Orthonormal polynomials

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Consider the initial polynomials $\{1, x, x^2, x^3, \ldots\}$, a region D of $\Re$, and a weight function $w(x) \geq 0$. The aim is to begin with this initial set and from it to construct an orthonormal set of polynomials $\{p_0, p_1, p_2, \ldots\}$. We work with the inner product defined by

$$(f, g) = \int_D f(x)g(x)w(x)dx$$

and use the Gram-Schmidt procedure. As is often done with this method, we first construct orthogonal polynomials $\{q_0, q_1, q_2, \ldots\}$, and then find the normalization constants $\{c_0, c_1, c_2, \ldots\}$ so that each $p_i = c_i q_i$ has norm 1, that is to say

$$\|p_i\| = (p_i, p_i)^{\frac{1}{2}} = 1.$$

If the region D is a 'symmetric' interval $[-a, a]$ and $w(x)$ is a symmetric function, then the even and odd polynomials are automatically orthogonal to each other and we can perform Gram-Schmidt separately in the even and odd spaces.

If we ignore the possibility of even-odd separation, we can describe the process in general as follows. Let $q_0(x) = 1$, and $c_0^2 = (1, 1)^{-1} = (\int_D w(x)dx)^{-1}$, then $p_0 = c_0 q_0.$. We now let $q_1(x)$ be the next term of the original set, with the part of it parallel to p_0 subtracted off, that is to say (with a slight abuse of the notation):

$$q_1(x) = x - (x, p_0)p_0(x).$$

We then compute the corresponding normalization constant $c_1^2 = (q_1, q_1)^{-1}$, and finally we have $p_1(x) = c_1 q_1(x)$. If we describe one more step, the whole process will be clear.

$$q_2(x) = x^2 - (x^2, p_1)p_1(x) - (x^2, p_0)p_0(x); \quad c_2^2 = (q_2, q_2)^{-1}, \quad p_2(x) = c_2 q_2(x).$$

As already mentioned, in symmetric cases, the even and odd spaces can be treated separately. Also, it is not necessary to normalize at each stage. There are many well-studied special cases such as: Legendre $[\,w(x) = 1, D = [-1, 1]\,]$, Hermite $[\,w(x) = e^{-x^2}, D = \Re]$, Laguerre $[\,w(x) = e^{-x}, D = [0, \infty)\,]$. More examples may be found in mathematical handbooks or, for example, in *Methods of Mathematical Physics* Vol. I by R. Courant and D. Hilbert. **Note:** A set of *functions* $\{u_i\}$ that are orthonormal with respect to the unweighted inner product $(f, g) = \int_D f(x)g(x)dx$ is given immediately by the recipe $u_i(x) = p_i(x)w(x)^{\frac{1}{2}}$.