with only a double pole at $P=Q$ (and symmetrically).*

4. *Biresidue normalization; if z is a local chart containing both points P,Q, with $z=z(P), z'=z(Q)$ then*

$$\Omega(P,Q) \underset{P\sim Q}{\simeq} \left[\frac{1}{(z-z')^2} + \frac{1}{6}S_B(z) + \mathcal{O}(z-z')\right] \mathrm{d}z\mathrm{d}z' \ , \tag{6.3.1}$$

where the very important quantity $S_B(\zeta)$ is the " Bergman projective connection" (it transforms like the Schwartzian derivative under changes of coordinates).

Exercise 6.3.1 *Using the above properties show that*

$$\oint_{P\in b_j} \Omega(P,Q) = 2i\pi\omega_j(Q) \ . \tag{6.3.2}$$

Proposition 6.3.1 (Fundamental bidifferential in terms of prime form) *The fundamental normalized bidifferential $\Omega(P,Q)$ is given by*

$$\Omega(P,Q) = \mathrm{d}_P\mathrm{d}_Q \ln E(P,Q) = \mathrm{d}_P\mathrm{d}_Q \ln \Theta_\Delta(\mathfrak{u}(P-Q)) \tag{6.3.3}$$

Proof. The logarithm of E is a murky object, since one takes the log of a half–differential; however the differentiations kill these terms. More clear is the last expression, which we now analyze.

The first observation is that the RHS is a single–valued differential since (see eq. 5.1.8) for $\gamma = \sum_{j=1}^g m_j a_j + n_j b_j \in H_1(\mathcal{M},\mathbb{Z})$

$$\ln \Theta_\Delta(\mathfrak{u}(P+\gamma-Q)) = 2i\pi \sum_{j=1}^g n_j \int_{Q^P} \omega_j + \ln \Theta_\Delta(\mathfrak{u}(P+\gamma-Q)) + \text{ constant} \tag{6.3.4}$$

and hence differentiatin w.r.t P,Q leaves a single–valued bidifferential.

The second observation is that the bidifferential on the RHS is also normalized since

$$\oint_{a_j} \Omega(P,Q) = \mathrm{d}_Q \ln \Theta_\Delta(\mathfrak{u}(P+a_j-Q)) - \mathrm{d}_Q \ln \Theta_\Delta(\mathfrak{u}(P-Q)) = 0 \tag{6.3.5}$$

since $\Theta_\Delta(\mathfrak{u}(P-Q))$ is periodic around the a–cycles.

Next, $F(P,Q) = \Theta_\Delta(\mathfrak{u}(P-Q))$ has simple zero at $P=Q$, so that $d_P d_Q F(P,Q)$ has a double pole at $P=Q$ without residue; indeed in a local coordinate $z=z(P), z'=z(Q)$ we have

$$F(P,Q) = (z-z')c(z,z') \ , \quad \partial_z\partial_{z'} \ln F = \frac{1}{(z-z')^2} + \mathcal{O}(1) \tag{6.3.6}$$

where $c(z,z') = c(z',z)$ is nonzero for $z=z'$.

Now $F(P,Q)$ has other $g-1$ zeroes at the divisor $\mathcal{D}^\Delta_{g-1}$ (with suitable multiplicity); if $R\in\Delta^\Delta_{g-1}$ with multiplicity k and $z(R)=0$ ($z=z(P)$) then

$$F(P,Q) = z^k(C(Q)+\mathcal{O}(z)) \quad \Rightarrow \quad \mathrm{d}_P\mathrm{d}_Q \ln F = \mathrm{d}_Q\left(\frac{k}{z} + \mathcal{O}(1)\right) = \mathcal{O}(1). \tag{6.3.7}$$

This shows that the RHS has the desired properties and hence is our fundamental bidifferential. **Q.E.D.**

6.3.1 Writing differentials of the second and third kind

Proposition 6.3.2 *The normalized differential of the third kind is*

$$\omega_{P_+P_-}(P) = \int_{P_-}^{P_+} \Omega(Q,P) = \sum_{j=1}^{g} \frac{\partial_{z_j}\Theta_\Delta(\mathfrak{u}(P)-\mathfrak{u}(S))\omega_j(P)}{\Theta_\Delta(\mathfrak{u}(P)-\mathfrak{u}(S))}\Bigg|_{S=P_-}^{S=P_+} = \mathrm{d}_P \ln \frac{\Theta_\Delta(\mathfrak{u}(P-P_+))}{\Theta_\Delta(\mathfrak{u}(P-P_-))} \tag{6.3.8}$$

Corollary 6.3.1 (Exchange formula) *For the normalized third kind differentials we have*

$$\int_B^A \omega_{PQ} = \int_Q^P \omega_{AB} \tag{6.3.9}$$

Proof. Indeed, from Prop. 6.3.2,

$$\int_B^A \omega_{PQ} = \ln \frac{\Theta_\Delta(\mathfrak{u}(A-P))\Theta_\Delta(\mathfrak{u}(B-Q))}{\Theta_\Delta(\mathfrak{u}(A-Q))\Theta_\Delta(\mathfrak{u}(B-P))} = \int_Q^P \omega_{AB} \tag{6.3.10}$$

The formula can be proved also directly without the explicit expression in terms of Theta funcitons, using Riemann bilinear identities (with some care, [1]). **Q.E.D.**

Proposition 6.3.3 *The normalized differential of the second kind w.r.t. a local parameter z, $z(P_0) = 0$ and pole of order $k+1$ at P_0 is given by*

$$\omega_{P_0,k}(P) = -\frac{1}{k} \operatorname*{res}_{Q=P_0} z(Q)^{-k}\Omega(P,Q) \tag{6.3.11}$$

Proof. One needs to check that the proposed expression is a normalized differential of the second kind and that the expansion at P_0 in the local coordinate z (the same used in the residue!) is

$$\omega_{P_0,k}(P) = \left(\frac{1}{z^{k+1}} + \mathcal{O}(1)\right)\mathrm{d}z\ . \tag{6.3.12}$$

The details are left as **exercise**. **Q.E.D.**

6.3.2 Differentials of the first kind for nonspecial divisors

Proposition 6.3.4 *Let $\xi = P_1 + \ldots + P_{g-1}$ be nonspecial (i.e. $i(\xi) = 1$ and let $\mathbf{e} = \mathfrak{u}(\xi) + \mathcal{K}$. Then the first kind differential in $\mathfrak{I}(\xi)$ is, up to nonzero constant*

$$\omega(P) \propto \det \begin{bmatrix} \omega_1(P) & \omega_2(P) & \ldots & \omega_g(P) \\ \omega_1(P_1) & \omega_2(P_1) & \ldots & \omega_g(P_1) \\ \vdots & & & \vdots \\ \omega_1(P_{g-1}) & \omega_2(P_{g-1}) & \ldots & \omega_g(P_{g-1}) \end{bmatrix} \propto \sum_{j=1}^{g} \partial_{z_j}\Theta(\mathbf{e})\omega_j(P) \tag{6.3.13}$$

The other $g-1$ zeroes $\xi' = P_1' + \ldots + P_{g-1}'$ satisfy

$$\mathfrak{u}(\xi+\xi') = -2\mathcal{K} \quad \Leftrightarrow \quad \mathfrak{u}(\xi) + \mathcal{K} = -\mathfrak{u}(\xi') - \mathcal{K}\ . \tag{6.3.14}$$

In the formula above the determinantal expression is valid only if the points are pairwise distinct; if the points have multiplicity then a similar determinant can be written (using derivatives of the first kind–differentials) but it is left as **exercise***.*

Proof. The determinantal expression is trivially verified to yield a *nonzero* first kind differential with zeroes at $P_1, \dots, P_{g-1}$ (if the divisor ξ is nonspecial), since the $(g-1) \times g$ matrix $[\omega_j(P_i)]_{i \le g-1, j \le g}$ is of maximal rank.

To prove the last part of the formula we consider

$$F(P, Q) = \Theta(\mathfrak{u}(P) - \mathfrak{u}(Q) - \mathbf{e}) \ . \tag{6.3.15}$$

Clearly $F(Q, Q) = \Theta(-\mathbf{e}) = 0$; consider

$$\omega(P) := \mathrm{d}_P F(P, Q)\Big|_{Q=P} = \sum_{j=1}^{g} \partial_{z_j} \Theta(-\mathbf{e}) \omega_j(P) = -\sum_{j=1}^{g} \partial_{z_j} \Theta(\mathbf{e}) \omega_j(P) \ . \tag{6.3.16}$$

We claim that it belongs to $\mathfrak{I}(\xi)$ and hence (by nonspecialty) spans it. First of all it is nonzero because (again by nonspecialty) not all partials of Θ at $\mathbf{z} = \mathbf{e}$ vanish (Thm. 5.2.1). We need to check that it vanishes at all points $\widetilde{P} \in \xi$ and to the correct order if the multiplicity of $\widetilde{P}$ is greater than one.

Now $F(P, Q)$ as a function of P has divisor of zeroes $Q + \xi$; indeed $Q + \xi$ is (generically for Q) nonspecial and hence –by Prop. 5.2.2– this is its divisor of zeroes.

Let $z(\widetilde{P}) = 0$ be a local parameter near the point $\widetilde{P} \in \xi$ and suppose that k is the multiplicity of $\widetilde{P}$ in ξ ; setting $z' = z(Q)$ we have

$$F(P, Q) = (z - z')z^k(C + \mathcal{O}(z, z')) \quad \Rightarrow \quad \mathrm{d}_P F(P, Q)|_{Q=P} = z^k(C + \mathcal{O}(z, z'))\mathrm{d}z \tag{6.3.17}$$

The position of the other $g - 1$ zeroes follows from the proof of Prop. 5.2.4. **Q.E.D.**

6.4 Fay identities

In this section we will follow the common usage and **omit the Abel map** when a point or a divisor appears in the argument of the Θ–function.

There are many identities due to J. Fay which appear in several guises in mathematical physics. One of the main identities is the following one, which is a generalization of the addition theorems for trigonometric functions.

Proposition 6.4.1 ([2] pag. 33) *Let $\mathbf{e} \in \mathbb{C}^g$ with $\Theta(\mathbf{e}) \neq 0$ and $P_1, \dots, P_N, Q_1, \dots Q_N$ be arbitrary points. Then*

$$\Theta\left(\sum P_j - \sum Q_j - \mathbf{e}\right) \Theta(\mathbf{e})^{N-1} \frac{\prod_{i<j} \Theta_\Delta(P_i - P_j)\Theta_\Delta(Q_i - Q_j)}{\prod_{i,j} \Theta_\Delta(P_i - Q_j)} = \det\left[\frac{\Theta(P_i - Q_j - \mathbf{e})}{\Theta_\Delta(P_i - Q_j)}\right]_{i,j \le N} \tag{6.4.1}$$

or, equivalently, (this is the original form of Fay's)

$$\Theta\left(\sum P_j - \sum Q_j - \mathbf{e}\right) \Theta(\mathbf{e})^{N-1} \frac{\prod_{i<j} E(P_i, P_j)E(Q_i, Q_j)}{\prod_{i,j} E(P_i, Q_j)} = \det\left[\frac{\Theta(P_i - Q_j - \mathbf{e})}{E(P_i, Q_j)}\right]_{i,j \le N} \tag{6.4.2}$$

Proof The equivalence of the two expressions follows in rather trivial way from the multilinearity of the determinant and by simple counting.

Let us consider eq. 6.4.2. First of all both sides are symmetric functions of $P_1, \ldots, P_N$ and $Q_1, \ldots, Q_N$ so that any statement made w.r.t. one point immediately applies to all other.

Consider the two sides of the equation as a function of P_1 (or any other P); using the periodicity properties of Theta (eqs. 5.1.2, 5.1.8) one verifies that both sides have the same multiplicative behavior under analytic continuation along any contour γ.

Moreover both sides vanish for $P_1 = P_j,\ j \neq 1$ and with the same order (simple if the points are pairwise distinct); both sides have poles for $P_1 = Q_j$ in eq. 6.4.2.

Therefore the ratio F :=LHS/RHS is a well–defined meromorphic function of P_1 with g poles and g zeroes, the zeroes coming from the first Theta in the LHS; we must prove that this funciton is actually a constant.

Indeed the pole–divisor $\mathcal{D}_\infty$ of F has Abel map

$$\mathfrak{u}(\mathcal{D}_\infty) = \mathbf{e} + \mathfrak{u}\left(\sum_{j=1}^{N} Q_j - \sum_{j\geq 2} P_j\right) - \mathcal{K}\ . \tag{6.4.3}$$

Since the points P_j, Q_j are arbitrary, we can choose them in a generic position so that $\mathfrak{u}(\mathcal{D}_\infty) \not\in (\Theta)$; in this case then $i(\mathcal{D}_\infty) = 0$ and hence there cannot be a nonconstant function with poles there, namely $F = const$.

To evaluate the constant we take the P_i, Q_j's in the same coordinate chart and set $z_i = z(P_i)$, $z' - j = Z(Q_j)$. Suppose $P_i \to Q_i$ while $Q_i \neq Q_j$; then the matrix in the determinant has poles only on the diagonal

$$\det\left[\frac{\Theta(P_i - Q_j - \mathbf{e})}{E(P_i, Q_j)}\right] = \det\begin{bmatrix} \frac{\Theta(-e)}{C(z_1 - z_1')} & \mathcal{O}(1) & \cdots & \\ \mathcal{O}(1) & \frac{\Theta(-\mathbf{e})}{C(z_2 - z_2')} & \cdots & \\ & & \ddots & \\ \mathcal{O}(1) & \cdots & & \frac{\Theta(-e)}{C(z_N - z_N')} \end{bmatrix} = \frac{\Theta(\mathbf{e})^N}{C^N \prod (z_i - z_i')}(1 + o(1)) \tag{6.4.4}$$

where the constant C is defined by $E(P_1, Q_1) = (z_1 - z_1')(C + \mathcal{O}(1))$. It is immediate to verify that the same constant appears in the leading singular behavior on the LHS. **Q.E.D.**

6.4.1 Cauchy kernel on Riemann–surfaces

We recall that in the ordinary case of $\mathbb{C}P^1$ the Cauchy kernel (in the coordinate z) is

$$C_0(w, z) = \frac{1}{z - w}\mathrm{d}z \tag{6.4.5}$$

where the subscript 0 stands for "genus 0". It is a function in the variable w and a differential in the variable z, with a simple pole at $z = w$ with residue 1; it has also a simple pole (as a differential!) at $z = \infty$. As a function of w it has a simple zero at $w = \infty$ and a simple pole at $w = z$.

The common usage of the Cauchy kernel is in Cauchy's theorem (!)

$$f(w) = \frac{1}{2i\pi} \oint_{|z-w|=\epsilon} \frac{f(z)\mathrm{d}z}{z-w} = \operatorname*{res}_{z=w} C_0(w,z) f(z) \tag{6.4.6}$$

Here $f(z)$ is any locally holomorphic function; if we restrict to meromorphic functions on $\mathbb{C}P^1$ (i.e. rational functions $f(z)$) then we can write (using that the sum of all residues is zero)

$$f(w) = -\operatorname*{res}_{z\in(f)_\infty} C_0(w,z) f(z) \tag{6.4.7}$$

where $(f)_\infty$ denotes the divisor of poles of f.

We seek to generalize the above kernel to a Riemann surface $\mathcal{M}$ of genus g, namely a kernel (i.e. a function/differential of two points) $C_g(P,Q)$ such that

- $C_g(P,Q;\infty)$ is a **single–valued** meromorphic function of P and a meromorphic differential in Q.
- $C_g(P,Q;\infty)$ as a differential in Q has a simple pole at $Q = P$ with residue $+1$ and a simple pole at $Q = \infty$ (here ∞ means some point on $\mathcal{M}$) and no other poles.
- $C_g(P,Q;\infty)$ as a function of P has a zero at $P = \infty$ and a pole at $P = Q$ and (necessarily) other poles.

The last "necessarily" is due to the fact that a meromorphic function on a R.S. of genus > 0 cannot have only one simple pole.

The advocated properties imply that $C_g(P,Q;\infty) = \omega_{P,\infty}(Q) + \sum_{j=1}^{g} c_j(P)\omega_j(Q)$, where $\omega_{P,\infty}$ is the normalized third kind differential.

We have

$$\omega_{P+a_j,\infty}(Q) = \omega_{P,\infty}(Q) \tag{6.4.8}$$

$$\omega_{P+b_j,\infty}(Q) = \omega_{P,\infty}(Q) + 2i\pi\omega_j(Q) \tag{6.4.9}$$

and therefore the normalized third kind differential alone cannot work.

Let us fix a **nonspecial** divisor $\mathcal{D}_g$ which –for simplicity only in writing some determinant formula– we assume made of distinct points $P_1, \ldots, P_g$.

Proposition 6.4.2 ([2], pag 24) *A Cauchy kernel in the sense above, subordinated to the choice of divisor $\mathcal{D}_g$ is*

$$
C_{g,\mathcal{D}_g}(P,Q;\infty) = \frac{\det \left[\begin{array}{ccccc} \omega_{P\infty}(Q) & \omega_{P\infty}(P_1) & \omega_{P\infty}(P_2) & \dots & \omega_{P\infty}(P_g) \\ \omega_1(Q) & \omega_1(P_1) & \omega_1(P_2) & \dots & \omega_1(P_g) \\ \omega_2(Q) & \omega_2(P_1) & \omega_2(P_2) & \dots & \omega_2(P_g) \\ \vdots & & & & \vdots \\ \omega_1(Q) & \omega_g(P_1) & \omega_g(P_2) & \dots & \omega_g(P_g) \end{array}\right]}{\det \left[\begin{array}{cccc} \omega_1(P_1) & \omega_1(P_2) & \dots & \omega_1(P_g) \\ \omega_2(P_1) & \omega_2(P_2) & \dots & \omega_2(P_g) \\ \vdots & & & \vdots \\ \omega_g(P_1) & \omega_g(P_2) & \dots & \omega_g(P_g) \end{array}\right]} \tag{6.4.10}
$$

or –equivalently–, with $\mathbf{e} = \mathfrak{u}(\mathcal{D}_g) + \mathcal{K}$,

$$
C_{g,\mathcal{D}_g}(P,Q;\infty) = \frac{\Theta(P-Q+\infty-\mathbf{e})\Theta(Q-\mathbf{e})E(P,\infty)}{\Theta(P-\mathbf{e})\Theta(\infty-\mathbf{e})E(P,Q)E(Q,\infty)} \tag{6.4.11}
$$

and it has the following divisor properties (which determine it uniquely)

- *As a function of P*

$$
(C_{g,\mathcal{D}_g}(P,Q;\infty))_P \geq \infty - \mathcal{D}_g - Q \tag{6.4.12}
$$

- *As a differential of Q*

$$
(C_{g,\mathcal{D}_g}(P,Q;\infty))_Q \geq -\infty + \mathcal{D}_g - P \tag{6.4.13}
$$

- *It is normalized by*

$$
\operatorname*{res}_{Q=P} C_{g,\mathcal{D}_g}(P,Q;\infty) = 1 \ . \tag{6.4.14}
$$

Proof. The determinantal expression has all the properties advocated in the bulleted list; note that the expression is **independent** of the local coordinates used in the evaluation of the differentials at the points of $\mathcal{D}_g$.

The uniqueness of a bidifferential with the properties (6.4.12, 6.4.13, 6.4.14) is proven considering that, since $i(\mathcal{D}_g) = 0$ then so must be, for generic Q for $i(\mathcal{D}_g + Q - \infty)$ and hence $r(\mathcal{D}_g + \infty - Q) = 1$ (and the generator cannot be a constant function);

Similarly there is a unique differential in $\mathfrak{I}(\mathcal{D}_g - P - \infty)$; this is so because $i(-P-\infty) = g+1$ and it is spanned by $\omega_1, \dots, \omega_g, \omega_{P,\infty}$. Adding the vanishing conditions at the nonspecial divisor $\mathcal{D}_g$ reduces the dimension by g.

Finally the normalization (6.4.14) fixes the multiplicative constant.

Similarly one has to check that the second expression (6.4.11) is a single–valued differential in Q, a single valued function in P and has the properties (6.4.12, 6.4.11, 6.4.14).

Note that $\Theta(\infty + \mathbf{e}) = \Theta(\infty - \mathcal{D}_g - \mathcal{K}) \neq 0$ because then it would be linearly equivalent to some divisor $\mathcal{D}'_{g-1}$ of degree $g-1$ and hence we should have

$$r(-\mathcal{D}_g + \infty) = i(\mathcal{D}_g - \infty) \geq 1 \ . \tag{6.4.15}$$

On the contrary the divisor $\mathcal{D}_g - \infty$ of degree $g-1$ has $i(\mathcal{D}_g - \infty) = 0$. Indeed

$$\mathfrak{I}(\mathcal{D}_g - \infty) = \mathfrak{I}(\mathcal{D}_g) = \{0\} \tag{6.4.16}$$

since a differential cannot have only one simple pole (the residue would necessarily be zero there!). **Q.E.D.**

We note that there are some "twists" that we can make; indeed it is possible to generalize the above Cauchy kernel by **twisting** the divisor $\mathcal{D}_g$ by an arbitrary (generic) divisor of degree 0, namely one can have a $\mathcal{D}_g$ of degree g but not necessarily positive. We do not pursue the matter here.

Instead we proceed to analyze the Cauchy kernel; we know already that

$$C_{g,\mathcal{D}}(P,Q;\infty) = \omega_{P\infty}(Q) + \sum_{j=1}^{g} c_j(P)\omega_j(Q) \ , \tag{6.4.17}$$

and we have expressed the coefficients $c_j(P)$ already from eq. (6.4.10) in terms of determinants. We seek an alternative description in terms of Theta functions directly. To this end we have

Proposition 6.4.3 ([2] Prop. 2.10) *The Cauchy kernel $C_{g,\mathcal{D}}(P,Q;\infty)$ is given ($\mathbf{e} = \mathfrak{u}(\mathcal{D}) + \mathcal{K}$)*

$$C_{g,\mathcal{D}}(P,Q;\infty) = \omega_{P\infty}(Q) + \sum_{j=1}^{g}\Big[\partial_j \ln\Theta(P - \mathbf{e}) - \partial_j \ln\Theta(\infty - \mathbf{e})\Big]\omega_j(Q) \tag{6.4.18}$$

Proof. The holomorphic differential in eq. 6.4.18 is (we consider P as a parameter)

$$\omega_{\mathcal{D}}(Q) := \sum_{j=1}^{g}\Big(\partial_j \ln\Theta(P - \mathcal{D} - \mathcal{K}) - \partial_j \ln\Theta(\infty - \mathcal{D} - \mathcal{K})\Big)\omega_j(Q) \tag{6.4.19}$$

The normalized third kind differential can be written also

$$\omega_{P\infty}(Q) = \mathrm{d}_Q \ln\frac{\Theta(Q - P + \mathbf{f})}{\Theta(Q - \infty + \mathbf{f})} = \sum_{j=1}^{g}\Big(\partial_j \ln\Theta(Q - P + \mathbf{f}) - \partial_j \ln\Theta(Q - \infty + \mathbf{f})\Big)\omega_j(Q) \tag{6.4.20}$$

for any $\mathbf{f} \in (\Theta)$ such that $\nabla\Theta(\mathbf{f}) \neq 0$ (usually we use $\mathbf{f} = -\Delta$); since $\mathcal{D}$ is nonspecial, we can choose generically enough so that removing **any** point $R \in \mathcal{D}$ yields a **nonspecial** divisor $\widehat{\mathcal{D}}_{(R)}$ (the subscript meaning the removed point). Divisors $\mathcal{D}$ with this property form still an open dense set amongst all divisors of degree g.

Choosing then $\mathbf{f} = \mathfrak{u}(\widehat{\mathcal{D}}_{(R)}) + \mathcal{K})$ one sees that $\omega_{P\infty}(R) = -\omega_{\mathcal{D}}(R)$ and this proves that the RHS of eq. 6.4.18 vanishes as a differential of Q precisely at the points of $\mathcal{D}$ (which was the most difficult part); from a density argument it follows that this is valid for any nonspecial divisor $\mathcal{D}$.

Clearly the LHS of eq. 6.4.18 has poles (as differential in Q) at P, ∞ with residues ± 1.

We analyze now its dependence on P; from eq. 6.4.20 and the periodicity properties of Θ (5.1.2) it follows that it is a single–valued meromorphic function of P with poles at $\mathcal{D}$; obviously from (6.4.18 it has also a zero at $P = \infty$ and $P = Q$. This proves that eq.6.4.18 satisfies all the properties (6.4.12, 6.4.13, 6.4.14) and hence it is equal to the Cauchy kernel. (The proof in [2] is much more involved) **Q.E.D.**

If we rewrite the above theorem using eq. 6.4.11 we find the equivalent description

$$\frac{\Theta(P-Q+\infty-\mathbf{e})\Theta(Q-\mathbf{e})E(P,\infty)}{\Theta(P-\mathbf{e})\Theta(\infty-\mathbf{e})E(P,Q)E(Q,\infty)} = \omega_{P\infty}(Q) + \sum_{j=1}^{g}\Big[\partial_j \ln\Theta(P-\mathbf{e}) - \partial_j\ln\Theta(\infty-\mathbf{e})\Big]\omega_j(Q) \quad (6.4.21)$$

Corollary 6.4.1 *The following formula is valid*

$$C_{g,\mathcal{D}}(P,Q;\infty)C_{g,\mathcal{D}}(Q,P;\infty) = \Omega(P,Q) + \frac{1}{2i\pi}\sum_{j,k=1}^{g}\frac{\partial^2\ln\Theta}{\partial z_j\partial z_k}(\mathbf{e}-\infty)\omega_j(Q)\omega_k(P) \quad (6.4.22)$$

Proof. Let us replace $\mathbf{e} \to \mathbf{e} + \mathfrak{u}(\infty)$ in eq. 6.4.21:

$$\frac{\Theta(P-Q-\mathbf{e})\Theta(Q-\mathbf{e})E(P,\infty)}{\Theta(P-\mathbf{e})\Theta(\mathbf{e})E(P,Q)E(Q,\infty)} = \omega_{P\infty}(Q) + \sum_{j=1}^{g}\Big[\partial_j\ln\Theta(P-\infty-\mathbf{e}) - \partial_j\ln\Theta(-\mathbf{e})\Big]\omega_j(Q) \quad (6.4.23)$$

Both sides of eq. 6.4.23 vanish to the first order when $P = \infty$; considering them as functions of ∞ and bringing $\infty \to P$ we get the first order in the Taylor expansion

$$-\frac{\Theta(Q-P-\mathbf{e})\Theta(P-Q-\mathbf{e})}{\Theta(\mathbf{e})^2E^2(P,Q)} = \Omega(P,Q) + \sum_{j,k=1}^{g}\partial_j\partial_k\ln\Theta(\mathbf{e})\omega_j(Q)\omega_k(P)\ . \quad (6.4.24)$$

On the other hand one has

$$C_{g,\mathcal{D}}(P,Q;\infty)C_{g,\mathcal{D}}(Q,P;\infty) = -\frac{\Theta(Q-P+\infty-\mathbf{e})\Theta(P-Q+\infty-\mathbf{e})}{\Theta(\mathbf{e}-\infty)^2E^2(P,Q)} \quad (6.4.25)$$

and hence the proof follows from 6.4.24 by substituting $\mathbf{e} \to \mathbf{e} - \mathfrak{u}(\infty)$. **Q.E.D.**

Let $\mathbf{e} = \mathfrak{u}(\mathcal{D}) + \mathcal{K}$ with $\mathcal{D} \in \mathcal{M}_g$ nonspecial: the differential

$$\eta(P) := \mathrm{d}_P\ln\Theta(P-\mathbf{e}) = \sum_{j=1}^{g}\partial_j\ln\Theta(P-\mathbf{e})\omega_j(P) \quad (6.4.26)$$

has simple poles at $\mathcal{D}$ It is **not single–valued** in that analytic continuation yields

$$\eta(P+a_j) = \eta(P)\ ,\quad \eta(P+b_j) = \eta(P) - 2i\pi\omega_j(P) \quad (6.4.27)$$

Chapter 7

Hyperelliptic surfaces, Thomæ formula

We have already defined hyperelliptic surfaces as the algebraic surface defined by an equation of the form

$$w^2 = P(x) \tag{7.0.1}$$

where $P(x)$ is any polynomial with distinct simple roots. We now seek a more intrinsic description of these surfaces, their Weierstrass points and we want to give an accoung of the classical Thomæ formulæ.

7.1 Intrinsic definition of hyperelliptic surfaces

Definition 7.1.1 *A surface of genus $g \geq 2$ is called hyperelliptic if there is a positive divisor $\mathcal{D}$ of degree 2 such that*

$$r(-\mathcal{D}) = 2\ . \tag{7.1.1}$$

In other words $\mathcal{M}$ is hyperelliptic if there is a meromorphic function x with only two poles (or one double pole). In either cases $x : \mathcal{M} \to \mathbb{C}P^1$ provides a double cover of the Riemann–sphere.

Lemma 7.1.1 *The map x has $2g+2$ branchpoints and they are all simple. Moreover the value of x at those points is distinct.*

Proof. Near a branchpoint there can be at most 2 preimages of the equation $x(p) = x$ and hence all branchpoints must be simple.

The degree of $\mathrm{d}x$ is $2g-2$ as for any differential. If x has two distinct poles ∞_+, ∞_- then $1/x$ provides a local coordinate near these points. Moroeover $\mathrm{d}x$ has two double poles at those points, and hence must have $2g+2$ zeroes.

If x has only one double pole ∞ then $1/\sqrt{x}$ is the local coordinate and hence (as a map to $\mathbb{C}P^1$ ∞ is a branchpoint. Moreover $\mathrm{d}x$ has a triple pole at ∞ and hence $2g+1$ other zeroes. Since ∞ is a simple branchpoint, the total degree of the branchlocus is still $2g+2$.

Finally if $P_1 \neq P_2$ are branchpoints of x and $x(P_1) = x(P_2)$ then x would have degree 4 or more, a contradiction. **Q.E.D.**

Lemma 7.1.2 *All branchpoints are Weierstrass points.*

Proof. If $\mathcal{D} = 2\infty$ then clearly ∞ is a Weierstrass point since $2 = r(-2\infty) \leq r(-g\infty)$ so that $g\infty$ is a special divisor.

If P_j is a branchpoint of x then $\frac{1}{x-x(P_j)}$ has a double pole at P_j and no other poles; hence P_j is a Weierstrass point by the same argument as above. **Q.E.D.**

In fact more is true (we don't prove it for brevity)

Proposition 7.1.1 *If $\mathcal{M}$ is hyperelliptic then the only Weierstrass points are the branchpoints of the map x and are $2g+2$. Moreover hyperelliptic surfaces are the only ones that have precisely $2g+2$ Weierstrass points, which is the minimum number possible.*

How unique is the map x? The next proposition answers the question

Proposition 7.1.2 *If $\mathcal{M}$ is hyperelliptic and $x, z : \mathcal{M} \to \mathbb{C}$ are two functions of degree 2 then they are related by a linear fractional transformation*

$$x = \frac{az+b}{cz+d} \ , \quad ad - bc \neq 0 \ . \tag{7.1.2}$$

Proof. Let $\mathfrak{X}, \mathfrak{Z}$ be the divisor of poles of x, z respectively (of degree 2). Let $\widetilde{P}$ be a Weierstrass point (and branchpoint of z and x); then $\mathfrak{X} \sim 2\widetilde{P} \sim \mathfrak{Z}$ (using the functions $\frac{1}{x-x(\widetilde{P})}, \frac{1}{z-z(\widetilde{P})}$). Therefore

$$\frac{z - z(\widetilde{P})}{x - x(\widetilde{P})} \tag{7.1.3}$$

has no pole at $\widetilde{P}$ (because $x - x(\widetilde{P})$ has a double zero as well as $z - z(\widetilde{P})$) and a pole divisor exceeding $-\mathfrak{Z}$; hence

$$\frac{z - z(\widetilde{P})}{x - x(\widetilde{P})} = cx + d \ \ \Rightarrow x - x(\widetilde{P}) = \frac{z - z(\widetilde{P})}{cz+d} \ \ \Rightarrow \ \ x = \frac{ax+b}{cx+d} \tag{7.1.4}$$

Q.E.D.

Consider the germ of analytic function **on the hyperelliptic curve**

$$y := \sqrt{\prod_{j=1}^{2g+2} (x - \alpha_j)} \qquad \alpha_j := x(P_j) \ . \tag{7.1.5}$$

defined in a neighborhood of a point $P \neq P_j$. We claim that it can be analytically continued to a **single valued** function on $\mathcal{M}$; indeed the only problems occur near the points P_j. However since $x - \alpha_j$ has a **double zero** at P_j, analytic continuation of the function around a loop encircling P_j yields the same germ of function.

Proposition 7.1.3 *Every surface of genus* 2 *is hyperelliptic.*

Proof. Let ω a first–kind differential, $(\omega) = P+Q$ be its divisor (of degree $2g-2 = 2$). Then $i(P+Q) = 1$ and

$$r(-P-Q) = i(P+Q) - 2 + 2 + 1 = 2\ . \tag{7.1.6}$$

This proves the assertion. **Q.E.D.**

Remark 7.1.1 *If we remove the requirement that $g \geq 2$ in Def. 7.1.1 then also curves of genus* 1 *are hyperelliptic (but usually they are called elliptic).*

The map x (or any fractional linear transformation thereof) allows us to construct an involutive map. Let $P^\star$ denote the point such that $x(P^\star) = x(P)$; since x is a double cover, we have clearly $P^{\star\star} = P$. If P_0 is not a ramification point for x then $P_0^\star \neq P_0$; locally a coordinate of both $P, P^\star$ in neighborhoods of $P_0, P_0^\star$ is $x - x(P) = x - x(P^\star)$, and also the map

$$\begin{aligned} J: \qquad & \mathcal{M} \to \mathcal{M} \\ & P \mapsto J(P) = P^\star \end{aligned}$$

is a holomorphic univalent map of these neigborhoods. If P_0 is a branchpoint, $x(P_0) = \alpha_0$ then a local coordinate is $z = \sqrt{x - \alpha_0}$ and we have $z(P^\star) = -z(P)$ (indeed $x = z^2 + \alpha_0$ takes the same value at the two points).

Thus the map $J : \mathcal{M} \to \mathcal{M}$ extends to a holomorphic involutive automorphism of $\mathcal{M}$; it fixes the $2g + 2$ branchpoints (and no other) by construction. Viceversa

Proposition 7.1.4 *The compact Riemann surface $\mathcal{M}$ of genus g is hyperelliptic iff it possesses an involutive automorphism J that fixes $2g + 2$ points.*

Proof. We need to show only the sufficiency. So let $J^2 = Id_{\mathcal{M}}$ be an involutive automorphism of $\mathcal{M}$ of genus g, with $2g + 2$ fixed points $P_1, \ldots, P_{2g+2}$.

Consider the quotient Riemann surface $\mathcal{M}/\mathbb{Z}^2$ where $\mathbb{Z}^2$ is generated by J; since it has $2g+2$ branchpoints of order 2, the Riemann–Hurwitz formula applied to the canonical projection implies that $\mathcal{M}/\mathbb{Z}^2$ has genus 0. In other words $\mathcal{M}$ carries a meromorphic function of degree 2. **Q.E.D.**

7.1.1 Canonical homology basis, special divisors, half–periods

Usually one represents the hyperelliptic surface by the equation

$$y^2 = \prod_{j=1}^{2g+2} (x - \alpha_j) \tag{7.1.7}$$

as a double cover of the x–plane, branched at the points α_j. The involutive automorphism J is the map $(x, y) \mapsto (x, -y)$.

Lemma 7.1.3 *The g differentials*

$$\eta_j := x^{j-1} \frac{\mathrm{d}x}{y} \ , \quad j = 1, \ldots, g \tag{7.1.8}$$

span $\mathcal{H}^1$, i.e. any holomorphic differential is of the form

$$\eta = P_{g-1}(x) \frac{\mathrm{d}x}{y} \ , \tag{7.1.9}$$

with P_{g-1} an arbitrary polynomial of degree $\leq g - 1$.

Exercise 7.1.1 *Prove the above lemma.*

It follows easily by direct inspection the classification of special divisors.

Proposition 7.1.5 *Let $\deg(\mathcal{D}) = g - 1$ be a positive divisor of distinct points $\mathcal{D} = \sum_{j=1}^{g-1} P_j$; then $i(\mathcal{D})$ is equal to the number of distinct pairs such that $x(P_i) = x(P_j)$ plus one.*

Proof (sketched). A holomorphic differential in $\mathfrak{I}(\mathcal{D})$ is of the form

$$\omega = P_{g-1}(x) \frac{\mathrm{d}x}{y} \ , \quad P_{g-1}(x(P_j)) = 0 \tag{7.1.10}$$

Since a holomorphic differential has $g - 1$ zeroes coming in "conjugate" pairs $P, P^\star$, the assertion follows from a simple counting of dimension of the space of polynomials of degree $g - 1$ vanishing at the given projections. **Q.E.D.**

A variation of the above is the following

Proposition 7.1.6 *Let $\mathcal{D}_\infty$ be the pole divisor of x, $\mathcal{D}_\infty = \infty_+ + \infty_-$; let $\mathcal{D}_{g+1} = \sum_{j=1}^{g+1} P_j$ be a positive divisor of degree $g + 1$. Then $i(\mathcal{D}_{g+1} - \mathcal{D}_\infty)$ is (again) the number of pairs of points in $\mathcal{D}_{g+1}$ with the same x–projection.*

Proof (sketched). Similarly to Prop. 7.1.5 we have $\mathfrak{I}(\mathcal{D}_{g+1} - \mathcal{D}_\infty) \subset \mathfrak{I}(-\mathcal{D}_\infty)$. Now from Riemann Roch (**exercise**) it follows $i(-\mathcal{D}_\infty) = r(\mathcal{D}_\infty) + g + 2 - 1 = g + 1$ and clearly this space is spanned by

$$\omega = P_g(x) \frac{\mathrm{d}x}{y} \ , \tag{7.1.11}$$

for an arbitrary polynomial of degree g. The rest of the proof is as in Prop. 7.1.5. **Q.E.D.**

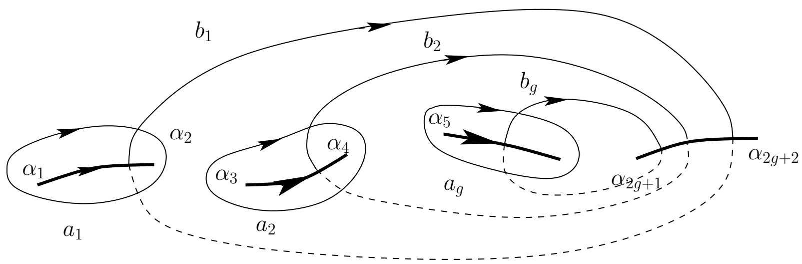

Figure 7.1: The cuts and (standard choice of) canonical basis for a hyperelliptic curve.

Proposition 7.1.7 *The divisors $\mathcal{D}_{g+1} - \mathcal{D}_\infty$ with $i(\mathcal{D}_{g+1} - \mathcal{D}_\infty)) = m \geq 1$ are linearly equivalent to a positive divisor; in particular if*

$$\mathcal{D}_{g+1} = \sum_{j=1}^{m} (P_j + P_j^\star) + \sum_{j=1}^{g+1-2m} Q_j \tag{7.1.12}$$

then $\mathcal{D}_{g+1} - \mathcal{D}_\infty$ is equivalent to any of the degree $g-1$ divisors $\mathcal{D}_{g-1}$ obtained by removing one of the pairs $P_j, P_j^\star$ from $\mathcal{D}_{g+1}$. It is also equivalent to the positive divisor

$$\sum_{j=1}^{g+1-2m} Q_j + (m-1)\mathcal{D}_\infty \ . \tag{7.1.13}$$

Proof. Use the function $1/(x - x(P_j))$ for the first assertion and the function $\prod_{j=1}^{m}(x - x(P_j))^{-1}$ for the last assertion. **Q.E.D.**

We order the branchpoints in some way (but fixed) and perform cuts as in Fig. 7.1. We choose the x–projection of a–cycles to go around the branchpoints $\alpha_{2j-1}, \alpha_{2j},\ j = 1, \ldots, g$ and the b–cycles to go around the points $\alpha_{2j}, \alpha_{2g+1},\ j = 1, \ldots, g$ as in Fig. 7.1.

Since the a–cycles are the simple loops embracing $\alpha_{2j-1}, \alpha_{2j},\ j = 1, \ldots, g$, the **normalized** holomorphic differentials are obtained by choosing an appropriate linear combination of the differentials in Lemma 7.1.3;

Lemma 7.1.4 *The normalized first kind differentials are*

$$\omega_j := \sum_{k=1}^{g} \sigma_{jk} x^k \frac{\mathrm{d}x}{y} \tag{7.1.14}$$

where

$$(\sigma^{-1})_{jk} = 2\int_{\alpha_{2k-1}}^{\alpha_{2k}} x^{j-1} \frac{\mathrm{d}x}{y} \ . \tag{7.1.15}$$

Proof. The only thing to note is that

$$\oint_{a_j} \eta_k = 2\int_{\alpha_{2j-1}}^{\alpha_{2j}} \eta_k \tag{7.1.16}$$

where the integral on the RHS has to be intended as running on the "left" of the corresponding cut as in Fig. 7.1. **Q.E.D.**

Let us denote by $R_1, \ldots, R_{2g+2}$ the Weierstrass points. We choose as **basepoint for the Abel map** $\mathfrak{u}$ one of the Weierstrass points, say R_{2g+2}; then

$$\begin{aligned}
&\mathfrak{u}(R_{2g+2}) = 0\\
&\mathfrak{u}(R_{2g+1}) = \int_{\alpha_{2g+2}}^{\alpha_{2g+1}} \vec{\omega} = -\left(\int_{\alpha_1}^{\alpha_2} - \ldots - \int_{\alpha_{2g-1}}^{\alpha_{2g}}\right)\vec{\omega} = -\frac{1}{2}\sum_{j=1}^{g} e^{(j)}\\
&\ldots\\
&\mathfrak{u}(R_2) = -\left(\int_{\alpha_2}^{\alpha_{2g+1}} + \sum_{j=2}^{g}\int_{\alpha_{2j-1}}^{\alpha_{2j}}\right)\vec{\omega} = -\frac{1}{2}\left(\tau^{(1)} + \sum_{j=2}^{g} e^{(j)}\right)\\
&\mathfrak{u}(R_1) = [......]
\end{aligned} \tag{7.1.17}$$

We see that the images of R_{2j+1} are **odd half–integer** characteristics (periods) while the images of R_{2j} are **even**.

Let $\mathcal{K} = \mathcal{K}_{R_{2g+2}}$ denote the vector of Riemann constants for the chosen basepoint;

Proposition 7.1.8 *The vector of Riemann constants is a half–period.*

Proof. We know that $-2\mathcal{K}$ is the Abel map of the divisor of any Abelian differential. Take $\omega = (x-\alpha_{2g+2})^{g-1}\frac{\mathrm{d}x}{y}$; we clearly have

$$(\omega) = (2g-2)R_{2g+2} \tag{7.1.18}$$

and hence $-2\mathcal{K} = (2g-2)\mathfrak{u}(R_{2g+2}) = 0$ (we are assuming $\alpha_{2g+2} \neq \infty$; we can always achieve that all the branchpoints are distinct from $\infty \in \mathbb{C}P^1$ by performing possibly a Möbius transformation). **Q.E.D.**

It is possible to compute exactly which half–period (see [1]).

Lemma 7.1.5 *We have* $\mathfrak{u}(\sum_{j=1}^{2g+2} R_j) = 0 \in J(\mathcal{M})$.

Proof. Indeed $\sum_{j=1}^{2g+2} R_j \sim (g+1)(\infty_+ + \infty_-) \sim (2g+2)R_1$ and since $\mathfrak{u}(R_1)$ is a half period and $2g+2$ is even, the assertion follows. **Q.E.D.**

Let $J \subset \{1, 2, \ldots, 2g+2\}$ and consider the divisor $\mathcal{D}_J := \sum_{j\in J} R_j$ of degree $\sharp J$. Clearly its image is a half-period and so is $\mathcal{D}_{J^c}$ the image ot of $\mathcal{D}_{J^c}$ for the subset $J^c = \{1, \ldots, 2g+2\} \setminus J$.

Lemma 7.1.6 *The image* $\mathfrak{u}(\mathcal{D}_J)$ *is in the singular locus of the Theta divisor if and only if* $\mathfrak{u}(\mathcal{D}_{J^c})$ *is.*

Proof. Since Θ is even, the singular locus (and its stratification) is symmetric around the origin. Now

$$0 = \mathfrak{u}(\mathcal{D}_J) + \mathfrak{u}(\mathcal{D}_{J^c}) \quad \Rightarrow \quad \mathfrak{u}(\mathcal{D}_J) = -\mathfrak{u}(\mathcal{D}_{J^c})\ . \tag{7.1.19}$$

Hence the order of vanishing of Θ at $\mathfrak{u}(\mathcal{D}_J)$ is exactly the same at $\mathfrak{u}(\mathcal{D}_{J^c})$. **Q.E.D.**

Lemma 7.1.7 *The Abel map (based at any of the Weierstrass points) of the pole divisor* $\mathcal{D}_\infty$ *of* x *is zero.*

Proof. It suffices to note that $\mathfrak{u}(P) = -\mathfrak{u}(P^\star)$ and that $\mathcal{D}_\infty = \infty + \infty^\star$. **Q.E.D.**

Proposition 7.1.9 *The even/odd characteristics correspond (via the Abel map) to the* **partitions** *of the set of Weierstrass points into complementary sets of* $g+1-2m, g+1+2m$ *points,* $\mathcal{R} = \mathcal{R}_{J_m} \cup \mathcal{R}^c_{J_m}$*, with* $\mathcal{R}_{J_m} := \{R_{\ell_j}\}_{j=1,\dots,g+1-2m}$ *as follows*

1. *The* **nonsingular, even half-periods** *correspond to the image of any partition* $\mathcal{R}_{J_0} \cup \mathcal{R}^c_{J_0}$ *into* $g+1$ *Weierstrass points; there are* $\frac{1}{2}\binom{2g+2}{g+1} = \binom{2g+1}{g}$ *such half–periods.*
2. *The* **nonsingular, odd half–periods** *correspond to the image of any partition* $\mathcal{R}_{J_1} \cup \mathcal{R}^c_{J_1}$ *into* $g-1, g+3$ *points; there are* $\binom{2g+2}{g-1}$ *such half–integer characteristics.*
3. *The other even/odd periods (characteristics) correspond to the paritions with* $m \geq 2$ *according to the parity of* m*. Moreover* m *is the order of vanishing of* Θ *at those periods (and so they all are singular).*

7.2 Variational formulæ

Suppose we have an hyperelliptic surface presented as

$$y^2 = P_{2g+2}(x) = \prod_{j=1}^{2g+2} (x - \alpha_j) \tag{7.2.1}$$

with the basis in $H_1(\mathcal{M}, \mathbb{Z})$ constructed in the previous sections, the normalized first kind differentials ω_j, the period matrix τ, etc.

Suppose now we move one of the branch–points α_j; if the motion is "small" we can assume that the x–image of the a, b–cycles is kept fixed in the process. Of course the first–kind differentials and the period matrix undergo a change; we set out to derive some formulæ for an infinitesimal change of this type. Note that this can be done in greater generality for any Riemann–surface, but it would require an excursus in the definition of Beltrami differentials, which is not possible in our timeframe.

Suppose we choose α_{j_0} and consider the one–parameter family of hyperelliptic curves

$$y^2 = P_t(x) = \prod_{j=1}^{2g+2} (x - \alpha_j + t\delta_{j,j_0}) \tag{7.2.2}$$

We use the subscript t to denote the objects corresponding to the member of the family and a dot to denote derivative w.r.t. t; the basis of first kind differentials is thus $\vec{\omega}_t$, etc.

We have

Lemma 7.2.1 *If $\vec{\omega}$ is the basis of first–kind differential then $\dot{\vec{\omega}}_0 \in \Im(-2R_{j_0})$, i.e. has at most a double pole at the ramification point R_{j_0}. Moreover*

$$\oint_{a_j} \dot{\vec{\omega}}_0 = 0 \ , \quad \forall j = 1, \ldots, g. \tag{7.2.3}$$

hence $\dot{\vec{\omega}}$ are **normalized** *second–kind differentials. In particular they are all proportional to*

$$\eta_{j_0}(P) := \Omega_0(P, R_{j_0}) \tag{7.2.4}$$

Proof. Near R_{j_0} the local parameter is $z = \sqrt{x - \alpha_{j_0} - t}$; since a holomorphic differential is of the form $\omega_t = P_t(x)\mathrm{d}x/y_t$ with P_t a family of polynomials of degree $\leq g - 1$, we have

$$\dot{\omega}_0 = \frac{\dot{P}_0(x) + \frac{P_0(x)}{2(x-\alpha_{j_0})}}{y_0}\mathrm{d}x \tag{7.2.5}$$

hence the first assertion. As for the other, since $\oint_{a_j} \omega_t$ is independent of t, differentiation shows that the derivative is a normalized second kind differential; then the normalized second kind differential is unique and clearly given by the proposed expression (where the evaluation of Ω_0 at R_{j_0} is done w.r.t. any local parameter). **Q.E.D.**

To nail down the proportionality constant we look carefully in a neighborhood of α_{j_0}; from eq. 7.2.5 we see that the **function**

$$F = \frac{\dot{\omega}_0}{\omega_0} \sim \frac{1}{2(x - \alpha_{j_0})} \tag{7.2.6}$$

Now the bidifferential Ω for $P, Q \sim R_{j_0}$ behaves like ($z = \sqrt{x - \alpha_{j_0}}$)

$$\Omega(P, Q) \sim \frac{1}{(z - z')^2}\mathrm{d}z\mathrm{d}z' \tag{7.2.7}$$

and hence (hereafter the evaluation is at $t = 0$)

$$\dot{\omega}_j = \left.\frac{\Omega(P, Q)\omega_j(Q)}{\mathrm{d}z(Q)^2}\right|_{Q=R_{j_0}} = \lim_{Q \to R_{j_0}} \frac{\Omega(P, Q)\omega_j(Q)2z}{\mathrm{d}z\mathrm{d}x} = \mathop{\mathrm{res}}_{Q=R_{j_0}} \frac{\Omega(P, Q)\omega_j(Q)}{\mathrm{d}x(Q)} \tag{7.2.8}$$

Note that $\mathrm{d}x = 2z\mathrm{d}z$ and in the last equality (and in others to follow) we have a used that the expression in the numerator is a **quadratic** differential at the point Q, namely something that transforms as $\mathrm{d}z^2$, while the denominator is an Abelian differential. The ratio transforms as $\mathrm{d}z$ and hence is an Abelian differential, of which we can evaluate residues.

We have just proved our first (and basic)

Proposition 7.2.1 *The following variational formula holds*

$$\partial_{\alpha_j}\vec{\omega}(P) = \mathop{\mathrm{res}}_{Q=R_j} \frac{\Omega(P, Q)\vec{\omega}(Q)}{\mathrm{d}x(Q)} \ . \tag{7.2.9}$$

Remark 7.2.1 *Note that we did not really use the fact that the curve is hyperelliptic.*

As a corollary we can compute the variation of the period matrix τ;

Corollary 7.2.1 (Rauch variational formula) *Under an infinitesimal deformation of a branchpoint we have*

$$\partial_j \tau_{k\ell} = \operatorname*{res}_{Q=R_j} \frac{\oint_{P\in b_\ell} \Omega(P,Q)\omega_k(Q)}{\mathrm{d}x(Q)} = \pi i \operatorname*{res}_{Q=R_j} \frac{\omega_k(Q)\omega_\ell(Q)}{\mathrm{d}x(Q)} \ . \tag{7.2.10}$$

7.3 Thomæ formula

We now prove the celebrated Thomæ formula, which expresses the Theta constants (i.e. the value of $\Theta[\mathbf{e}]$ at some characteristics) in terms of the branchpoints.

Proposition 7.3.1 *For $\mathcal{M}$ hyperelliptic with brachpoints $\{\alpha_j\}_{j=1,\dots,2g+2}$ we have the following formula due to Thomæ*

$$\Theta[\mathbf{e}]^8(0) = \frac{1}{(\det\sigma)^4} \prod_{\substack{k,\ell=1\\k<\ell}}^{g+1} \left(\alpha_{i_\ell} - \alpha_{i_k}\right)^2 \left(\alpha_{j_\ell} - \alpha_{j_k}\right)^2 \tag{7.3.1}$$

where $\mathbf{e}$ *is a* **nonsingular even half–characteristics** *and* $\mathcal{R} = \{\alpha_{i_\ell}\}_{\ell=1,\dots,g+1} \cup \{\alpha_{j_k}\}_{k=1,\dots,g+1}$ *is the partition corresponding to this characteristics. The matrix σ is the matrix entering the form of the first kind differentials as in Lemma 7.1.4.*

Proof (almost complete). Let us denote by $\mathcal{M}_\alpha$ the hyperelliptic curve with branchpoints $\alpha_1,\dots,\alpha_{2g+2}$ (i.e. a $2g+2$ dimensional family of hyperelliptic curves). Let $\mathbf{e} \in \mathbb{C}^g$ be a fixed vector outside of the Theta divisor; we now consider variational formulæ w.r.t. α_1 but we could replace α_1 by any of the branchpoints with similar considerations.

$$\partial_{\alpha_1} \ln\Theta[\mathbf{e}](0) = \sum_{k\le\ell} \frac{\partial\Theta[\mathbf{e}](0)}{\partial\tau_{k\ell}} \partial_{\alpha_1}\tau_{k\ell} \tag{7.3.2}$$

Using the heat equation (5.1.3) and the variational formula of Cor. 7.2.1 we have

$$\partial_{\alpha_1} \ln\Theta[\mathbf{e}](0) = \frac{1}{2i\pi} \sum_{k\le\ell} \frac{\partial^2\Theta[\mathbf{e}](0)}{\partial z_k \partial z_\ell} \operatorname*{res}_{Q=R_1} \frac{\omega_j\omega_\ell}{\mathrm{d}x} \tag{7.3.3}$$

Consider the degree g divisor of poles $\mathcal{D}_\alpha = \sum_{s=1}^g T_s$ of the expression

$$F_k(P) := \partial_{z_k} \ln\Theta[\mathbf{e}](\mathfrak{u}(P-R_1)) \tag{7.3.4}$$

which has simple poles at $\mathcal{D} := \sum_{s=1}^g T_s$ (and $R_1 \notin \mathcal{D}$); note that

$$F_k(P + a_s) = F_k(P) \tag{7.3.5}$$

$$F_k(P + b_s) = F_k(P) - 2i\pi\delta_{sk} \ . \tag{7.3.6}$$

Hence the logarithmic differential $\mathrm{d}_P F_k$ gives a differential with **double** poles at $\mathcal{D}$, **residueless** and **normalized** (**exercise!**) which therefore can be written as

$$\mathrm{d}F_k = \sum_{\ell=1}^{g} \partial_{z_\ell}\partial_{z_k} \ln\Theta[\mathbf{e}](\mathfrak{u}(P-R_1))\omega_\ell(P) = \sum_{s=1}^{g} \Omega(P,T_s)C_{s,k} \tag{7.3.7}$$

for some matrix $C_{s,k}$ that we now compute using the reciprocity theorem 3.4.2. Indeed

$$-2i\pi\delta_{k\ell} = \oint_{b_\ell} \mathrm{d}F_k = \sum_{s=1}^{g} \oint_{b_\ell} \Omega(P,T_s)C_{s,k} = 2i\pi\omega(T_s)C_{s,k}\ . \tag{7.3.8}$$

Therefore $C_{s,k} = -[\omega_s(T_k)]^{-1}$ (meaning the matrix inverse, which exists since the divisor $\mathcal{D}$ is nonspecial). Using this in eq. 7.3.3 we can rewrite it as

$$\begin{aligned}\partial_{\alpha_1} \ln\Theta[\mathbf{e}](0) &= \frac{1}{2i\pi}\sum_{k\leq\ell} \frac{\partial^2\Theta[\mathbf{e}](0)}{\partial z_k\partial z_\ell} \mathop{\mathrm{res}}_{Q=R_1} \frac{\omega_j\omega_\ell}{\mathrm{d}x} = \\ &= \frac{1}{2i\pi}\sum_{s,\ell} \left[\omega_s(T_\ell)\right]^{-1} \mathop{\mathrm{res}}_{P=R_1} \frac{\Omega(P,T_s)\omega_\ell(P)}{\mathrm{d}x(P)}\end{aligned} \tag{7.3.9}$$

Note that eq. 7.3.9 is invariant under change of local parameters at the points T_s and that, by the variational formula in Prop. 7.2.1, the residue expression above is

$$\mathop{\mathrm{res}}_{P=R_1} \frac{\Omega(P,T_s)\omega_\ell(P)}{\mathrm{d}x(P)} = 2\partial_{\alpha_1}\omega_\ell(T_s) \tag{7.3.10}$$

namely the variation of the matrix $\omega_\ell(T_s)$ w.r.t. α_1. Hence

$$\partial_{\alpha_1} \ln\Theta[\mathbf{e}](0) = -\frac{1}{2}\partial_{\alpha_1} \ln\det[\omega_\ell(T_s)]_{s,\ell} \tag{7.3.11}$$

Now

$$\det[\omega_\ell(T_s)]_{s,\ell} = \det\sigma \frac{\det[x_s^{j-1}]_{s,j\leq g}}{y(T_s)} = \det\sigma \frac{\prod_{r<s}(x_j-x_s)}{\prod_{s=1}^{g} y(T_s)}\ ,\quad x_s := x(T_s)\ , \tag{7.3.12}$$

where we have used the local coordinate $x-x_s$ in the evaluation (assuming none of the T_s is a branchpoint). Summarizing so far we have proved

$$4\partial_{\alpha_1} \ln[\mathbf{e}](0) = -2\partial_{\alpha_1} \ln\det\sigma \frac{\prod_{r<s}(x_j-x_s)}{\prod_{s=1}^{g} y(T_s)} \tag{7.3.13}$$

The problem is that the points T_s depend on α_1 if $\mathbf{e}$ is chosen arbitrarily; instead we are going to specify $\mathbf{e}$ to a nonsingular half–period corresponding to the partition

$$\{R_1, R_{i_1}, \ldots, R_{i_g}\} \cup \{R_{j_1}, \ldots, R_{j_{g+1}}\} \tag{7.3.14}$$

This implies that then $T_s = R_{i_s}$ and hence their x–projection is independent of α_1; the above formula needs to be computed in the local coordinates $\sqrt{x-\alpha_{i_s}}$ and gives

$$8\partial_{\alpha_1}\ln[\mathbf{e}](0) = -\partial_{\alpha_1}\ln(\det\sigma)^4 \frac{\prod_{r<s}(\alpha_{i_r}-\alpha_{i_s})^4}{\prod_{r=1}^{g}\prod_{\substack{k=1\\k\neq i_r}}^{2g+2}(\alpha_{i_r}-\alpha_k)^2} = -\partial_{\alpha_1}\ln(\det\sigma)^4 \frac{1}{\prod_{r=1}^{g}(\alpha_{i_r}-\alpha_1)^2\prod_{r=1}^{g}\prod_{s=1}^{g+1}(\alpha_{i_r}-\alpha_{j_s})^2} =$$

$$= -\partial_{\alpha_1}\ln(\det\sigma)^4\prod_{r=1}^{g}(\alpha_{i_r}-\alpha_1)^{-2} \tag{7.3.15}$$

where we have dropped the terms that do not depend on α_1 (implicitly and explicitly). Letting $\alpha_{i_{g+1}} := \alpha_1$ we rewrite more symmetrically

$$8\partial_{\alpha_1}\ln[\mathbf{e}](0) = -\partial_{\alpha_{i_{g+1}}}\ln(\det\sigma)^4\prod_{r=1}^{g}(\alpha_{i_r}-\alpha_{i_{g+1}})^{-2} \tag{7.3.16}$$

Clearly we could have chosen any of the α_{i_ℓ}'s obtaining a symilar equation so that

$$8\partial_{\alpha_{i_\ell}}\ln[\mathbf{e}](0) = -\partial_{\alpha_{i_\ell}}\ln(\det\sigma)^4\prod_{r<s}(\alpha_{i_r}-\alpha_{i_s})^{-2}\ . \tag{7.3.17}$$

Note now that

$$\mathbf{e} = \sum_{\ell=1}^{g+1}\mathfrak{u}(R_{i_\ell}) + \mathcal{K} = -\sum_{r=1}^{g+1}\mathfrak{u}(R_{j_r}) + \mathcal{K} = \sum_{r=1}^{g+1}\mathfrak{u}(R_{j_r}) + \mathcal{K}\ , \tag{7.3.18}$$

where we have used that $\sum_{j=1}^{2g+2}\mathfrak{u}(R_j)\equiv 0$ and that all the terms are half–periods. Thus, what has been said for the α_{i_ℓ}'s can be repeated verbatim for the α_{j_r}'s obtaining thus the symmetric formula

$$8\partial_{\alpha_k}\ln[\mathbf{e}](0) = -\partial_{\alpha_k}\ln(\det\sigma)^4\prod_{r<s}^{g}(\alpha_{i_r}-\alpha_{i_s})^{-2}(\alpha_{j_r}-\alpha_{j_s})^{-2}\ ,\quad \forall k=1,\ldots,2g+2. \tag{7.3.19}$$

which means that

$$\Theta[\mathbf{e}](0)^8 = C\frac{1}{(\det\sigma)^4}\prod_{\substack{k,\ell=1\\k<\ell}}^{g+1}\left(\alpha_{i_\ell}-\alpha_{i_k}\right)^2\left(\alpha_{j_\ell}-\alpha_{j_k}\right)^2 \tag{7.3.20}$$

where the constant C is **independent of all the branchpoints** and hence is some universal constant (it *may* depend on the genus, however).

The rest of the proof would consist in computing the constant (which turns out to be 1 independently of the genus); we do not do this but refer to [2] Pag. 48 for the details. In short, the constant is computed in the limit where the branchpoints coalesce in pairs to give a rational curve of the type $y^2 = (P_{g+1}(x))^2$.
Q.E.D.

Chapter 8

Degeneration of Riemann surfaces

8.1 A good pinch

By *degeneration* we mean the situation in which the (compact) surface is "squeezed" or "pinched" along a cycle. There are two ways of pinching a RIemann–surface

(Hom0) pinching a cycle homologous to zero (i.e. a curve that separates the surfaces in two disconnected components, on the left in Fig. 8.1);

(Hom1) pinching a non homologically trivial cycle (on the right in Fig. 8.1).

In either cases the local model of the pinch is an hyperboloid with "thin waist"

$$x^2 - y^2 = \epsilon \quad \Leftrightarrow y^2 = x^2 - \epsilon \tag{8.1.1}$$

where $\epsilon \to 0$ and the **vanishing cycle** is the homology class of the loop around the cut of the x–plane (this is a local coordinate only!) from $-\sqrt{\epsilon}$ to $\sqrt{\epsilon}$. It should be clear from the pictures that the limiting situation is either

1. two Riemann surfaces whose genera **adds up** to the genus of the pinched surface, with one point in each identified with the corresponding point on the other;

2. one Riemann surface with two points identified, with genus one less than the genus of the pinched surface.

We can construct the **family** of pinched surfaces, parametrized by ϵ in a neighborhood of $\epsilon = 0$ as follows.

In case **(Hom0)** we take two surfaces $\mathcal{M}_{1,2}$ of genera $g_{1,2}$ with two points $P_{1,2} \in \mathcal{M}_{1,2}$ and local parameters $z_{1,2}$ on neighborhoods $U_{1,2}$ around them.

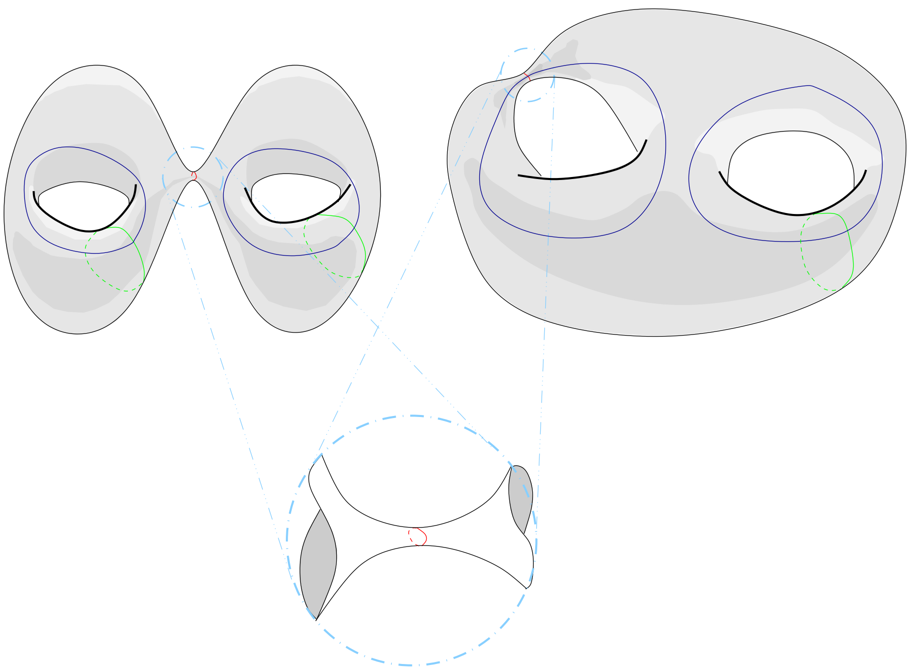

Figure 8.1: Two ways of pinching a surface of genus 2; the local model around the pinch is the same, i.e. an hyperboloid with thin waist.

We perform two identical cuts on the charts $z_{1,2}$ from $z_j = -\sqrt{\epsilon}$ to $z_j = \sqrt{\epsilon}$ and identify the two rims as usual. In other words the upper sheet of eq. 8.1.1 is identified as

$$(y_+, x) \sim z_1(P) = x \qquad (y_-, x) \sim z_2(P) = x; \tag{8.1.2}$$

In case **(Hom1)** the construction is similar but using two points $P_{1,2}$ and local parameters $z_{1,2}$ of the *same* Riemann surface $\mathcal{M}$ of genus g. We denote by $\mathcal{M}_\epsilon$ the resultin Riemann–surface and call $U_1 \cup U_2 / \sim$ the **pinching region**.

Note that, outside of the pinching region, the coordinates of $\mathcal{M}_\epsilon$ are provided by the original atlas of $\mathcal{M}$ (or $\mathcal{M}_1, \mathcal{M}_2$), namely they are "independent" of ϵ.

Definition 8.1.1 *An Abelian differential ω on $\mathcal{M}_\epsilon$ is said to depend smoothly (holomorphically) on ϵ if*

- *outside of the pinching region it is represented by $\omega = f(z, \epsilon)\mathrm{d}z$ with $f(z, \epsilon)$ depending smoothly (holomorphically) on ϵ;*
- *within the pinching region it is represented by*

$$\omega = f(x, y, \epsilon)\frac{\mathrm{d}x}{y} = f(x, y, \epsilon)\frac{\mathrm{d}x}{\sqrt{x^2 - \epsilon}} \tag{8.1.3}$$

with $f(x, y, \epsilon)$ holomorphic in x, y and smooth (holomorphic) in ϵ.

We can choose the canonical basis in the homology as follows

1. In case **(Hom0)** since the vanishing cycle separates the surface into disconnected components, we can choose symplectic bases in the two parts outside of the pinching region. We will choose them so that in the limit $\epsilon = 0$ we have $a_\ell^{(j)}, b_\ell^{(j)},\ \ell = 1, \dots, g_j,\ j = 1, 2$ spanning the respective homology groups (refer to Fig. 8.1).

2. In case **(Hom1)** there are two cycles that necessarily intersect the pinching region; one of them (which we choose as the a_{g+1} cycle and the cycle that traverses it will be the b_{g+1} cycle. The remaining $2g$ cycles can be chosen so as not to intersect the pinching region.

Let ω_ϵ be a holomorphic differential on $\mathcal{M}_\epsilon$ depending holomorphically on ϵ. Then, in the pinching regions we have

$$\omega_\epsilon = f(x, y, \epsilon)\frac{\mathrm{d}x}{y} = A(x;\epsilon)\mathrm{d}x + B(x;\epsilon)\frac{\mathrm{d}x}{y} = \sum_{\nu=0}^{\infty} A_\nu(\epsilon)x^\nu \mathrm{d}x + \sum_{\nu=0}^{\infty} B_\nu(\epsilon)x^\nu \frac{\mathrm{d}x}{\sqrt{x^2-\epsilon}} \ . \tag{8.1.4}$$

In the case **(Hom0)** we must have

$$\oint_\gamma \omega_\epsilon \equiv 0 \ , \tag{8.1.5}$$

where γ is a loop encircling the cut. We have

$$\oint_\gamma \omega_\epsilon = \sum_{\nu=0}^{\infty} \oint_\gamma B_\nu(0)x^\nu \frac{\mathrm{d}x}{\sqrt{x^2-\epsilon}} = \sum_{k=0}^{\infty} B_{2k}(\epsilon)\oint_\gamma \frac{x^{2k}\mathrm{d}x}{\sqrt{x^2-\epsilon}} = \sum_{k=0}^{\infty}(-)^k \binom{-1/2}{k} B_{2k}(\epsilon)\epsilon^k \tag{8.1.6}$$

and hence $B_0(0) = 0 = B_0'(0) + \frac{1}{2}B_2(0)$.

In both cases we have, expanding the square-root as a Taylor series in ϵ[1]

$$\begin{aligned}
\omega_\epsilon \quad &= \pm \sum_{\nu=0}^{\infty} B_\nu(\epsilon) \sum_{k=0}^{\infty} (-\epsilon)^k x^{\nu-2k-1} \binom{-1/2}{k} + \sum_\nu A_\nu(\epsilon)x^\nu \mathrm{d}x = \\
&= \underbrace{\left(\frac{B_0(0)}{x} + \sum_\nu \big(A_\nu(0) \pm B_{\nu+1}(0)\big)x^\nu\right)\mathrm{d}x}_{=\omega_0} + \\
&\pm\epsilon\left(\frac{B_0(0)}{2x^3} + \frac{B_1(0)}{x^2} + \frac{B_0'(0)+1/2B_2(0)}{x} + \mathcal{O}(1)\right)\mathrm{d}x + \mathcal{O}(\epsilon^2) \ .
\end{aligned} \tag{8.1.7}$$

where the $\pm$ depends on which sheet of $y = \sqrt{x^2-\epsilon}$ we evaluate (i.e. whether we are in a neighborhood of P_1 on $\mathcal{M}_1$ or of P_2 in $\mathcal{M}_2$).

[1] The expansion is valid for our purposes in all compact regions not containing $x = 0$; the details of the functional analysis are left to your good sense.

8.1.1 Case of a homologically trivial vanishing cycle

We take a basis $\omega_\ell^{(j)}$, $\ell = 1, \dots, g_j$, $j = 1, 2$ of first kind differentials normalized along the choice of contours above. We denote by $\Omega_j(P, Q)$ the fundamental bidifferentials on $\mathcal{M}_j$ and recall that P_1, P_2 are the points by which $\mathcal{M}_1, \mathcal{M}_2$ are attached in the limit.

Proposition 8.1.1 *The basis of holomorphic differentials has the expansion in* ϵ

$$\omega_\ell^{(1)}(P;\epsilon) = \left\{ \begin{array}{ll} \omega_\ell^{(1)}(P) + \frac{\epsilon}{2} \mathop{\mathrm{res}}_{Q=P_1} \frac{\Omega_1(P,Q)\omega_\ell^{(1)}(Q)}{x\mathrm{d}x(Q)} + \mathcal{O}(\epsilon^2) & P \in \mathcal{M}_1 \\ \frac{\epsilon}{2} \mathop{\mathrm{res}}_{Q=P_2} \frac{\Omega_2(P,Q)\omega_\ell^{(2)}(Q)}{x\mathrm{d}x(Q)} + \mathcal{O}(\epsilon^2) & P \in \mathcal{M}_2 \end{array} \right. \tag{8.1.8}$$

$$\omega_\ell^{(2)}(P;\epsilon) = \left\{ \begin{array}{ll} \omega_\ell^{(2)}(P) + \frac{\epsilon}{2} \mathop{\mathrm{res}}_{Q=P_2} \frac{\Omega_2(P,Q)\omega_\ell^{(2)}(Q)}{x\mathrm{d}x(Q)} + \mathcal{O}(\epsilon^2) & P \in \mathcal{M}_2 \\ \frac{\epsilon}{2} \mathop{\mathrm{res}}_{Q=P_1} \frac{\Omega_1(P,Q)\omega_\ell^{(1)}(Q)}{x\mathrm{d}x(Q)} + \mathcal{O}(\epsilon^2) & P \in \mathcal{M}_1 \end{array} \right. \tag{8.1.9}$$

Proof. Let $\omega(P, \epsilon)$ be one of the $\omega_\ell^{(j)}(P)$, let us say $\omega_\ell^{(1)}$ (the situation is completely symmetric) and let us expand it in ϵ

$$\omega(P;\epsilon) = \omega_0(P) + \epsilon\omega_1(P) + \dots \tag{8.1.10}$$

Now $\omega_0(P)$ must have $\oint_{\alpha_\ell^{(1)}} \omega_0(P) = 1$ and all other a–periods vanishing. Hence it is zero on $\mathcal{M}_2$ and it is $\omega_\ell^{(1)}$ on $\mathcal{M}_1$. On the other hand, the expansion in eq. 8.1.7 shows that near P_1 or P_2 (since $B_0(0) = 0 = B_0'(0) + 1/2B_2(0)$) the order ϵ differential has a double pole without residue. Necessarily ω_1 must have all vanishing a–periods and hence is a normalized second kind differential with a double pole at P_1, P_2 when restricted to the two disconnected components $\mathcal{M}_1, \mathcal{M}_2$.

Moreover, again eq. 8.1.7, tells us that

$$\lim_{\substack{Q\to P_1 \\ Q\in\mathcal{M}_1}} \frac{\omega_0(Q)}{\mathrm{d}x(Q)} = A_0(0) + B_1(0) = \lim_{\substack{Q\to P_1 \\ Q\in\mathcal{M}_1}} \frac{\omega_\ell^{(1)}}{\mathrm{d}x} \tag{8.1.11}$$

$$\lim_{\substack{Q\to P_2 \\ Q\in\mathcal{M}_2}} \frac{\omega_0(Q)}{\mathrm{d}x(Q)} = A_0(0) - B_1(0) = 0 \tag{8.1.12}$$

and hence

$$A_0(0) = B_1(0) = \frac{1}{2}\frac{\omega_\ell^{(1)}}{\mathrm{d}x}\bigg|_{P_1}. \tag{8.1.13}$$

The evaluation of the function (defined in the coordinate chart $x = z_1$ of P_1 only!) $\omega_\ell^{(1)}/\mathrm{d}x$ can be put in a residue form as in the statement of the Prop.. **Q.E.D.**

As a corollary one can find the ϵ expansion of the period matrix $\tau_{i,j}$; the computation is straightforward using Prop. 8.1.1 and the appropriate reciprocity theorem. It is left as **exercise**

Corollary 8.1.1 *The period matrix of $\mathcal{M}_\epsilon$ has the expansion*

$$\tau_\epsilon = \left[\begin{array}{c|c} \tau_1 & 0 \\ \hline 0 & \tau_2 \end{array}\right] + \frac{\epsilon}{2}\vec{R}\cdot\vec{R}^t + \mathcal{O}(\epsilon^2) \tag{8.1.14}$$

where the **column** *vector $\vec{R}$ of dimension $g_1 + g_2$ has components*

$$\vec{R} = \frac{1}{\mathrm{d}x}(\omega_1^{(1)}, \ldots, \omega_{g_1}^{(1)}, \omega_1^{(2)}, \ldots, \omega_{g_2}^{(2)})^t \bigg|_{x=0} \tag{8.1.15}$$

and $\tau_{1,2}$ are the period matrices for the two surfaces $\mathcal{M}_{1,2}$.

8.1.2 Case of a homologically non-trivial vanishing cycle

We take a basis $\omega_1, \ldots \omega_{g+1}$ of $\mathcal{M}_\epsilon$ (of genus $g+1$); recall that the a_{g+1} cycle is chosen as the vanishing cycle (the red one in Fig. 8.1). Much of the computation is the same as the previous case; the main difference is now that for

$$1 \equiv \oint_{a_{g+1}} \omega_\ell(\epsilon) = \lim_{\epsilon\to 0}\oint_{a_{g+1}} \omega_\ell(\epsilon) = 2i\pi B_0(0)\delta_{\ell,g+1} \ . \tag{8.1.16}$$

In other words, in the limit the ω_{g+1} first–kind differential becomes the $\frac{1}{2i\pi}$ multiple of the **normalized third kind differential** with poles at $P_1, P_2 \in \mathcal{M}_0$.

We note that we have a Torelli marked surface $\mathcal{M}_\epsilon$; however in this case the marking (i.e. the choice of homology basis) **is not "single valued"** as we perform a small loop in the ϵ–space around the origin. The reason is that the as $\arg(\epsilon) \mapsto \arg(\epsilon) + 2\pi$ the cut in the x–plane makes a π–radians turn; while this does not affect the a_{g+1} cycle, it does the b_{g+1} cycle, which gets changed to (see FIg. 8.2)

$$b_{g+1} \mapsto b_{g+1} \pm a_{g+1} \ . \tag{8.1.17}$$

Therefore the expansions that we are doing are valid only in a punctured neighborhood of $\epsilon = 0$ cut so that it is simply connected It also follows that the function

$$B(\epsilon) := \oint_{b_{g+1}} \omega_{g+1} \tag{8.1.18}$$

has the same multivaluedness of $\frac{1}{2i\pi}\ln\epsilon$ and hence

$$B(\epsilon) - \frac{1}{2i\pi}\ln(\epsilon) \tag{8.1.19}$$

is **analytic** in a punctured neighborhood of $\epsilon = 0$ (in fact we will see that it is holomorphic at $\epsilon = 0$). Considerations quite similar to the previous ones lead to

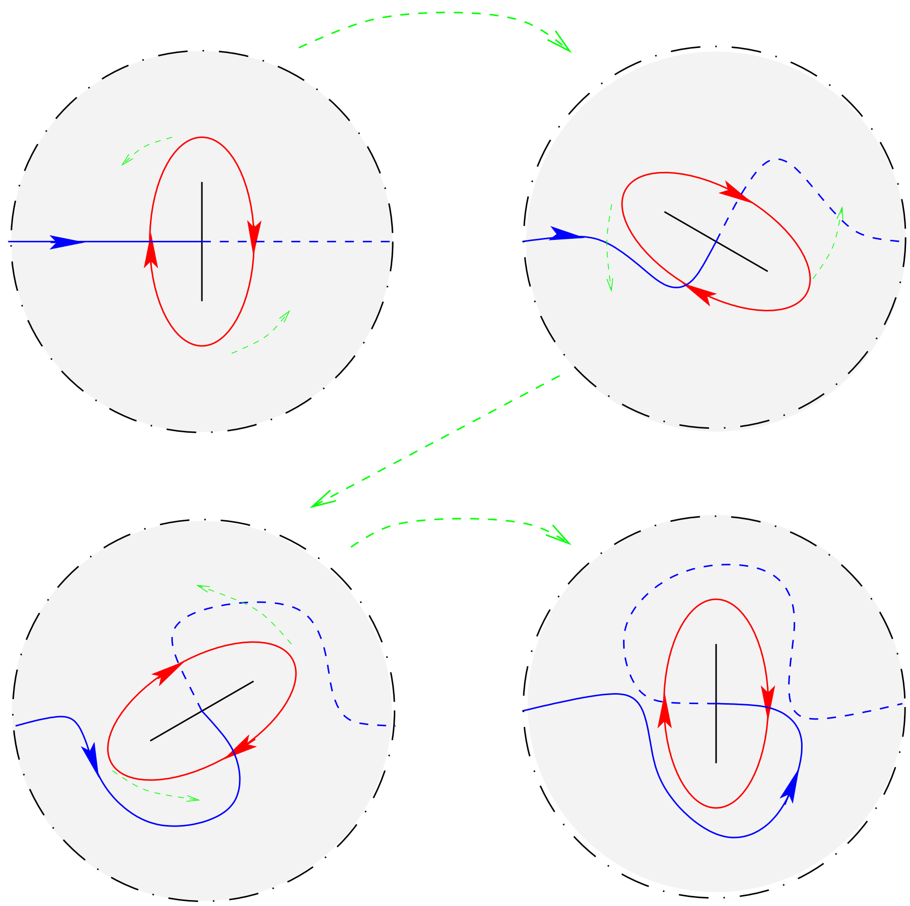

Figure 8.2: The change in b_{g+1} as ϵ goes around the origin and the branchcut extending between $\pm\sqrt{\epsilon}$ makes half a turn. The final b_{g+1} cycle (blue) has acquired the class of the red cycle (which is the a_{g+1} cycle).

Proposition 8.1.2 *The basis of holomorphic differentials has the expansion in* ϵ

$$\omega_\ell(P;\epsilon) = \omega_\ell(P) + \frac{\epsilon}{2}\left(\frac{\omega_\ell}{\mathrm{d}z_1}\bigg|_{P_1} - \frac{\omega_\ell}{\mathrm{d}z_2}\bigg|_{P_2}\right)\Omega(P,P_1) - \Omega(P,P_2) + \mathcal{O}(\epsilon^2) \tag{8.1.20}$$

$$\omega_{g+1}(P;\epsilon) = \frac{1}{2i\pi}\omega_{P_2,P_1}(P) + \epsilon\widetilde{\omega}_{g+1}(P) + \mathcal{O}(\epsilon^2)\ , \tag{8.1.21}$$

where $\widetilde{\omega}_{g+1}(P)$ *is a second–kind differential with poles of order* 3 *at* P_1, P_2 *and expansion in the pinching coordinates*

$$\widetilde{\omega}_{g+1}(P) = \left(-\frac{1}{2z_j^3} + \frac{\beta}{z_j^2} + \mathcal{O}(1)\right)\mathrm{d}z_j \tag{8.1.22}$$

and

$$2\beta = \frac{\mathrm{d}_P \ln\Theta_\Delta(\mathfrak{u}(P-P_1))}{\mathrm{d}z_2}\bigg|_{P_2} - \frac{\mathrm{d}_P \ln\Theta_\Delta(\mathfrak{u}(P-P_2))}{\mathrm{d}z_1}\bigg|_{P_1} \tag{8.1.23}$$

As before an immediate corollary is the expansion of the period matrix

Corollary 8.1.2 *The period matrix of* $\mathcal{M}_\epsilon$ *has the expansion*

$$\tau(\epsilon) = \left[\begin{array}{c|c} \tau_0 + \epsilon\sigma & \vec{A} + \epsilon\vec{B} \\ \hline \vec{A}^t + \epsilon\vec{B}^t & \ln\epsilon + c_1 + c_2\epsilon \end{array}\right] + \mathcal{O}(\epsilon^2). \tag{8.1.24}$$

where τ_0 *is the* $g \times g$ *period matrix of* $\mathcal{M}_0$*,*

$$\vec{A} = \mathfrak{u}(P_2 - P_1) \tag{8.1.25}$$

$$\vec{B} = ... \tag{8.1.26}$$

The constant c_1, c_2 *are not interesting to write (but it is possible with some work).*

Example: quasirational torus

A torus, or elliptic curve can be always written as

$$y^2 = \prod_{j=1}^{4}(x - \alpha_j) \tag{8.1.27}$$

Using a Möbius transformation we can always recast it in the form

$$y^2 = x(x-1)(x+1)(x-t) \tag{8.1.28}$$

A degeneration of the above torus into a $\mathbb{C}P^1$ with two points identified occurs when $t \to= 0, 1, -1$. For example, if $t = 0$ we have

$$y^2 = x^2(x^2 - 1) \tag{8.1.29}$$

and we can find a **rational parametrization** of y, x as follows

$$x = z + \frac{1}{4z} \tag{8.1.30}$$

$$y = z^2 - \frac{1}{16z^2} = \frac{16z^4 - 1}{16z^2}\ . \tag{8.1.31}$$

The points $z = 0, \infty$ are mapped to the two points above $x = \infty$. The two points $z = \pm\frac{i}{2}$ are mapped to the **same point** $x = 0, y = 0$; On the elliptic surface for $t \neq 0$ the unique holomorphic differential is

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}x}{\sqrt{x(x-t)(x^2-1)}} \tag{8.1.32}$$

and we choose as the a–cycle the vanishing cycle that loops around $x = 0, x = t$ so that the **normalized** holomorphic diffferential is

$$\omega = \frac{1}{2\int_0^t \left(\xi(\xi - t)(\xi^2-1)\right)^{-\frac{1}{2}} \mathrm{d}\xi} \frac{\mathrm{d}x}{y}\ . \tag{8.1.33}$$

As $t \to 0$ the normalizing factor becomes (by choosing appropriately the branches of the square root and the orientation of the loop γ around $x = 0$)

$$N = \lim_{t\to 0} \int_\gamma \frac{\mathrm{d}x}{x(x-t)(x^2-1)} = \int_\gamma \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = 2\pi \tag{8.1.34}$$

So that

$$\begin{aligned} \omega \ \longrightarrow \qquad & \frac{\mathrm{d}x}{\pi x\sqrt{x^2-1}} = \frac{(4z^2-1)16z^2\mathrm{d}z}{z^2(16z^4-1)} = \\ & = \frac{16\mathrm{d}z}{\pi(4z^2+1)} = \frac{1}{2\pi}\frac{\mathrm{d}z}{(z-i/2)(z+i/2)} = \frac{1}{2i\pi}\left(\frac{1}{z-i/2} - \frac{1}{z+i/2}\right)\mathrm{d}z\ , \end{aligned} \tag{8.1.35}$$

namely the third kind differntial with poles at $\pm i/2$. As for the b–cycle we can choose the loop embracing $x = -1, 0$ so that

$$\begin{aligned} &\frac{1}{2}\oint_b \frac{\mathrm{d}x}{\sqrt{x(x-t)(x^2-1)}} = \int_{-1}^0 \frac{1}{i\sqrt{x(x-t)}} + \int_{-1}^0 \left(\frac{1}{\sqrt{x(x-t)(x^2-1)}} - \frac{1}{i\sqrt{x(x-t)}}\right)\mathrm{d}x \sim \\ &\sim \int_{-1}^0 \frac{i - \sqrt{x^2-1}}{x\sqrt{x^2-1}} + \int_{-1}^0 \frac{1}{i\sqrt{x(x-t)}} = -\ln(2) + \ln\frac{t}{t+2-2\sqrt{t+1}} \sim \ln t + \ln(2) + \mathcal{O}(t) \end{aligned} \tag{8.1.36}$$

(the constants in the expansion can be computed but they are unimportant here) This shows that

$$\tau(t) \sim 2\ln t + \ln(4) + \mathcal{O}(t). \tag{8.1.37}$$

The (unique) theta function

$$\Theta(z, \tau) = \sum_{n\in\mathbb{Z}} \exp\left(i\pi n^2 \tau\right) \tag{8.1.38}$$

Index