1 Notation.

Rings: $\widetilde{R} := R \otimes_V \overline{V}$ and $\widetilde{R}_n = \widetilde{R} \otimes_R R_n$ and $\widetilde{R}_\infty = \cup_{n\geq 0} \widetilde{R}_n$.

Galois groups: $\mathcal{G}_R = \pi_1(R[\frac{1}{p}])$, $\mathrm{G}_R = \pi_1(\widetilde{R}[\frac{1}{p}])$, $\mathrm{G}_K = \mathrm{Gal}(\overline{K}/K) = \mathcal{G}_R/\mathrm{G}_R$. Also $\mathcal{H}_R = \pi_1(R_\infty[\frac{1}{p}]), \mathrm{H}_R = \pi_1(\widetilde{R}_\infty[\frac{1}{p}])$. Moreover let $\Gamma_R = \mathcal{G}_R/\mathcal{H}_R = \mathrm{Gal}(R_\infty[\frac{1}{p}]/R[\frac{1}{p}])$ and let $\widetilde{\Gamma}_R := \mathrm{G}_R/\mathrm{H}_R = \mathrm{Gal}(\widetilde{R}_\infty[\frac{1}{p}]/\widetilde{R}[\frac{1}{p}])$.

Modules: let M be a $\mathbf{Z}_p$-representation of $\mathcal{G}_R$, then let: $\mathfrak{D}(M) := (M \otimes_{\mathbf{Z}_p} \mathbf{A})^{\mathcal{H}_R}$ be the associated (φ, Γ)-module. Let $\mathrm{D}(M) := (M \otimes_{\mathbf{Z}_p} \mathbf{A})^{H_R}$ be the module over $\widetilde{\mathbf{A}} := (\mathbf{A})^{H_R}$ as in your notes.

2 Comparison.

2.1 Definition. *Let* D *be a* $(\varphi, \widetilde{\Gamma}_R)$*-module. Define* $\mathcal{C}^\bullet(\widetilde{\Gamma}_R, \mathrm{D})$ *the complex of continuous cochains with values in* D. *Denote by* $d\colon \mathcal{C}^{n-1}(\widetilde{\Gamma}_R, \mathrm{D}) \to \mathcal{C}^n(\widetilde{\Gamma}_R, \mathrm{D})$ *the differential.*

Let $\mathcal{T}^\bullet(\mathrm{D})$ *be the mapping cone associated to* $\varphi\colon \mathcal{C}^\bullet(\widetilde{\Gamma}_R, \mathrm{D}) \to \mathcal{C}^\bullet(\widetilde{\Gamma}_R, \mathrm{D})$.

2.2 Theorem. *There is an isomorphism of* δ*-functors from* $\mathrm{H}^i(\mathrm{G}_R, \_)$ *to* $\mathrm{H}^i\big(\mathcal{T}^\bullet(\mathrm{D}(\_))\big)$ *as functors from the category of torsion* $\mathbf{Z}_p$*-representations of* $\mathcal{G}_R$ *to the category of torsion* G_K*-modules.*

Proof:

2.3 The map $\mathrm{H}^i\big(\mathcal{T}^\bullet(\mathrm{D}(\_))\big) \to \mathrm{H}^i(\mathrm{G}_R, \_)$. Let V be a torsion $\mathbf{Z}_p$-representation of $\mathcal{G}_R$. Let (α, β) be a n-cocycle in $\mathcal{T}^\bullet(\mathrm{D}(V))$ i. e., (α, β) lies in $\mathcal{C}^{n-1}(\widetilde{\Gamma}_R, \mathrm{D}(V)) \times \mathcal{C}^n(\widetilde{\Gamma}_R, \mathrm{D}(V))$ and satisfies $(-1)^n(\varphi - 1)\beta + d\alpha = 0$ and $d\beta = 0$.

Remark that $\varphi - 1$ is surjective on $\mathbf{A}$ and $\mathbf{A}^{\varphi - 1} = \mathbf{Z}_p$. Thus, $\varphi - 1$ is surjective on $\mathrm{D}(V) \otimes_{\widetilde{\mathbf{A}}} \mathbf{A} \cong V \otimes_{\mathbf{Z}_p} \mathbf{A}$ and its kernel is precisely V. We may then choose $\gamma \in \mathcal{C}^{n-1}(\mathrm{G}_R, V \otimes_{\mathbf{Z}_p} \mathbf{A})$ such that $(\varphi - 1)\gamma = \alpha$. Define

$$c_{\alpha,\beta} := (-1)^n \beta + d\gamma \in \mathcal{C}^n(\mathrm{G}_R, V \underset{\mathbf{Z}_p}{\otimes} \mathbf{A}).$$

Then, $dc_{\alpha,\beta} = 0$ and $(\varphi - 1)c_{\alpha,\beta} = (-1)^n(\varphi - 1)\beta + d(\varphi - 1)\gamma = (-1)^n(\varphi - 1)\beta + d\alpha = 0$. Thus, $c_{\alpha,\beta}$ is an n-cocycle in $\mathcal{C}^n(\mathcal{G}_R, V)$.

If we choose a γ' instead of γ then $(\varphi - 1)(\gamma' - \gamma) = 0$ so that $\gamma' - \gamma$ lies in $\mathcal{C}^{n-1}(\mathrm{G}_R, V)$. Thus, $c_{\alpha,\beta}$ changes by the coboundary $d(\gamma' - \gamma)$.

Let $(\alpha,\beta)=((-1)^{n-1}(\varphi-1)b+da,db)\in\mathcal{C}^{n-1}(\widetilde{\Gamma}_R,\mathrm{D}(V))\times\mathcal{C}^n(\widetilde{\Gamma}_R,\mathrm{D}(V))$ be a n-coboundary in $\mathcal{T}^\bullet(\mathrm{D}(V))$. Let γ' in $\mathcal{C}^{n-1}(\mathrm{G}_R,V\otimes_{\mathbf{Z}_p}\mathbf{A})$ be such that $(\varphi-1)\gamma'=a$. Let $\gamma:=(-1)^{n-1}b+d\gamma'$. Then, $c_{\alpha,\beta}=(-1)^n db+(-1)^{n-1}db+d^2\gamma'=0$.

We thus get a G_K-equivariant map, functorial in V:

$$\rho_i\colon \mathrm{H}^i\big(\mathcal{T}^\bullet(\mathrm{D}(V))\big)\longrightarrow \mathrm{H}^i(\mathrm{G}_R,V).$$

2.4 The maps ρ_i are isomorphism. Consider the exact sequence

$$0\longrightarrow V\longrightarrow \mathrm{D}(V)\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A}\xrightarrow{\varphi-1}\mathrm{D}(V)\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A}\longrightarrow 0$$

of $\mathcal{G}_R$-modules. Recall that $\mathrm{D}(V)\otimes_{\widetilde{\mathbf{A}}}\mathbf{A}\xrightarrow{\sim}V\otimes_{\mathbf{Z}_p}\mathbf{A}$. We consider cochains of G_R with values in $V\otimes_{\mathbf{Z}_p}\mathbf{A}$ which (1) are continuous for topology induced on $V\otimes_{\mathbf{Z}_p}\mathbf{A}$ from the natural topology on $\mathbf{A}$, see [????], and (2) when restricted to H_R they are continuous for the discrete topology on $V\otimes_{\mathbf{Z}_p}\mathbf{A}$. We then get an exact sequence

$$\mathrm{H}^{n-1}(\mathrm{G}_R,\mathrm{D}(V)\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A})\xrightarrow{\delta}\mathrm{H}^n(\mathrm{G}_R,V)\to\mathrm{H}^n(\mathrm{G}_R,\mathrm{D}(V)\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A})\xrightarrow{\varphi-1}\mathrm{H}^n(\mathrm{G}_R,\mathrm{D}(V)\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A}).$$

Thanks to 2.5 the inflation map $\mathrm{Inf}_n\colon\mathrm{H}^n(\widetilde{\Gamma}_R,\mathrm{D}(V))\to\mathrm{H}^n(\mathrm{G}_R,\mathrm{D}(V)\otimes_{\widetilde{\mathbf{A}}}\mathbf{A})$ is an isomorphism. Since $\mathcal{T}^\bullet(\mathrm{D})$ is a mapping cone, we get an exact sequence

$$\mathrm{H}^{n-1}(\widetilde{\Gamma}_R,\mathrm{D}(V))\longrightarrow\mathrm{H}^n(\mathcal{T}^\bullet(\mathrm{D}(V))\longrightarrow\mathrm{H}^n(\widetilde{\Gamma}_R,\mathrm{D}(V))\xrightarrow{\varphi-1}\mathrm{H}^n(\widetilde{\Gamma}_R,\mathrm{D}(V)).$$

We claim that the two exact sequences are compatible via ρ_n and the inflation maps Inf_n and Inf_{n-1}. Indeed, take a cocycle τ in $\mathcal{C}^{n-1}(\widetilde{\Gamma}_R,\mathrm{D}(V))$. One constructs $\delta\big(\mathrm{Inf}_{n-1}(\tau)\big)$ as follows. Take $\gamma\in\mathcal{C}^{n-1}(\mathrm{G}_R,\mathrm{D}(V)\otimes_{\widetilde{\mathbf{A}}}\mathbf{A})$ such that $(\varphi-1)\gamma=\mathrm{Inf}_{n-1}(\tau)$. Then, γ satisfies conditions (1) and (2) above. Indeed, since V is torsion, $\mathrm{Inf}_{n-1}(\tau)$ viewed as a cochain for H_R factors via a finite quotient G of H_R i. e., has values in $V\otimes_{\mathbf{Z}_p}A'$ for a finite étae and Galois extension $\widetilde{\mathbf{A}}\subset A'$ of group G. Then, inverse image of $V\otimes_{\mathbf{Z}_p}A'$ via $(\varphi-1)$ lies in $V\otimes_{\mathbf{Z}_p}A''$ with $A'\subset A''$ étale and Galois with group $\mathbf{Z}/p\mathbf{Z}$. Furthermore, $\delta\big(\mathrm{Inf}_{n-1}(\tau)\big)=d\gamma$. Hence, $\delta\big(\mathrm{Inf}_{n-1}(\tau)\big)$ coincides with the image of τ via ρ_n as wanted.

We then deduce that the map $\rho_n\colon\mathrm{H}^n\big(\mathcal{T}^\bullet(\mathrm{D}(V))\big)\longrightarrow\mathrm{H}^n(\mathrm{G}_R,V)$ is an isomorphism. We conclude proving the following:

2.5 Lemma. *Let V be a torsion $\mathbf{Z}_p$-representation of $\mathcal{G}_R$. Then,*

$$\mathrm{H}^i(\mathrm{H}_R,\mathrm{D}(V)\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A})=\begin{cases}\mathrm{D}(V) & \text{if } i=0\\ 0 & \text{otherwise}\end{cases}.$$

Proof: Note that $\mathrm{D}(V)\otimes_{\widetilde{\mathbf{A}}}\mathbf{A}\cong V\otimes_{\mathbf{Z}_p}\mathbf{A}$. By devissage we may assume $pV=0$ so that $\mathrm{D}(V)$ is a free $\widetilde{\mathbf{A}}/p\widetilde{\mathbf{A}}$-module of rank equal to the dimension of V as $\mathbf{F}_p$-vector space. Note that $\mathbf{A}/p\mathbf{A}=\mathbf{E}$ and $\widetilde{\mathbf{A}}/p\widetilde{\mathbf{A}}=\mathbf{E}_R\otimes_{\mathbf{E}_V}\mathbf{E}_V^{\mathrm{sep}}$; see [????]. Since we are interested in the H_R-cohomology groups of $\mathrm{D}(V)\otimes_{\widetilde{\mathbf{A}}}\mathbf{A}$, it suffices to prove that

$$\mathrm{H}^i\left(\mathrm{H}_R,\mathbf{E}\right)=\begin{cases}\mathbf{E}_R\otimes_{\mathbf{E}_V}\mathbf{E}_V^{\mathrm{sep}} & \text{if } i=0\\ 0 & \text{otherwise}\end{cases}.$$

The equality $\mathbf{E}^{\mathrm{H}_R}=\mathbf{E}_R\otimes_{\mathbf{E}_V}\mathbf{E}_V^{\mathrm{sep}}$ follows from [???]. We are left to prove that $\mathrm{H}^i\left(\mathrm{H}_R,\mathbf{E}\right)$ is zero for $i\geq 1$.

Since we are computing continuous cohomology where the topology on $\widetilde{\mathbf{E}}$ is the π_0-adic topology, the natural map

$$\lim_{n\leftarrow\infty}\mathrm{H}^i\left(\mathrm{H}_R,\frac{1}{\pi_0^n}\mathbf{E}^+\right)\longrightarrow\mathrm{H}^i\left(\mathrm{H}_R,\mathbf{E}\right)$$

is surjective. Since multiplication by π_0 is an isomorphism on $\mathbf{E}$, it suffices to prove that $\mathrm{H}^i\left(\mathrm{H}_R,\mathbf{E}^+\right)$ is π_0-torsion for $i\geq 1$. Recall that $\mathbf{E}^+$ is the direct limit $\lim\overline{B}$ over all normal and finite extensions $\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}\subset\overline{B}$ which are étale and Galois after inverting π_0 and H_R is then identified with $\mathrm{Gal}\big(\mathbf{E}/\mathbf{E}_R\otimes_{\mathbf{E}_V}\mathbf{E}_V^{\mathrm{sep}}\big)$; see [????]. For any such extension $\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}\subset\overline{B}$ we denote by $\mathrm{G}_{\overline{B}}$ the Galois group. Since we are considering cochains for H_R which factor via finite quotients of H_R, the inflation maps induce an isomorphism

$$\lim_{\overline{B}}\mathrm{H}^i\left(\mathrm{G}_{\overline{B}},\overline{B}\right)\xrightarrow{\sim}\mathrm{H}^i\big(\mathrm{H}_R,\mathbf{E}^+\big).$$

We are reduced to prove that that $\mathrm{H}^i\left(\mathrm{G}_{\overline{B}},\overline{B}\right)$ is π_0-torsion for $i\geq 1$. Since the extension $\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}\subset\overline{B}$ is finite and étale after inverting π_0, there exists $\ell\in\mathbf{N}$ such that $\pi_0^\ell\mathfrak{e}\in\overline{B}\otimes_{\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}}\overline{B}$. Consider the $\overline{B}\left[\mathrm{G}_{\overline{B}}\right]$-submodule

$$T_{\overline{B}}\hookrightarrow\overline{B}\underset{\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}}{\otimes}\overline{B}$$

generated by $\pi_0^\ell\mathfrak{e}$ (the structure as $\overline{B}$-module is given identifying $\overline{B}\subset\overline{B}\otimes_{\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}}\overline{B}$ via $a\mapsto a\otimes 1$). Then,

$$T_{\overline{B}}\cong\overline{B}\left[\mathrm{G}_{\overline{B}}\right]\cong\mathrm{Hom}_{\mathbf{E}_R^+\otimes_{\mathbf{E}_V^+}\mathbf{E}_V^{+,\mathrm{sep}}}\left(\mathbf{E}_R^+\underset{\mathbf{E}_V^+}{\otimes}\mathbf{E}_V^{+,\mathrm{sep}}\left[\mathrm{G}_{\overline{B}}\right],\overline{B}\right).$$

In particular, $T_{\overline{B}}$ has trivial $\mathrm{G}_{\overline{B}}$-cohomology. Since $\sum_{g\in\mathrm{G}_{\overline{B}}}g(\pi_0^\ell\mathfrak{e})=\pi_0^\ell\sum_{g\in\mathrm{G}_{\overline{B}}}g(\mathfrak{e})=\pi_0^\ell\cdot 1$, we have a $\mathrm{G}_{\overline{B}}$-equivariant inclusion

$$\pi_0^\ell\overline{B}\hookrightarrow T_{\overline{B}}.$$

The composite with the multiplication map $T_{\overline{B}} \subset \overline{B} \otimes_{\mathbf{E}_R^+ \otimes_{\mathbf{E}_V^+} \mathbf{E}_V^{+,\mathrm{sep}}} \overline{B} \to \overline{B}$ is the inclusion $\pi_0^\ell \overline{B} \subset \overline{B}$. Thus, multiplication by π_0^ℓ annihilates $\mathrm{H}^i\big(\mathrm{G}_{\overline{B}}, \overline{B}\big)$ for every $i \geq 1$. The lemma follows.