Math 380 Sec AA Final Exam April 1996

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Instructions: *Please answer all 5 questions, which carry equal marks. Calculators are permitted. Lined booklets are required.*

1. Consider the Euclidean space E^3 along with its tangent spaces $T_p(E^3)$. Suppose $f = ze^{xy}$ is a real-valued function, $\phi = x^2dx + xzdz$ is a 1-form, $W = (y-x)U_2 - zx^2U_3$ is a vector field, $\mathbf{p} = (2, 0, -1) \in E^3$, and $\mathbf{v} = (2, -1, 1)$ is a tangent vector at $\mathbf{p}$.
   (a) Find the value of $\phi(\mathbf{v})$, and also an expression for the real-valued function $\phi(W) : E^3 \to R$.
   (b) Find the 1-form $\psi = df$ and the value $\psi(\mathbf{v})$.
   (c) Find the wedge product $\phi \wedge \psi$.
   (d) Find a *general* expression for $d(g\theta)$, where g is a function and θ is a 1-form. In particular find $d(f\phi)$.
   (e) Find an expression for the the vector field $V = (\nabla_W \nabla f)$ and also the value $V(\mathbf{p})$.

2. Suppose that $F : E^3 \to E^3$ is the mapping $F = T_{\mathbf{a}} \circ C$, where $T_{\mathbf{a}}(\mathbf{p}) = \mathbf{a} + \mathbf{p}$, $C(\mathbf{p}) = \mathbf{Cp}$, $\mathbf{a}$ is a fixed vector, and $\mathbf{C}$ is a fixed orthogonal matrix.
   (a) Define an *isometry* in E^3 and prove that F is an isometry.
   (b) Define the derivative of a mapping; in particular find the derivative F_* of F in the case that C represents a rotation by θ about the x-axis.
   (c) Let α be a curve in E^3. Suppose that $\kappa \neq 0$ and that the ratio $\tau/\kappa =$ constant. Describe the curves α and $\beta = F(\alpha)$. Justify your answers.

3. Consider a unit-speed curve α in E^3 with $\kappa > 0$ and $\tau \neq 0$. If α lies on a sphere centre $\mathbf{c}$ of radius r, prove the following:
   (a) $\alpha - \mathbf{c} = -\rho N - \rho' \sigma B$, where $\rho = 1/\kappa$ and $\sigma = 1/\tau$.
   (b) $(r\kappa)^2 = 1 + (\rho' \sigma \kappa)^2$.

4. Consider $M = \mathbf{x}(D) \subset E^3$ defined by the single Monge patch $\mathbf{x}(D) = (u, v, uv)$, where $D = \{(u,v) \in E^3 | \quad |u+v| + |u-v| < 2\}$. Suppose that the function f is defined by $f = z\cos(x+y)$ restricted to M.
   (a) Find the derivative $\mathbf{x}_*$ and show that $\mathbf{x}$ is a proper patch. Conclude that M is a surface in E^3.
   (b) Find the coordinate expression $\mathbf{x}^* f : D \to \mathbb{R}$ for f.
   (c) Define the coordinate expression $\mathbf{x}^* \phi$ of a 1-form ϕ on M. In particular find the coordinate expression $\mathbf{x}^* df$ for the differential df on M.

5. Prove *one* of the following two theorems:
   (a) If α and β are unit speed curves in E^3 and $\kappa_\alpha = \kappa_\beta$ and $\tau_\alpha = \pm \tau_\beta$, then α and β are congruent, that is to say, there exists an isometry F such that $\beta = F(\alpha)$.
   (b) Stokes's Theorem: if ϕ is a 1-form on a surface M, and $\mathbf{x} : R \to M$ is a 2-segment, then
$$\int\int_{\mathbf{x}} d\phi = \int_{\partial \mathbf{x}} \phi.$$