Some notes on derivatives (23 October 2003)

(a) The derivative as a limit The usual definition of the derivative $f'(x)$ of the function $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1}$$

We get the derivative at a particular point $x = a$ by replacing x by a in (1). Thus

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{2}$$

We can also choose to write h as $h = x - a$, in which case we have from (2)

$$f'(a) = \lim_{x \to a} \frac{f(a + (x-a)) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{3}$$

The new expression (3) is sometimes useful, and it leads to nice examples.

[**a1**] Suppose in (3) we take $f(x) = cos(x)$, and $a = 3\pi$, Then, since $f(a) = cos(3\pi) = -1$, we have in this case

$$f'(a) = cos'(3\pi) = \lim_{x \to 3\pi} \frac{cos(x) + 1}{x - 3\pi} = 0. \tag{4}$$

The result is zero because $cos'(x) = -sin(x)$, so $cos'(3\pi) = -sin(3\pi) = 0$.

[**a2**] Suppose in (3) we take $f(x) = \ln(x)$, and $a = 1$, then (since $ln(1) = 0$)

$$f'(a) = \ln'(1) = \lim_{x \to 1} \frac{\ln(x) - \ln(1)}{x - 1} = \lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1. \tag{5}$$

The result is 1 because $ln'(x) = 1/x$, and hence $\ln(1) = 1$. Since the limit is

1, we know that near $x=1$, $\ln(x)$ 'looks like' its tangent line $y=x-1$.

(b) Some logarithmic deriviatives The basic result $y=e^x \Rightarrow y'=e^x$ leads immediately (via the chain rule) to examples such as $y=e^{bx} \Rightarrow y'=be^x$, and $y=e^{\cos(x)} \Rightarrow y'=-\sin(x)e^{\cos(x)}$. For $y=a^x$, we can write $y=\exp(x\ln(a)) \Rightarrow y'=\ln(a)\exp(x\ln(a))=\ln(a)a^x$. However if, in this example, a also depends on x, we have to think again. One very effective method to treat such cases is first to 'take logs', as shown in the following examples:

[b1] Suppose more generally that $y=f(x)^{r(x)}$, then $\ln(y)=r(x)\ln f(x)$, and we have $y'/y=r'\ln(f)+rf'/f$. Consequently we have

$$y=(f)^r \quad \Rightarrow \quad y'=(f)^r\left(r'\ln(f)+\frac{rf'}{f}\right).$$

When 'towers' of exponentiation are involved, one can take logs repeatedly, until a simple form is reached. It is first necessary to recall the elementary result $y=\ln(\ln(x))$ for which $y'=1/(x\ln(x))$, by the chain rule.

[b2]

$$y=2^{3^{x^2}} \quad \Rightarrow \quad \ln(y)=\ln(2)3^{x^2} \quad \Rightarrow \quad \ln(\ln(y))=\ln(\ln(2))+\ln(3)x^2.$$

Upon differentiation, the constant term vanishes and we first obtain

$$y=2^{3^{x^2}} \quad \Rightarrow \quad \frac{y'}{y\ln(y)}=2x\ln(3)$$

and finally

$$y'=\ln(2)\ln(3)2x3^{x^2}2^{3^{x^2}}.$$

One can get the same answer by using the chain rule repeatedly; taking logs twice may be a little 'safer'. The temptation to involve the number 8 must be resisted.

(c) An nth derivative formula Suppose $f(x) = \sqrt{x}$, and we seek the nth derivative $f^{(n)}(x)$. We first make a few steps, and then try to generalize.

$$f'(x) = \left(\frac{1}{2}\right) x^{-\frac{1}{2}}$$

$$f''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) x^{-\frac{3}{2}}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-\frac{5}{2}}$$

$$\ldots$$

$$f^{(n)}(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2n-3}{2}\right)\ x^{-\frac{2n-1}{2}}$$

$$f^{(n)}(x) = (-1)^{n-1}\,\frac{1\times 3\times 5\times \ldots \times (2n-3)}{2^n}\ x^{-n+\frac{1}{2}}, \quad n \geq 2. \qquad (c1)$$

This general formula works nicely for $n > 1$, and this is satisfactory since we don't need a formula for $n = 1$. The idea to replace the product in (c1) by a ratio of factorial functions and powers of 2 is caprice made here merely 'for fun'. We now work on this product: the difficulty is to count correctly so that the products begin and end as they should.

$$1\times 3\times \ldots \times (2n-3) = \frac{1\times 2\times 3\times 4\times \ldots \times (2n-3)}{2\times 4\times \ldots \times (2n-4)}$$

$$= \frac{(2n-3)!}{2^{n-2}\times 1\times 2\times \ldots \times (n-2)}$$

$$= \frac{(2n-3)!}{2^{n-2}(n-2)!}$$

Replacing the sum in (c1) we find finally

$$f^{(n)}(x) = (-1)^{n-1}\,\frac{(2n-3)!}{2^{2n-2}(n-2)!}\ x^{-n+\frac{1}{2}}, \quad n \geq 2.$$

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